ESSAYS IN REGIME SWITCHING POLICY AND ADAPTIVE LEARNING IN DYNAMIC STOCHASTIC GENERAL EQUILIBRIUM

by

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A DISSERTATION

Presented to the Department of Economics
and the Graduate School of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

June 2018

DISSERTATION APPROVAL PAGE

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Title: Essays in Regime Switching Policy and Adaptive Learning in Dynamic Stochastic General Equilibrium

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Degree awarded June 2018.

DISSERTATION ABSTRACT

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This dissertation studies monetary-fiscal policy interactions and adaptive learning applications in regime-switching DSGE models. A common thread through my research is understanding how policymakers may be affected by the interaction of policy regime change and agents' beliefs about past, current or future policy in general equilibrium. The work I present in this dissertation shows that conventional and unconventional policy outcomes, as well as the existence, uniqueness and expectational stability of rational expectations solutions, depend heavily on the expectational effects of time-varying policy. These findings suggest that uncertainty over future fiscal policy may curb the effectiveness of monetary policy, or otherwise constrain the actions of central bankers. In carrying out this research agenda, my work also examines the relationship between determinacy and expectational stability in a general class of Markov-switching DSGE models.

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ACKNOWLEDGEMENTS

I thank Professors George Evans, Bruce McGough and Jeremy Piger for their assistance in the preparation of this manuscript, and for their guidance over the years. I also thank Professors Shankha Chakraborty and David Evans for helpful comments and advice. This research was supported in part by a Kleinsorge Research Fellowship at the University of Oregon Department of Economics.

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CHAPTER I

INTRODUCTION

This dissertation studies monetary-fiscal policy interactions and adaptive learning applications in regime-switching DSGE models. A common thread through my research is understanding how policymakers may be affected by the interaction of policy regime change and agents' beliefs about past, current or future policy in general equilibrium. The work I present in this dissertation shows that conventional and unconventional policy outcomes, as well as the existence, uniqueness and expectational stability of rational expectations solutions, depend heavily on the expectational effects of time-varying policy. These findings suggest that uncertainty over future fiscal policy may curb the effectiveness of monetary policy, or otherwise constrain the actions of central bankers. In carrying out this research agenda, my work also examines the relationship between determinacy and expectational stability in a general class of Markov-switching DSGE models.

Chapter 2 of my dissertation generalizes McCallum (2007) and is the first to address the relationship between determinacy and E-stability in Markov-switching Dynamic Stochastic General Equilibrium (MS-DSGE) models with lagged endogenous variables. I prove that the sufficient conditions for determinacy in Cho (2016) imply the E-stability of the forward solution in MS-DSGE models with lagged endogenous variables when agents condition their expectations of future endogenous variables on current endogenous and exogenous variables. The class of models studied in this paper is very general, and nests a wide array of models that are frequently studied in modern macroeconomics.

In Chapter 3, I study the impact of expansionary forward guidance in a simple New Keynesian model with recurring or permanent active fiscal policy. This work addresses and offers a potential solution to the simple New Keynesian model's prediction that expansionary forward guidance can generate an implausibly large stimulus. I find that the introduction of permanent or recurring active fiscal policy dampens the response of output and inflation to forward guidance in the New Keynesian model. Moreover, the presence of regime-switching policy introduces expectational effects that cause forward guidance to be less stimulative in our regime-switching model's active money, passive fiscal policy regime. Finally, the introduction of long-term debt affects the magnitude of the stimulus resulting from forward guidance in models with active fiscal policy.

In Chapter 4, I explore determinacy and E-stability in a New Keynesian model with switching fiscal and monetary policy. Here I present three categories of results. First, the maturity structure of government debt matters for determinacy and the existence of stable equilibria in our switching model, which is not true in the analogous fixed coefficient model. I use two numerical solution techniques to show that maturity affects both the multiplicity of stable solutions, and the existence of sunspot equilibria. Second, determinacy generally implies E-stability when agents do not observe contemporaneous observable variables, but not for certain arguably unrealistic regions of the model parameter space. Third, this chapter presents conditions for stability under infinite-horizon learning in Markov-switching DSGE models and compares stability under infinite horizon and one-step-ahead learning. To the best of my knowledge, this is the first paper to derive these stability conditions in a model with switching coefficients.

Finally, Chapter 5 examines the performance and robustness of simple monetary policy rules in models with learning agents subject to: (1) permanent or occasionally active fiscal policy; and/or (2) the presence of long-term government debt. My analysis indicates that the "global" response of the fiscal policymaker to debt determines the optimal monetary policy response. When fiscal policy is globally passive or globally active the optimal monetary policy rule typically features time-invariant coefficients with high inflation reaction coefficients in globally passive models and interest rate pegs in globally active models. In cases where fiscal policy features balanced or strong switching between active and fiscal policy stances, the optimal monetary policy rule features switching coefficients. These results extend to models with adaptive learning, including a hidden Markov model of learning never seen before in the literature.

CHAPTER II

E-STABILITY VIS-A-VIS DETERMINACY IN MS-DSGE MODELS

Introduction

Under rational expectations, a given macroeconomic model may have multiple equilibria. When studying models that admit a multiplicity of rational expectations equilibria, researchers must confront the issue of equilibrium selection: which, if any, of the equilibria are economically reasonable? To that end, the notion of "determinacy" is used extensively as a selection criterion. When a model is determinate only one equilibrium exists, and this eliminates the need to choose between equilibria. Alternative criteria advance robustness to bounded rationality as a means of selecting plausible equilibria. In the adaptive learning literature the "learnability" or E-stability of a specific equilibrium is viewed as a criterion for the selection of a rational expectations equilibrium. Under adaptive learning, rational expectations are replaced with a forecasting model that assumes the same functional form as the equilibrium law of motion, and agents are assumed to update the parameters of their model each period. If, in the limit, agents' parameter estimates converge to the parameters consistent with a given rational expectations equilibrium, then this equilibrium is said to be "E-stable" or "stable under learning."

Despite its popularity, determinacy has some weaknesses as an equilibrium selection criterion. First, determinacy only guarantees the existence of a unique stable rational expectations solution, and it is well known that determinate models may have explosive equilibria (see Cochrane (2007)). Second, determinacy does not explain how agents coordinate on a unique rational expectations equilibrium. The E-stability criterion mitigates these two problems as follows. First, the determinate equilibrium, but not explosive solutions in determinate models, tend to be stable under learning as shown by McCallum (2009). Second, learning explicitly examines how and when boundedly agents can learn to become rational forecasters and therefore coordinate on a rational expectations equilibrium. Because an E-stable determinate equilibrium is robust to these weaknesses of determinacy as a selection criterion, it is important to study and characterize the relationship between determinacy and E-stability.

The relationship between determinacy and E-stability has been extensively explored in general classes of linear models with different assumptions about agents' information sets and the horizons over which agents form expectations.[1] In models that feature Markov-switching parameters, relatively little is known about determinacy and E-stability, and for the following reasons: (1) tractable necessary conditions for determinacy in Markov-switching models are not known; (2) variation in the set of current and past Markov states that agents use when forming expectations generates distinct classes of equilibria. As shown in Branch, Davig, and McGough (2013), this second limitation makes it hard to establish whether an equilibrium is unique to the model, or merely to its class of equilibria. Despite these limiting factors, some research has successfully isolated general relationships between determinacy and E-stability in purely-forward looking Markov-switching rational expectations (MS-DSGE) models.

In this paper, we explore the relationship between determinacy and E-stability in a very general class of Markov-switching rational expectations models with lagged endogenous variables. Specifically, we demonstrate that a set of tractable sufficient conditions for determinacy from Cho (2016) imply the learnability of the unique

[1]The following papers, among others, have studied the relationship between determinacy and E-stability: McCallum (2007, 2009), Cochrane (2009), Ellison and Pearlman (2011), Bullard and Eusepi (2014)

mean-square stable rational expectations solution if agents know current endogenous variables when forming one-period-ahead expectations. This result contributes to the relevant literature in three ways. First, this result extends McCallum (2007), which finds that determinacy implies E-stability in a general class of linear rational expectations models. Second, this is the first study to explore the relationship between determinacy and E-stability in MS-DSGE models where agents know contemporaneous variables. Third, our result applies to models with lagged endogenous variables, whereas previous research focused exclusively on classes of purely-forward looking models.

The paper is organized as follows: first, we review the literature on determinacy and E-stability in linear and Markov-switching models, as well as papers from the fiscal theory of the price level literature that are relevant for applications; second, we define the class of models, model equilibria and determinacy and E-stability conditions under consideration; third, we provide the main analytical result; fourth, we present applications to models of monetary-fiscal policy interactions with Markov-switching policy parameters; finally, we conclude.

Brief Literature Review

Determinacy and E-stability in LRE and MS-DSGE Models

This section explores three strands of the literature surrounding the connection between determinacy and E-stability in linear and Markov-switching rational expectations models. First, a robust literature examines the relationship between determinacy and E-stability in linear rational expectations (LRE) models. Second, a recent research program seeks both necessary and sufficient conditions for determinacy in rational expectations models with Markov-switching. Third, a couple of papers have explicitly examined the connection between determinacy and E-stability in MS-DSGE models. We conclude with a review of the fiscal theory of the price level literature that emphasizes New Keynesian models with Markov-switching monetary and fiscal policy parameters and establishes a foundation for the applications in 2.5.

McCallum (2007) offers what is perhaps the most general analytical connection between determinacy and E-stability. The paper employs a broad class of linear rational expectations models with lagged endogenous variables. These models are populated with agents who observe contemporaneous endogenous and exogenous variables when forecasting tomorrow's endogenous variables, and who utilize a forecasting model that shares a functional form with the minimal state variable solution. In these settings, necessary and sufficient conditions for determinacy imply that the determinate equilibrium is stable under learning. This result does not condition on any other assumptions [2], but is not robust to alternative assumptions about agents' information sets. For example, McCallum is not able to isolate restrictions that guarantee the abovementioned implication when agents are unable to observe contemporaneous endogenous variables–an issue that Ellison and Pearlman (2011) further studies.

Ellison and Pearlman (2011) provides a "completely general" link between determinacy and E-stability by requiring agents to use a saddlepath learning rule. Their approach departs from McCallum's in one notable way: agents' forecasting model assumes the same form as the saddlepath relationship. Econometrically, this approach imposes zeros on the coefficients on lags of non-predetermined variables in the regression model used in McCallum (2007). When agents know time-t variables, the same relationship between determinacy and E-stability holds with either learning

[2] Except for a few regularity assumptions that McCallum dismisses as innocuous

rule. However, the saddlepath learning rule requires agents to condition future endogenous variables on estimates of current predetermined variables. This restriction, they argue, ensures that the unique solution in any determinate model will also be E-stable when agents don't know the time-t endogenous variables (i.e. only know the time-t exogenous variables). Moreover, Ellison and Pearlman prove that only one minimal state variable solution will be "iteratively" E-stable in indeterminate models when agents use a saddlepath learning rule. This notion of "iterative" E-stability is a discretization of the E-stability conditions in Evans and Honkapohja (2001), and without it, McCallum could not obtain general E-stability results in indeterminate models.

Bullard and Eusepi (2014) extends Ellison and Pearlman (2011) and McCallum (2007) in two directions: (1) they allow for richer lag structures in the information set of agents; (2) they permit agents to form expectations over infinite-horizons. Their results are clear: determinacy does not generally imply E-stability under infinite-horizon (IH) learning or finite-horizon (FH) learning. In particular, the presence of delays in the information set breaks the McCallum (2007) relationship between determinacy and E-stability in models of finite-horizon learning. They illustrate this using a simple New Keynesian model with Calvo pricing, a cash-in-advance constraint and a basic Taylor-type interest rate rule. When the model under study is determinate and agents do not know contemporaneous endogenous variables and the monetary policy rule, the unique solution is generally E-unstable. Abstracting from the New Keynesian setting, Bullard and Eusepi identify additional sources of E-instability in determinate models. For instance, they find that IH and FH learning rules give us stability conditions that differ only in magnitude, with magnitude being determined by "discount factors" that "capture reduced-form discounting of future variables in the equilibrium dynamics of the model." Due to these differences in magnitude, a change in the underlying decision rule may render an E-stable solution E-unstable and vice-versa. As with the papers before, these results apply to linear DSGE models.

Attempts to study determinacy and E-stability in MS-DSGE models have been stymied, in part, by a lack of useful necessary and sufficient conditions for determinacy. However, numerous papers have made great strides in the direction of these conditions. These papers are best categorized by the concept of stability they employ. Papers by Davig and Leeper (2007) and Branch, Davig and McGough (2013) use the familiar concept of bounded stability and identify conditions under which a purely-forward looking model is determinate.[3] In the other camp, Farmer, Waggoner, and Zha [FWZ] (2009, 2011) and Cho (2016) use mean-square stability–a notion of stability often used in engineering–to formulate their own conditions for determinacy.

One of the earliest papers in this literature, Davig and Leeper (2007), studies a standard New Keynesian model with a Markov-switching Taylor-type interest rate rule, and finds that monetary policy can occasionally switch into an indeterminate ("passive") regime without generating a multiplicity of stable equilibria. This occurs provided that monetary policy satisfies the "long-run Taylor principle" (LRTP), which allows monetary policy to be *sometimes* passive if the Taylor principle is satisfied sufficiently often. While this result by itself has significant policy implications, the methods they employ constitute a separate contribution to the MS-DSGE literature. In particular, they demonstrate how to render the original non-linear system of equations linear by "conditioning the structural form of the model" on the underlying Markov state. When the model is written in this form, a straightforward application of the

[3]Determinate in the sense that a unique regime-dependent equilibrium (RDE) exists, where a RDE is an equilibrium in which agents condition expectations on the current Markov state, and no lags of the Markov state.

Blanchard and Khan (1980) conditions yields the LRTP. Branch, Davig and McGough (2013) develops an analogous condition in a more general purely-forward looking model, that they call the "conditionally linear determinacy condition" or CLDC. The CLDC is easy to use, and generalizes conditions for determinacy in LRE models to a class of MS-DSGE model, but it does not apply to models with lagged endogenous variables. Additionally, Branch, Davig and McGough (2013) are able to show that the CLDC only guarantees the existence of a unique regime-dependent equilibrium; if agents condition expectations on past states of the economy, they may generate additional "history-dependent equilibria" when indeterminate regimes exhibit negative feedback and the CLDC is satisfied. Similarly, FWZ (2010) identifies the potential for indeterminacy in a New Keynesian model that satisfies the LRTP.

Because of the shortcomings of the LRTP, FWZ (2009) abandons the concept of bounded stability in favor of mean-square stability. This decision to work with mean-square stability also allowed them to borrow necessary-and-sufficient conditions for mean-square stability from a rich engineering literature on the subject. In their paper, they show how to write any rational expectations solution to a purely forward looking MS-DSGE model as the sum of a fundamentals and non-fundamentals component. The task of identifying determinacy is then twofold: (1) isolate conditions under which the fundamentals component is a mean-square stable process; (2) isolate conditions under which all non-fundamental components are mean-square unstable processes. One can complete these tasks by minimizing the spectral radius of a single matrix with respect to the dimensions spanned by the non-fundamental solutions. This amounts to identifying the full set of sunspot solutions–a task that becomes drastically more difficult as the number of states and equations increases–and restricting these solutions to be unstable. As a result, this approach is very difficult to use in practice. Moreover, the approach only applies to a class of MS-DSGE models that do not have lagged endogenous variables.

Whereas FWZ (2009) seeks to identify determinacy by finding all sunspot solutions, FWZ (2011) presents a numerical algorithm that may detect indeterminacy. Their method applies to forward-looking models with lagged endogenous variables, and is able to detect indeterminacy if the algorithm converges to more than one fundamental solution of the model under study. There is no proof that their method finds all fundamental solutions to a model. As such, their method's inability to detect multiple fundamental solutions does not preclude indeterminacy.

Foerster et al. (2016) introduces a very general, and tractable perturbation approach for finding higher-order solutions to MS-DSGE models. In their approach, they only perturb time-varying parameters that affect the model's steady state. In doing so, they perturb the smallest permissible set of time-varying parameters, and they show how to obtain reasonable approximations around the ergodic mean of the time-varying parameters that impact steady state. Foerster et al. (2016) also advocates for the use of a Gröbner basis approach to solving MS-DSGE models. The theory of Gröbner bases is particularly useful in the MS-DSGE framework because MS-DSGE equilibrium coefficients solve quadratic polynomials that typically cannot be solved using standard solution techniques, and because we can obtain the full set of minimum state variable (MSV) solutions to a MS-DSGE model using a Gröbner basis. As such, Foerster et al (2016) offers a means for researchers to study issues of uniqueness and existence in the class of MSV solutions. Importantly, their methods only study MSV solutions to the model; they do not consider sunspot solutions in their analysis.

While the approach of FWZ (2009) and (2011) are limited in terms of tractability and sufficiency, respectively, and Foerster et al. (2016) is limited with respect to its treatment of sunspot solutions, Cho (2016) offers a set of implementable sufficient conditions for determinacy and indeterminacy. Cho's approach builds on prior work by McCallum (2007) and Cho and Moreno (2011) in a manner that simplifies the task of restricting all sunspot solutions to be unstable. Unlike FWZ (2009), which requires one to solve a complicated minimization problem to identify the full set of sunspot solutions, Cho's method solves a model forward for a unique solution, and computes the spectral radii of two matrices. Cho's method is discussed in greater detail later in this chapter.

A much sparser literature examines the link between determinacy and E-stability in MS-DSGE models. The biggest contribution along these lines is Branch, Davig and McGough (2013). They prove that satisfaction of the CLDC implies E-stability of the minimal state variable solution in purely forward looking MS-DSGE models. They also obtain an intriguing result about the learnability of history-dependent equilibria: if the CLDC is satisfied, then a history-dependent equilibrium can be learned when agents condition their expectations on a "sunspot variable which captures the self-fulfilling serial correlation in the equilibrium." Reed (2015) extends their results by proving that satisfaction of the conditions for determinacy in the mean-square stability sense implies E-stability of the minimal state variable solution when agents do not know time t endogenous variables. This is because the conditions in Cho are stronger than the CLDC in purely forward looking models. The results in Reed only apply to purely forward looking models–which I attempt to expand on in this section.

Fiscal Theory of the Price Level and New Keynesian Models

MS-DSGE models commonly feature Markov-switching fiscal and monetary policy rules. This is even true to the extent that many, if not most, MS-DSGE models in the New Keynesian literature would be LRE models without Markov-switching policy rules. In this chapter, we focus on New Keynesian MS-DSGE models that allow fiscal policy and monetary policy rules to switch between various configurations of "active" and "passive" fiscal and monetary policy. Models that feature these switching rules are often associationg with the Fiscal Theory of the Price Level (FTPL) literature. To better motivate the significance of these models, we briefly introduce the FTPL and discuss some of the salient papers in this literature below.

The FTPL is developed in a host of papers that study how fiscal and monetary policy jointly determine inflation. This complements the more canonical view of inflation, which is rooted in the Quantity Theory and often abstracts away from fiscal policy by assuming Ricardian equivalence. It's important to emphasize that the FTPL complements–and not contrasts–the canonical view, because even the canonical view expresses inflation as the outcome of a joint fiscal-monetary policy configuration in which fiscal policy stabilizes the real market value of debt and does not affect prices. While the FTPL has faced considerable criticism, it raises interesting questions about the role of fiscal policy in price-level determination. In particular, the FTPL argues that inflation is a fiscal phenomenon in the absence of Ricardian fiscal policy. Given that growing evidence challenges Ricardian equivalence, the FTPL is arguably quite relevant.

The FTPL largely begins with Leeper (1991). This seminal paper introduces a dichotomy in the conduct of monetary and fiscal policy: a "passive" policymaker is "constrained to stabilize debt", whereas an "active" policymaker does not act to stabilize government debt. To formalize this distinction, Leeper identifies regions of the parameter space consistent with passive and active policymaking in a stylized endowment economy with a monetary authority that adjusts interest rates in response to inflation and a fiscal authority that adjusts lump-sum taxes in response to real debt. Since each policymaker is in one of two disjoint parameter regions, there are a total of four disjoint regions of the parameter spaces. Of these four, one yields indeterminacy (passive fiscal and monetary policy), one yields explosive solutions (active fiscal and monetary policy), and two of them yield determinacy: (1) the active monetary and passive fiscal policy configuration; (2) the active fiscal and passive monetary configuration.

Economists typically choose parameters from the first of these determinate parameter regions for conventional macroeconomic models of inflation and output. In this region, the monetary authority responds aggressively to inflation, and the fiscal policymaker executes a Ricardian policy. Because a Ricardian fiscal policy does not affect consumption and inflation, economists can ignore the fiscal authority when studying the determination of inflation and corresponding equilibrium dynamics of the model. In the second region, however, the fiscal authority does not raise enough new revenue to service the interest and pay down the principal on newly issued debt. This allows increases in debt to have first-order wealth effects that raise household consumption and inflation since forward-looking agents do not expect governments to tax away the benefits of the new debt wealth in the future. If the monetary authority responds aggressively to the run up in inflation caused by debt issuance in this environment, it will raise real interest rates, and therefore generate higher debt service costs that an active fiscal authority will, again, not pay for. This generates explosive model dynamics. A stable debt path therefore requires the monetary authority to respond weakly to inflation–that is, a monetary authority must be "accommodative." Henceforth we embrace the terminology from Leeper and Leith (2016) and refer to an active fiscal, passive monetary regime configuration as a "Regime F" configuration, and an active monetary and passive fiscal policy configuration as a “Regime M” configuration.

Woodford (1995) and Sims (1994) shed further light on the relationship between fiscal policy and inflation by studying endowment economies where monetary authorities have direct control over the money supply. In these two papers, Woodford and Sims build on Leeper (1991) by introducing an equilibrium condition sometimes referred to as a “bond valuation” equation that determines prices in relation to outstanding debt and the discounted present value of expected future real surpluses.[4] Sims (1994) show that the same holds in a similar model. Sims also uncovers more nuanced dependencies of monetary policy efficacy on fiscal policy. For example, in cases where exogenous money growth does not keep up with agents' discounting, the fiscal authority must credibly back the value of the currency or else inflationary sunspot equilibria may arise. In cases where the money supply rule targets an interest rate peg, a larger menu of fiscal policy rules–including a constant tax rate–ensure a unique, stable equilibrium.

Woodford (1998a, 1998b, 2001) are among the earlier papers to extend the Leeper (1991) framework to include endogenous output and nominal rigidities. In this framework, Woodford (1998a) clearly shows that a fiscal inflation may arise through the private behavior of households–irrespective of monetary policy. One can illustrate this point in a variety of ways, including: when nominal bond holding changes, real wealth changes against sluggish prices, and these wealth effects vary aggregate demand. This contrasts Sargent and Wallace (1981) which argues that fiscal policy causes inflation only insofar as the monetary authority monetizes runaway debt. Under

[4]In contrast, Leeper (1991) studies bounded solutions to systems of linearized equilibrium conditions. These conditions, when combined with a transversality condition, imply the bond valuation equation, but Leeper (1991) does not explicitly derive this equation

Woodford's interpretation, however, fiscal inflation is not simply the result of monetary policy, and it is this point that helps separate the fiscal theory from previous work on the relationship between inflation and fiscal policy. These papers also study fiscal-monetary policy interactions in the presence of debt ceilings, the zero lower bound, and bounded rationality. For instance, Woodford (1998a) finds that a debt ceiling of the kind proposed in the Maastrict Treaty constrains fiscal policy to be Ricardian. Woodford (2001) suggests that the post-war bond-price support regime is an example of Regime F behavior in U.S. data, as policymakers defended an interest rate peg. In this analysis he further examines the zero lower bound as a subcase of interest rate pegs.

Though Woodford (2001) describes the role of maturity structure in the fiscal theory, Cochrane (2001) thoroughly characterizes how and when variation in the maturity of debt affects the relationship between fiscal policy and inflation. Cochrane studies a simple frictionless model of consumer optimization combined with a flow constraint on fiscal policy, and a present value condition that requires the real value of debt to equal the discounted present value of expected primary fiscal surpluses. He calls this last condition the "equilibrium valuation equation" because it follows from market-clearing and no-arbitrage conditions, and is therefore true in equilibrium.[5] This condition explicitly relates fiscal policy to inflation when debt is nominal: if surpluses change exogenously, either nominal debt or price must change to restore the valuation equation. If all debt is short-term debt, then debt is predetermined, so that a variation in surpluses necessitates a price adjustment. To illustrate the complexity of this relationship in the presence of long-term debt, Cochrane engages in two comparative statics exercises. The first exercise studies the effects of debt on price. With a richer

[5]Conchrane's valuation equation generalizes the aforementioned bond valuation equation from Woodford (1995) and Sims (1994) to environments with rich maturity structures of government debt

maturity structure, debt variations cause outstanding debt to be revalued, which creates a second channel for restoring the present value condition. This implies that policymakers can trade current inflation for future inflation by lenthening the maturity structure of debt. The second exercise looks at the effects of surplus on price, and finds that price is more responsive to current surpluses as the maturity lengthens. This is because outstanding long-term debt is not a claim to the current surplus; variation in the current surplus cannot revalue outstanding debt. Similarly, variation in future surpluses revalues outstanding long-term debt, thereby reducing the need for price changes today. The paper concludes with an optimal policy exercise that minimizes the variance of inflation with respect to the scale and composition debt. Cochrane finds that longer maturities are optimal when the present value of surpluses is more volatile than current surpluses. The intuition follows from the second comparative static exercise.

The preceding works assume that policy is fixed, but a growing body of empirical research suggests that regime switches in policy are the norm. In this vein, Clarida et al (2000) identifies a regime change in monetary policy under Volcker. They estimate a forward-looking Taylor-type policy reaction function[6] combined with the assumption that the Federal Reserve partially adjusts the Funds rate to its target. They then estimate the reaction function using GMM and find that pre-Volcker policy is passive and post-Volcker policy is active. This suggests, as they demonstrate in a New Keynesian model, that the Great Moderation occurred because post-Volcker policymakers pursued stabilizing policies, while pre-Volcker regimes permitted sunspot instability. Lubik and Schorfheide (2004) helps to confirm these results.

[6]the authors point out that Taylor (1993) proposes a rule where policy is backward-looking, but if lagged inflation or a linear combination of lagged inflation and output"is a sufficient statistic for forecasting future inflation" then their rule nests the Taylor rule

While Clarida et al. (2000) offers evidence of a one-time regime change, Cogley and Sargent (2002, 2005) find evidence of drift in monetary policy rules using a random coefficients model. Evidence of recurring structural change appears in Sims and Zha (2006). They estimate a backward-looking Markov-switching model that allows for switching in monetary policy parameters and shock volatilities, and conclude that switches in shock variances explain the Great Moderation.

Davig and Leeper (2006, 2011) combine Markov-switching reduced-form estimation of policy rules and MS-DSGE models to examine the general equilibrium implication of switching in policy parameters. In the first of these two papers, they estimate a standard Markov-switching Taylor-type interest rate rule and a lump-sum transfers rule that responds to real debt, government purchases, and output. Each rule follows a two-state Markov process and they are estimated separately to determine when each authority exhibits active or passive behavior. These rules are then embedded in an on otherwise standard New Keynesian model with government purchases. We emphasize three results. First, they uncover evidence of recurring Regime F policy regimes in U.S. economic history. Second, they find that aggregate demand responds to lump-sum transfer shocks, which gives additional reason to question the assumption of Ricardian equivalence in models of the U.S. economy. In the model, for example, a $1 tax cut increases the discounted present value of output by 76 to 102 cents. Third, the U.S. can be described as a single unique equilibrium. This conclusion breaks with Clarida et. al (2000) and Lubik and Schorfheide (2004), which both suppose that the economy unexpectedly jumps between regions of determinacy, indeterminacy and explosiveness, so that agents always expect permanent regimes. When agents expect regime change, as is assumed by Davig and Leeper, the economy can temporarily pass through indeterminate and explosive regimes without undermining the determinacy of the model.

Additional attempts to extend and complement the work of Davig and Leeper come from Bianchi (2012, 2013), Bianchi and Ilut (2017), and Bianchi and Melosi (2013).[7] Bianchi (2012) and Bianchi and Ilut (2017) estimate a regime-switching New Keynesian model using Bayesian techniques and techniques from FWZ (2011) and Kim and Nelson (1999). They assume a "circular structure" for transition probabilities that forces the economy to move from Regime F to an active-active regime to either Regime M or Regime F.[8] Unlike Davig and Leeper (2006, 2011), they jointly estimate policy rules and model parameters. They find that the economy was in Regime F pre-Volcker, in the explosive regime during the early 1980s, and then transitioned into Regime M.

Bianchi and Ilut (2017) juxtaposes the impulse response functions from a similar rational expectations model and impulse response functions from a counterfactual model in which agents maintain different assumptions about the persistence and number of regimes. These comparisons of "actual" and "counterfactual" impulse response functions suggest that the magnitude of fiscal inflation under an active fiscal regime depends on the horizon over which agents expect active fiscal policy to persist. They conclude that the Great Inflation of the 1970s would not have occurred if agents either believed fiscal policy was passive, or felt that a return to passive fiscal policy was imminent. Specifically, their estimated model indicates that monetary policy switched from passive monetary to active monetary under Volcker in 1979, but fiscal policy did not switch from active to passive until 1981. Their counterfactual exercises

[7] Other papers not mentioned here include Bhattarai et al (2012), Bhattarai et al (2014), Bhattarai et al (2016), Gonzalez-Astudillo (2013), Chung et al (2007)

[8] Bianchi (2012) employs a similar circular structure that forces the economy to transition from active-active regimes to Regime M

predicts that a coordinated monetary-fiscal regime change in 1979 would have reduced inflation sooner. Similary, Bianchi (2013) conducts counterfactuals that suggest the Great Inflation might not have occurred had agents believed that the economy could have switched to a very active monetary regime ("Eagle" regime) in the 1980s and on.

These results also echo findings from Bianchi and Melosi (2013), in which policy regimes are hidden from rational agents who experience recurring deviations from passive fiscal policy to two active fiscal regimes that are identical apart from their average persistence. As agents observe more deviations from passive or "virtuous" policy regime, they grow more pessimistic about the probability of being in the more persistent regime. In this environment, the growing pessimism causes inflation to become more responsive to fiscal policy, and this means that once "dormant", persistent shocks to government debt can have creeping inflationary effects as agents become more pessimistic.

Though Bianchi and Melosi study Bayesian learning in MS-DSGE models of fiscal-monetary policy interactions, little research has studied adaptive learning in this class of models. A number of papers, however, approach adaptive learning in fixed regime models of the fiscal theory of prices. Namely, Evans and Honkapohja (2007) examines stability of the "fiscalist" and "monetarist" solutions in McCallum (2001), and the Regime M and Regime F equilibria in Leeper (1991). To this end, they employ a model of an endowment economy with constant government purchases, money in utility, and policy rules from the two aforementioned papers. They find that the explosive fiscalist equilibrium obtained in a non-stochastic model with point expectations is unstable under learning when agents use a one-period ahead learning rule. Using the same rule, the fixed price, zero-debt monetarist solution is E-stable. In the context of the Leeper (1991) model, they find that Regime M equilibria are stable while only a subregion of the parameter space consistent with Regime F equilibria yield learnable solutions. These results lend some credence to Regime F equilibria.

Evans and Honkapohja (2005) studies learning and fiscal-monetary policy interactions in the presence of a liquidity trap. They first examine a nonlinear frictionless endowment economy to demonstrate the existence of a desired high inflation steady state and a low inflation liquidity trap steady state. Under passive fiscal policy, a unique high steady state equilibrium exists, under active policy, a unique low steady state exists. The stability of these equilibria under learning are examined using two learning rules. First, agents engage in a process of steady state learning under which the high state is E-stable, and the low state is E-unstable. In the second learning model, agents use a perceived law of motion that shares a functional form with the equilibrium law of motion. Under this second learning scheme, the high state is E-stable given passive fiscal policy, and the low state is learnable when policy is active. However, the basin of attraction at the lower steady state is small, and this presents the potential for a cumulative deflation. If the economy is pushed into such a deflation, the monetary authority must pursue an aggressive policy to regain the desirable high steady state. In other words, active fiscal policy is not enough. This subject is further studied in a production economy with sticky prices in Evans, Guse and Honkapohja (2008).

Eusepi and Preston tackle a number of adaptive learning questions in New Keynesian models of monetary-fiscal policy interactions with nominal government debt. Their analysis in Eusepi and Preston (2012) shows that a restricted subset of the Regime F equilibria in a simple New Keynesian model with Leeper-style policy rules are E-stable when agents are uncertain about the prevailing monetary-fiscal policy mix. However, the removal of this uncertainty renders some of these previously E-unstable equilibria stable under learning. In Eusepi and Preston (2011), the relationship between expectational stability and the scale and composition of debt is examined. To capture scale effects, they vary the steady state level of debt, and to capture composition effects, they vary the rate of decay in the government's geometrically decaying bond portfolio. They find that models with very short or very long average maturities tend to be the most expectationally stable. This is because variation in maturity has two competing effects. First, a lengthening in maturity structure subjects debt to changes in inflation expectations through revaluation effects. This means that a shock to expectations that revalues debt may feedback to agents' expectations in a destabilizing way. Second, a lengthening in maturity reduces the percentage of outstanding debt that must be rolled over in each period. This makes it easier to finance new debt should an unexpected exogenous shock place pressure on government finances. These effects combine to make expectations least stable at medium-to-long average maturities. Across all maturities, expectations are most stable when the fiscal policy is active against an interest rate peg.

For more information on the FTPL, consult Leeper and Leith (2016). Other relevant papers not introduced in this section are discussed in Chapters 3 and 5 of this dissertation.

MS-DSGE Models

General Class of MS-DSGE Models

In this paper, we consider Markov-Switching Rational Expectations (MS-DSGE) models of the following form:

$$X_t = M(s_t)E_t(X_{t+1}) + N(s_t)X_{t-1} + Q(s_t)U_t \tag{2.1}$$

where X_t is $n \times 1$ vector of endogenous variables, U_t is $m \times 1$ vector of exogenous variables that follows

$$U_t = \rho U_{t-1} + \epsilon_t$$

where ρ is a diagonal matrix and ϵ_t is a white noise process. By assumption, U_t is a covariance stationary process. s_t is a S-state Markov Chain and $p_{ij} = Pr(s_{t+1} = j | s_t = i)$ is the (i, j)-th element of the transition probability matrix, P. From Proposition 1 in Cho (2016), any rational expectations solution to (2.1) can be written as a linear combination of a minimal state variable solution that depends on X_t, s_t, and U_t and a non-fundamental solution component, w_t, as:

$$X_t = \Omega(s_t) X_{t-1} + \Gamma(s_t) U_t + w_t \quad (2.2)$$

$$w_t = E_t(F(s_t) w_{t+1}) \quad (2.3)$$

where the coefficient matrices satisfy the following conditions for all realizations of the Markov Chain:

$$\Omega(s_t) = \{I_n - E_t[M(s_t)\Omega(s_{t+1})]\}^{-1} N(s_t) \quad (2.4)$$

$$\Gamma(s_t) = \{I_n - E_t[M(s_t)\Omega(s_{t+1})]\}^{-1} Q(s_t) + E_t(F(s_t)\Gamma(s_{t+1})\rho) \quad (2.5)$$

$$F(s_t) = \{I_n - E_t[M(s_t)\Omega(s_{t+1})]\}^{-1} M(s_t) \quad (2.6)$$

A minimal state variable solution takes the form given in (2.2) with $w_t = 0_{n\times 1}$:[9]

$$X_t = \Omega(s_t) X_{t-1} + \Gamma(s_t) U_t \quad (2.7)$$

[9]Non-fundamental solutions arise with $w_t \neq 0_{n\times 1}$ where w_t satisfies (2.3)

Any $\Omega(s_t)$ and $\Gamma(s_t)$ that satisfy (2.4) and (2.5) give us a minimal state variable solution of the form given in (2.7). Moreover, $\Gamma(s_t)$ is uniquely determined by $\Omega(s_t)$ and can be obtained by vectorizing (2.5) for each state and stacking the vectorized equations as follow:

$$vec\begin{pmatrix}\Gamma(1)\\ \vdots \\ \Gamma(S)\end{pmatrix} = vec\begin{pmatrix}\Xi(1)^{-1}Q(1)\\ \vdots \\ \Xi(S)^{-1}Q(S)\end{pmatrix} + ((\oplus_{j=1}^{S}\rho' \otimes F(j))(P \otimes I_{n\times m}))vec\begin{pmatrix}\Gamma(1)\\ \vdots \\ \Gamma(S)\end{pmatrix}$$

where $\oplus_{j=1}^{S}\rho' \otimes F(j) = diag(\rho' \otimes F(1), \ldots, \rho' \otimes F(S))$ and

$$\Xi(i) = I_n - E_t[M(s_t)\Omega(s_{t+1})] \tag{2.8}$$

Also, P denotes the transition probability matrix

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1S} \\ p_{21} & p_{22} & \cdots & p_{2S} \\ \vdots & & \ddots & \vdots \\ p_{S1} & p_{S2} & \cdots & p_{SS} \end{pmatrix}$$

where p_{ij} is the probability of transitioning from state i to state j.

The Forward Method for Solving MS-DSGE Models

In this section, we discuss the forward method for solving MS-DSGE models developed in Cho (2016) and Cho and Moreno (2011). When the forward method works, we obtain the "forward solution" to the MS-DSGE model given in (2.1). We discuss this method and the "forward solution" for two reasons: (1) the determinacy results used in this paper only apply to the forward solution; (2) the presence of lagged endogenous variables in MS-DSGE models generates an infinite regress problem that often precludes use of standard solution techniques such as the method of undetermined coefficients. As stated in Cho (2016), the forward method "simply amounts to text-book style 'solving the model'" forwards. To that end, consider the following proposition, which is Proposition 3 in Cho (2016):

Proposition 1

Consider (2.1). For given s_t, X_t and X_{t-1} there exists a unique sequence of real-valued matrices $(\Omega_k(s_t), \Gamma_k(s_t), F_k(s_t))$ for $k = 1, 2, 3, ...$ such that

$$X_t = E_t[M_k(s_t, s_{t+1}, ..., s_{t+k})X_{t+k}] + \Omega_k(s_t)X_{t-1} + \Gamma_k(s_t)U_t \tag{2.9}$$

where $\Omega_1(s_t) = N(s_t)$, $\Gamma_1(s_t) = Q(s_t)$, $F_1(s_t) = M_1(s_t) = M(s_t)$,

$$\Omega_k(s_t) = \{I_n - E_t[M(s_t)\Omega_{k-1}(s_{t+1}]\}^{-1}N(s_t) \tag{2.10}$$

$$\Gamma_k(s_t) = \{I_n - E_t[M(s_t)\Omega_{k-1}(s_{t+1})]\}^{-1}Q(s_t) + E_t(F_k(s_t)\Gamma_{k-1}(s_{t+1})P) \tag{2.11}$$

$$F_k(s_t) = \{I_n - E_t[M(s_t)\Omega_{k-1}(s_{t+1})]\}^{-1}M(s_t) \tag{2.12}$$

and

$$M_k(s_t, s_{t+1}, ..., s_{t+k}) = F_k(s_t)M_{k-1}(s_{t+1}, ..., s_{t+k})$$

Proof: See Appendix D in Cho (2016)

Definition 1

(2.1) satisfies the forward convergence condition if the coefficients of the state variables, $(\Omega_k(s_t), \Gamma_k(s_t), F_k(s_t))$ in (2.10)-(2.12) converge as k

goes to infinity for every s_t. Let $\Omega^*(s_t) = \lim_{k\to\infty} \Omega_k(s_t)$ and $\Gamma^*(s_t) = \lim_{k\to\infty} \Gamma_k(s_t)$.

Then the model implies:

$$X_t = \lim_{k\to\infty} E_t(M_k(s_t, ..., s_{t+k})X_{t+k}) + \Omega^*(s_t)X_{t-1} + \Gamma^*(s_t)U_t \tag{2.13}$$

Now, since $\Omega^*(s_t)$ and $\Gamma^*(s_t)$ satisfy (2.4) and (2.5). The following equation describes a rational expectations solution of (2.1):

$$X_t = \Omega^*(s_t)X_{t-1} + \Gamma^*(s_t)U_t \tag{2.14}$$

Equation (2.14) is the forward solution to (2.1). (2.13) and (2.14) jointly imply that $E_t(M_k(s_t, ..., s_{t+k})X_{t+k}) = 0_{n\times 1}$, and Cho (2016) shows that (2.14) is the unique solution obtained via the forward method that satisfies the No Bubble Condition (NBC): $E_t(M_k(s_t, ..., s_{t+k})X_{t+k}) = 0_{n\times 1}$. Because $\Gamma^*(s_t)$ does not exist if $\Omega^*(s_t)$ does not exist, one should first check to see if $\Omega_k(s_t)$ converges. If $\Omega^*(s_t)$ exists, and the spectral radius of $(\oplus_{j=1}^{S}\rho' \otimes F(j))(P \otimes I_{n\times m})$ is less than one, then $\Gamma^*(s_t)$ exists and can be recovered from the following equation:

$$vec\begin{pmatrix}\Gamma^*(1)\\ \vdots \\ \Gamma^*(S)\end{pmatrix} = vec\begin{pmatrix}\Xi^*(1)^{-1}Q(1)\\ \vdots \\ \Xi^*(S)^{-1}Q(S)\end{pmatrix} + ((\oplus_{j=1}^{S}\rho' \otimes F^*(j))(P \otimes I_{n\times m}))vec\begin{pmatrix}\Gamma^*(1)\\ \vdots \\ \Gamma^*(S)\end{pmatrix}$$

where we have substituted $\Omega^*(s_t)$ for $\Omega(s_t)$ in (2.6) and (2.8) to form $F^*(s_t)$ and $\Xi^*(s_t)$, respectively.

E-Stability

In this paper, we focus on the stability of the forward solution of (2.1) under adaptive learning. Specifically, we show that a set of sufficient conditions for determinacy in this class of models imply that the forward solution of (2.1) is stable under learning. This result depends on the information that agents have at their disposal. Suppose agents observe s_t, P, U_t, and X_t at time t. Agents have the following perceived law of motion:

$$X_t = A(s_t) + B(s_t)X_{t-1} + C(s_t)U_t$$

where $A(i)$ is $n \times 1$, $B(i)$ is $n \times n$ and $C(i)$ is $n \times m$. Consistent with the PLM, the learning agent believes that if $s_t = i$ then $X_t = A(i) + B(i)X_{t-1} + C(i)U_t$ for $i = 1, 2, ..., S$. The coefficient matrices of the PLM may be estimated using a recursive least squares procedure that updates $A(i)$, $B(i)$, and $C(i)$ each time $s_t = i$. Notice that we allow agents to learn steady state values by including a constant term in the PLM. In this section, we solve for agents' state-contingent expectations and derive the state-contingent T-map. If $s_t = i$ then

$$\begin{aligned} E_t(X_{t+1}) &= E(X_{t+1}|s_t = i; X_t, U_t) \\ &= \sum_{j=1}^{S} E(X_{t+1}|s_{t+1} = j, s_t = i; X_t, U_t) \\ &= \sum_{j=1}^{S} p_{ij}(A(j) + B(j)E_tX_t + C(j)E_tU_{t+1})) \\ &= \sum_{j=1}^{S} p_{ij}(A(j) + B(j)X_t + C(j)\rho U_t)) \end{aligned}$$

Substituting $E_t(X_{t+1})$ into (2.1) yields the actual data generating process:

$$\begin{aligned}
X_t &= \{I - M(i)(\sum_{j=1}^{S} p_{ij}B(j))\}^{-1}M(i)(\sum_{j=1}^{S} p_{ij}A(j)) \\
&+ \{I - M(i)(\sum_{j=1}^{S} p_{ij}B(j))\}^{-1}N(i)X_{t-1} \\
&+ \{I - M(i)(\sum_{j=1}^{S} p_{ij}B(j))\}^{-1}(M(i)(\sum_{j=1}^{S} p_{ij}C(j))\rho + Q(i))U_t
\end{aligned}$$

If we define $B = (B(1)\ B(2) \cdots B(S))$ and $\Xi(i,B) = \{I - M(i)(\sum_{j=1}^{S} p_{ij}B(j))\}$ then the state-contingent T-map is:

$$\begin{aligned}
A(i) &\to \Xi(i,B)^{-1}M(i)\sum_{j=1}^{S} p_{ij}A(j) \\
B(i) &\to \Xi(i,B)^{-1}N(i) \\
C(i) &\to \Xi(i,B)^{-1}(M(i)\sum_{j=1}^{S} p_{ij}C(j)\rho + Q(i))
\end{aligned}$$

Notice that the forward solution is a fixed point of the T-map. The block of the T-map associated to $B = (B(1)\ B(2) \cdots B(S))$ decouples from the other equations. This block is given by:

$$T_B(B) = \begin{pmatrix} (\Xi(1,B)^{-1}N(1))' \\ (\Xi(2,B)^{-1}N(2))' \\ \vdots \\ (\Xi(S,B)^{-1}N(S))' \end{pmatrix}' = \begin{pmatrix} T_B^1(B)' \\ T_B^2(B)' \\ \vdots \\ T_B^S(B)' \end{pmatrix}'$$

To assess E-stability, we begin by examining the following differential equation:

$$\frac{dB}{dt} = T_B(B) - B$$

Let $DT_B(\bar{B})$ denote the Jacobian of T_B evaluated at the forward solution, $\bar{B} = (\Omega^*(1)\ \Omega^*(2) \cdots \Omega^*(S))$. Since T_B is continuously differentiable, Proposition 5.6 in Evans and Honkapohja (2001) tells us that $\bar{B}$ is locally asymptotically stable if the eigenvalues of $DT_B(\bar{B})$ have real parts less than one. As demonstrated in Appendix A. this process yields the following Jacobian:

$$\begin{aligned} DT_B(\bar{B}) &= \begin{pmatrix} p_{11}\Omega^*(1)' \otimes F^*(1) & p_{12}\Omega^*(1)' \otimes F^*(1) & \cdots & p_{1S}\Omega^*(1)' \otimes F^*(1) \\ p_{21}\Omega^*(2)' \otimes F^*(2) & p_{22}\Omega^*(2)' \otimes F^*(2) & \cdots & p_{2S}\Omega^*(2)' \otimes F^*(2) \\ \vdots & & \ddots & \vdots \\ p_{S1}\Omega^*(S)' \otimes F^*(S) & p_{S2}\Omega^*(S)' \otimes F^*(S) & \cdots & p_{SS}\Omega^*(S)' \otimes F^*(S) \end{pmatrix} \\ &\equiv (\oplus_{j=1}^{S} \Omega^{*'}(j) \otimes F^*(j))(P \otimes I_{n^2}) \end{aligned}$$

E-stability requires the real parts of $(\oplus_{j=1}^{S} \Omega^{*'}(j) \otimes F^*(j))(P \otimes I_{n^2})$ to be less than one. It is important to note that our derivation of the E-stability conditions hinges on the following: $\Xi(i, \bar{B})^{-1} N(i) = \{I - M(i)(\sum_{j=1}^{S} p_{ij}\Omega^*(j))\}^{-1} N(i) = \{I - M(i)(E_t(\Omega^*(s_{t+1}))\}^{-1} N(i) = \Omega^*(i)$ and $\Xi(i, \bar{B})^{-1} M(i) = \{I - M(i)(\sum_{j=1}^{S} p_{ij}\Omega^*(j))\}^{-1} M(i) = \{I - M(i)(E_t(\Omega^*(s_{t+1}))\}^{-1} M(i) = F^*(i)$ where E_t denotes conditional expectations here. We now turn to the equation for $A = (A(1)'\ A(2)' \cdots A(S)')'$:

$$T_A(A) = \begin{pmatrix} \Xi(1,B)^{-1}M(1)(\sum_{j=1}^{S} p_{1j}A(j)) \\ \Xi(2,B)^{-1}M(2)(\sum_{j=1}^{S} p_{2j}A(j)) \\ \vdots \\ \Xi(S,B)^{-1}M(S)(\sum_{j=1}^{S} p_{Sj}A(j)) \end{pmatrix}$$

$$= \begin{pmatrix} p_{11}\Xi(1,B)^{-1}M(1) & p_{12}\Xi(1,B)^{-1}M(1) & \cdots & p_{1S}\Xi(1,B)^{-1}M(1) \\ p_{21}\Xi(2,B)^{-1}M(2) & p_{22}\Xi(2,B)^{-1}M(2) & \cdots & p_{2S}\Xi(2,B)^{-1}M(1) \\ \vdots & & \ddots & \vdots \\ p_{S1}\Xi(S,B)^{-1}M(S) & p_{S2}\Xi(S,B)^{-1}M(S) & \cdots & p_{SS}\Xi(S,B)^{-1}M(S) \end{pmatrix} A$$

Using the same methods as before we obtain the following Jacobian evaluated at the REE where $\bar{A} = 0_{Sn\times 1}$:

$$DT_A(\bar{A},\bar{B}) = \begin{pmatrix} p_{11}F^*(1) & p_{12}F^*(1) & \dots & p_{1S}F^*(1) \\ p_{21}F^*(2) & p_{22}F^*(2) & \dots & p_{2S}F^*(2) \\ \vdots & & \ddots & \vdots \\ p_{S1}F^*(S) & p_{S2}F^*(S) & \dots & p_{SS}F^*(S) \end{pmatrix} \equiv (\oplus_{j=1}^{S} F^*(j))(P \otimes I_n)$$

E-stability requires the real parts of $((\oplus_{j=1}^{S} F^*(j))(P \otimes I_n))$ to be less than one. Finally, we consider the equation for $C = (C(1)'\ C(2)' \cdots C(S)')'$:

$$T_C(B,C) = \begin{pmatrix} \Xi(1,B)^{-1}(M(1)(\sum_{j=1}^S p_{1j}C(j))\rho + Q(1)) \\ \Xi(2,B)^{-1}(M(2)(\sum_{j=1}^S p_{2j}C(j))\rho + Q(2)) \\ \vdots \\ \Xi(S,B)^{-1}(M(S)(\sum_{j=1}^S p_{Sj}C(j))\rho + Q(S)) \end{pmatrix}$$

$$= \begin{pmatrix} p_{11}\Xi(1,B)^{-1}M(1) & p_{12}\Xi(1,B)^{-1}M(1) & \cdots & p_{1S}\Xi(1,B)^{-1}M(1) \\ p_{21}\Xi(2,B)^{-1}M(2) & p_{22}\Xi(2,B)^{-1}M(2) & \cdots & p_{2S}\Xi(2,B)^{-1}M(1) \\ \vdots & & \ddots & \vdots \\ p_{S1}\Xi(S,B)^{-1}M(S) & p_{S2}\Xi(S,B)^{-1}M(S) & \cdots & p_{SS}\Xi(S,B)^{-1}M(S) \end{pmatrix} C\rho$$

$$+ \begin{pmatrix} \Xi(1,B)^{-1}Q(1) \\ \Xi(2,B)^{-1}Q(2) \\ \vdots \\ \Xi(S,B)^{-1}Q(S) \end{pmatrix}$$

Using the same methods as before we obtain the following Jacobian evaluated at the REE where $\bar{C} = (\Gamma^*(1)'\ \Gamma^*(2)' \cdots \Gamma^*(S)')'$:

$$DT_C(\bar{B},\bar{C}) = \rho \otimes \begin{pmatrix} p_{11}F^*(1) & p_{12}F^*(1) & \dots & p_{1S}F^*(1) \\ p_{21}F^*(2) & p_{22}F^*(2) & \dots & p_{2S}F^*(2) \\ \vdots & & \ddots & \vdots \\ p_{S1}F^*(S) & p_{S2}F^*(S) & \dots & p_{SS}F^*(S) \end{pmatrix} \equiv \rho \otimes ((\oplus_{j=1}^S F^*(j))(P \otimes I_n))$$

The REE solution $\bar{A}, \bar{B}, \bar{C}$ is E-stable if:

i. all the eigenvalues of $(\oplus_{j=1}^{S} \Omega^{*'}(j) \otimes F^*(j))(P \otimes I_{n^2})$ have real parts less than 1,

ii. all the eigenvalues of $(\oplus_{j=1}^{S} F^*(j))(P \otimes I_n)$ have real parts less than 1, and,

iii. all the eigenvalues of $\rho \otimes ((\oplus_{j=1}^{S} F^*(j))(P \otimes I_n))$ have real parts less than 1

The solution is not E-stable if any of these three conditions fail with eigenvalues strictly greater than one.

Mean-Square Stability

As noted in the literature review, the MS-DSGE literature utilizes two notions of stability: (1) bounded stability; (2) mean-square stability. We focus on mean-square stability in this paper, but offer definitions for both mean-square stability and bounded stability below.

Definition 2

An $n \times 1$ stochastic process y_t is mean-square stable (MSS) if there exists an $n \times 1$ vector $\bar{y}$ and $n \times n$ matrix Q s.t. $\lim_{t\to\infty}(E[y_t]) = \bar{y}$ and $\lim_{t\to\infty}(E[y_t y_t']) = Q$.

Intuitively, a stochastic process is MSS if its first and second moments converge to well-defined limits. The concept of mean-square stability is closely related to other often used concepts of stability. For example, asymptotic covariance stationarity implies mean-square stability in general, and is even equivalent to mean-square stability in models with exogenous asymptotic covariance stationary shock processes.

Definition 3

An n-dimensional stochastic process y_t is bounded if there exists a real number N such that $||y_t|| < N$ for all t

where $|| \bullet ||$ is any well-defined norm, including the uniform norm. In LRE models with bounded shocks, determinacy in the mean-square stability sense and determinacy in the bounded stability sense are equivalent equilibrium selection criteria. In models with unbounded shocks–such as the normal or lognormal exogenous shocks commonly used in applications–the criteria differ; a mean-square stable process can have unbounded support while bounded processes cannot. Of course, this paper, and other papers in the MS-DSGE literature, work with models that are linearized around a nonstochastic steady state, and therefore function best with small shocks. This substantially reduces the meaningfulness of embracing a concept of stability that allows for unbounded shock processes. Even with bounded shocks, the two concepts are not equivalent in MS-DSGE models. For example, Farmer, Whaggoner and Zha (2009) argues that MS-DSGE models with persistent unstable regimes may admit unbounded MSS rational expectations equilibria. This means that bounded stability may rule out equilibria of theoretical importance in certain applications (i.e. equilibria in economies that experience recurring hyperinflation). These reasons notwithstanding, we work with mean-square stability because there are not tractable conditions for determinacy in the boundedly stable sense in MS-DSGE models with lagged endogenous variables.

Though Definition 2 both formalizes the concept of mean-square stability, and offers helpful intuition, it does not help us identify mean-square stability. To that end, Costa et al. (2005) developed a set of tractable conditions for identifying whether or not a given process is MSS. We state these conditions by considering a very general process:

$$y_{t+1} = D(s_t, s_{t+1})y_t + H(s_{t+1})\eta_{t+1} \tag{2.15}$$

where $D(s_t, s_{t+1})$ is $n \times n$, $H(s_{t+1})$ is $n \times l$ and η_{t+1} is some $l \times 1$ mean-square stable (MSS) process. $D(s_t, s_{t+1})$ can depend on s_t alone, s_{t+1} alone, or both s_t and s_{t+1}. Given certain conditions on U_t, the forward solution is a stochastic process of this form [10]:

$$X_t = \Omega^*(s_t)X_{t-1} + \Gamma^*(s_t)U_t \tag{2.16}$$

We can also form a stochastic process of this form by forcing the coefficient matrices in the forward solution to depend on last period's state:

$$X_t = \Omega^*(s_{t-1})X_{t-1} + \Gamma^*(s_{t-1})U_t \tag{2.17}$$

Theorem 3.34 in Costa et al. (2005) allows us to focus solely on the homogeneous part of (2.15): $y_{t+1} = D(s_t, s_{t+1})y_t$. Consider the following matrices:

$$\bar{\Psi}_D = \begin{pmatrix} p_{11}D(1,1) & p_{21}D(2,1) & \cdots & p_{S1}D(S,1) \\ p_{12}D(1,2) & p_{22}D(2,2) & \cdots & p_{S2}D(S,2) \\ \vdots & & \ddots & \vdots \\ p_{1S}D(1,S) & p_{2S}D(2,S) & \cdots & p_{SS}D(S,S) \end{pmatrix}$$

$$\Psi_D = \begin{pmatrix} p_{11}D(1,1) & p_{12}D(1,2) & \cdots & p_{1S}D(1,S) \\ p_{21}D(2,1) & p_{22}D(2,2) & \cdots & p_{2S}D(2,S) \\ \vdots & & \ddots & \vdots \\ p_{S1}D(S,1) & p_{S2}D(S,2) & \cdots & p_{SS}D(S,S) \end{pmatrix}$$

[10] If U_t consists of m independent covariance stationary $AR(1)$ processes then $\Gamma(s_t)U_t$ will be MSS and these conditions are met

$$\bar{\Psi}_{D\otimes D} = \begin{pmatrix} p_{11}D(1,1)\otimes D(1,1) & p_{21}D(2,1)\otimes D(2,1) & \cdots & p_{S1}D(S,1)\otimes D(S,1) \\ p_{12}D(1,2)\otimes D(1,2) & p_{22}D(2,2)\otimes D(2,2) & \cdots & p_{S2}D(S,2)\otimes D(S,2) \\ \vdots & & \ddots & \vdots \\ p_{1S}D(1,S)\otimes D(1,S) & p_{2S}D(2,S)\otimes D(2,S) & \cdots & p_{SS}D(S,S)\otimes D(S,S) \end{pmatrix}$$

$$\Psi_{D\otimes D} = \begin{pmatrix} p_{11}D(1,1)\otimes D(1,1) & p_{12}D(1,2)\otimes D(1,2) & \cdots & p_{1S}D(1,S)\otimes D(1,S) \\ p_{21}D(2,1)\otimes D(2,1) & p_{22}D(2,2)\otimes D(2,2) & \cdots & p_{2S}D(2,S)\otimes D(2,S) \\ \vdots & & \ddots & \vdots \\ p_{S1}D(S,1)\otimes D(S,1) & p_{S2}D(S,2)\otimes D(S,2) & \cdots & p_{SS}D(S,S)\otimes D(S,S) \end{pmatrix}$$

Definition 4

Let $r_\sigma(X) = max_{1\leq i\leq n}(|\lambda_i|)$, where $\lambda_1, ..., \lambda_n$ are the eigenvalues of the $n \times n$ matrix X.

Theorem 1

The process (2.15) is mean-square stable if and only if $r_\sigma(\bar{\Psi}_{D\otimes D}) < 1$.

Proof: See Proposition 3.9 in Costa et al. (2005).

To derive this result, Costa et al. develop first-order difference equations that describe the evolution of the first and second moments of an arbitrary stochastic process. They show that $\bar{\Psi}_{D\otimes D}$ governs the evolution of the second moment such that the second moment equation converges in the limit if and only if the spectral radius of $\bar{\Psi}_{D\otimes D}$ is less than one. Conveniently, this condition is also sufficient for the convergence of the first moment process, whose evolution is governed by $\bar{\Psi}_D$ (see Theorem 2). As a result, $r_\sigma(\bar{\Psi}_{D\otimes D}) < 1$ is necessary and sufficient for the mean-square stability of the process in (2.15). The following is also a useful result from Costa et al (2005).

Theorem 2

If $r_\sigma(\bar{\Psi}_{D\otimes D}) < 1$ then $r_\sigma(\bar{\Psi}_D) < 1$

Proof: See Proposition 3.6 in Costa et al. (2005)

Corollary: If $r_\sigma(\Psi_{D\otimes D}) < 1$ then $r_\sigma(\Psi_D) < 1$

Proof: Since $r_\sigma(\Psi_{D\otimes D}) = r_\sigma(\bar{\Psi}_{D'\otimes D'}) < 1$, Theorem 2 implies that $r_\sigma(\bar{\Psi}_{D'}) = r_\sigma(\Psi_D) < 1$.

The stochastic processes that we study in this paper have homogeneous component matrices that strictly depend on s_t or s_{t+1}, but not both. For example, let $D(s_t, s_{t+1}) = \Omega^*(s_{t+1})$ as in (2.16). In this case: $\bar{\Psi}_{D\otimes D} = (\oplus_{j=1}^{S}\Omega^*(j)\otimes\Omega^*(j))(P'\otimes I_{n^2})$; $\bar{\Psi}_D = (\oplus_{j=1}^{S}\Omega^*(j))(P'\otimes I_n)$; $\Psi_{D\otimes D} = (P\otimes I_{n^2})(\oplus_{j=1}^{S}\Omega^*(j)\otimes\Omega^*(j))$; $\Psi_D = (P\otimes I_n)(\oplus_{j=1}^{S}\Omega^*(j))$. If $D(s_t, s_{t+1}) = \Omega^*(s_t)$ as in (2.17), then: $\bar{\Psi}_{D\otimes D} = (P'\otimes I_{n^2})(\oplus_{j=1}^{S}\Omega^*(j)\otimes\Omega^*(j))$; $\bar{\Psi}_D = (P'\otimes I_n)(\oplus_{j=1}^{S}\Omega^*(j))$; $\Psi_{D\otimes D} = (\oplus_{j=1}^{S}\Omega^*(j)\otimes\Omega^*(j))(P\otimes I_{n^2})$; $\Psi_D = (\oplus_{j=1}^{S}\Omega^*(j))(P\otimes I_n)$

Cho (2016) Sufficient Conditions for Determinacy

As shown in Cho (2016), the following two conditions are sufficient for determinacy in the mean square stability sense:

1. $r_\sigma((\oplus_{j=1}^{S}\Omega^*(j)\otimes\Omega^*(j))(P'\otimes I_{n^2})) < 1$
2. $r_\sigma((\oplus_{j=1}^{S}F^*(j)\otimes F^*(j))(P\otimes I_{n^2})) \leq 1$

where all coefficient matrices are taken from the forward solution to the model.

Determinacy and E-Stability

In this section, we show that the abovementioned sufficient conditions for determinacy imply E-stability in the class of MS-DSGE models under consideration in this paper. Simply put, determinacy implies $r_\sigma(((\oplus_{j=1}^{S} F^*(j))(P \otimes I_n))) < 1$ and $r_\sigma((\oplus_{j=1}^{S} \Omega^{*'}(j) \otimes F^*(j))(P \otimes I_{n^2})) < 1$. $r_\sigma((\oplus_{j=1}^{S} \Omega^{*'}(j) \otimes F^*(j))(P \otimes I_{n^2})) < 1$ implies that condition (i) for E-stability holds, and $r_\sigma((\oplus_{j=1}^{S} F^*(j))(P \otimes I_n)) < 1$ implies that conditions (ii) and (iii) are satisfied as well.

E-Stability and Cho (2016) Conditions

Theorem 3

If the MSV solution to a model of the form (2.1) satisfies the sufficient conditions for determinacy in Cho (2016), then the MSV solution is stable under learning.

Proof: The proof consists of two parts. First, we demonstrate that both determinacy conditions imply $r_\sigma((\oplus_{j=1}^{S} \Omega^{*'}(j) \otimes F^*(j))(P \otimes I_{n^2})) < 1$, which implies the first sufficient condition for E-stability. Then, we demonstrate that the second determinacy condition implies $r_\sigma((\oplus_{j=1}^{S} F^*(j))(P \otimes I_n)) < 1$, which implies the second and third conditions for E-stability . We begin by considering the following proposition:

Proposition 2

Consider the two arbitrary stochastic processes with the following homogeneous components:

$$\begin{aligned} X_{t+1} &= G(s_t)X_t \\ \omega_{t+1} &= H(s_t)'\omega_t \end{aligned}$$

where X and ω are both $n \times 1$. If $r_\sigma((\oplus_{j=1}^S H(j) \otimes H(j))(P \otimes I_n)) < 1$ and $r_\sigma((P' \otimes I_n)(\oplus_{j=1}^S G(j) \otimes G(j))) < 1$ then $r_\sigma((\oplus_{j=1}^S G^{'}(j) \otimes H(j))(P \otimes I_{n^2})) < 1$.

Proof: See proof of Lemma 1 in Cho (2016). Set $F'(s_{t-1}, s_t) = H(s_{t-1})'$ and $G(s_{t-1}, s_t) = G(s_{t-1})$ and Proposition 2 follows. ■

Now suppose $G(s_t) = \Omega^*(s_t)$ and $H'(s_t) = F^*(s_t)'$ so that $(\oplus_{j=1}^S G^{'}(j) \otimes H(j))(P \otimes I_{n^2}) = (\oplus_{j=1}^S \Omega^{*'}(j) \otimes F^*(j))(P \otimes I_{n^2})$. If the sufficient conditions for determinacy imply that we can form MSS processes with the following homogeneous components:

$$X_{t+1} = \Omega^*(s_t) X_t \tag{2.18}$$

$$w_{t+1} = F^*(s_t)' \omega_t \tag{2.19}$$

then $r_\sigma((\oplus_{j=1}^S \Omega^{*'}(j) \otimes F^*(j))(P \otimes I_{n^2})) < 1$ by Proposition 2. We now prove this implication. From the first condition for determinacy, $r_\sigma((\oplus_{j=1}^S \Omega^*(j) \otimes \Omega^*(j))(P' \otimes I_{n^2})) < 1$. Since $(\oplus_{j=1}^S \Omega^*(j) \otimes \Omega^*(j))(P' \otimes I_{n^2})$ and $(P' \otimes I_{n^2})(\oplus_{j=1}^S \Omega^*(j) \otimes \Omega^*(j))$ have the same characteristic equation, the first condition for determinacy implies $r_\sigma((P' \otimes I_{n^2})(\oplus_{j=1}^S \Omega^*(j) \otimes \Omega^*(j))) < 1$. From Theorem 1, it follows that (2.18) is a MSS homogenous component. Now consider the second condition for determinacy: $r_\sigma((\oplus_{j=1}^S F^*(j) \otimes F^*(j))(P \otimes I_n)) < 1$. Since $((\oplus_{j=1}^S F^*(j) \otimes F^*(j))(P \otimes I_n))' = (P' \otimes I_n)(\oplus_{j=1}^S F^*(j)' \otimes F^*(j)')$, the second condition for determinacy also implies $r_\sigma((P' \otimes I_n)(\oplus_{j=1}^S F^*(j)' \otimes F^*(j)')) < 1$. It follows from Theorem 1 that (2.19) is also a MSS homogeneous component. Together, the stability of (2.18) and (2.19) give us $r_\sigma((\oplus_{j=1}^S \Omega^{*'}(j) \otimes F^*(j))(P \otimes I_{n^2})) < 1$ which satisfies the first E-stability condition

Now, we show that conditions (ii) and (iii) are satisfied. As demonstrated, determinacy implies $r_\sigma((P' \otimes I_n)(\oplus_{j=1}^S F^*(j)' \otimes F^*(j)')) < 1$. From Theorem 2 and its corollary, $r_\sigma((P' \otimes I_n)(\oplus_{j=1}^S F^*(j)')) = r_\sigma((\oplus_{j=1}^S F^*(j))(P \otimes I_n)) < 1$ follows.

Therefore, if the forward solution satisfies the sufficient conditions for determinacy in Cho (2016), then E-stability condition (ii) is satisfied. To see that (iii) is satisfied, observe the following: $r_\sigma(\rho \otimes ((\oplus_{j=1}^{S} F^*(j))(P \otimes I_n))) = r_\sigma(\rho) r_\sigma((\oplus_{j=1}^{S} F^*(j))(P \otimes I_n))$. Since ρ is a diagonal matrix and all of its entries are bounded below by 0 and above by 1, $r_\sigma(\rho)$ is less than one. Hence, $r_\sigma(((\oplus_{j=1}^{S} F^*(j))(P \otimes I_n))) < 1$ implies condition (iii). We conclude that the sufficient conditions for determinacy imply E-stability of the forward solution for any S-state Markov chain. ■

McCallum (2007): A Special Case

Though we implicitly pursue this result in order to further our understanding of nondegenerate Markov-switching models (i.e. models with $S > 1$), we draw attention to the fact that linear rational expectations models are nested in the class of models we examine. This means that our methods should agree with the main result in McCallum (2007), which shows that determinacy implies the learnability of the minimal state variable solution when agents use an information set that is identical to the one used in this paper. To see what our approach says about linear models we let $S = 1$. When this is true, the forward solution $\bar{A}, \bar{B}, \bar{C}$ is E-stable if:

i. all the eigenvalues of $(\Omega^*)' \otimes F$ have real parts less than 1,

ii. all the eigenvalues of F^* have real parts less than 1, and,

iii. all the eigenvalues of $\rho \otimes F^*$ have real parts less than 1

Likewise, the sufficient conditions for determinacy become:

1. $r_\sigma(\Omega^* \otimes \Omega^*) < 1$

2. $r_\sigma(F^* \otimes F^*) < 1$

A well known result in matrix algebra says that for any matrix, A, $r_\sigma(A \otimes A) < 1$ if and only if $r_\sigma(A) < 1$. Therefore, we can use the following equivalent conditions for determinacy:

1. $r_\sigma(\Omega^*) < 1$

2. $r_\sigma(F^*) < 1$

McCallum (2007) shows that these last two determinacy conditions imply the three E-stability conditions above. Hence, our results agree with McCallum (2007).

Numerical Example

We apply our results to a basic New Keynesian model of the kind Woodford (1998a) uses. This model is augmented to allow for Markov-switching in fiscal and monetary policy parameters, and features a representative household and firm, monopolistic competition in the production of intermediate goods, and price stickiness a la Calvo (1983) according to which $1-\theta$ fraction of firms can change their prices each period. The model also allows government to sell one-period nominal bonds, B_t, at a price that equals the inverse of the monetary policy instrument, $1+i_t$. The government collects lump-sum taxes in accordance with an endogenous primary surplus rule, τ_t and government purchases are assumed to equal 0, so that income, Y_t, equals C_t in equilibrium.

The model is linearized around the non-stochastic steady state with zero inflation. Let $\hat{z}_t \equiv ln(z_t) - ln(\bar{z})$ where $\bar{z}$ is the value of z in steady state. The behavior of households and firms then reduces to two-equations [11]:

$$\hat{y}_t = E_t\hat{y}_{t+1} - \sigma^{-1}(\hat{i}_t - E_t\hat{\pi}_{t+1}) \tag{2.20}$$

$$\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + \kappa\hat{y}_t \tag{2.21}$$

where β is the household discount factor, σ^{-1} is the intertemporal elasticity of substitution and κ is defined in Appendix B. β, κ, and σ are positive by assumption, and β is also bounded above by 1. Monetary policy follows a standard interest rate rule of the form:

$$\hat{i}_t = \alpha(s_t)\hat{\pi}_t + z_{mt} \tag{2.22}$$

$$z_{mt} = \rho_m z_{m,t-1} + \epsilon_{mt} \tag{2.23}$$

where s_t follows a 2-state Markov chain, z_{mt} is an AR(1) exogenous monetary policy shock, and ϵ_m is an exogenous i.i.d. mean-zero innovation. Fiscal policy is characterized by the following linearized rule for primary surpluses:

$$\hat{\tau}_t = \gamma(s_t)(\hat{b}_{t-1} - \hat{i}_{t-1}) + z_{ft} \tag{2.24}$$

$$z_{ft} = \rho_F z_{f,t-1} + \epsilon_{ft} \tag{2.25}$$

where $\hat{b}_t$ is the percentage deviation of real bonds from steady state, z_{ft} is an exogenous fiscal policy shock, and ϵ_f is an exogenous mean-zero i.i.d innovation. γ is the fiscal authority's policy parameter and it follows the same Markov process as α.

[11]See Appendix B. for full-derivation of (2.20), (2.21), and policy equations

Fiscal policy must also satisfy the following budget constraint:

$$\hat{b}_t - \hat{i}_t + (\beta^{-1} - 1)\hat{s}_t = \beta^{-1}(\hat{b}_{t-1} - \hat{\pi}_t) \tag{2.26}$$

To reduce the number of equations in the system, we substitute (2.24) into (2.26), which yields:

$$\hat{b}_t - \hat{i}_t + \beta^{-1}\pi_t = (\beta^{-1} - \gamma(s_t)(\beta^{-1} - 1))\hat{b}_{t-1} + \gamma(s_t)(\beta^{-1} - 1)\hat{i}_{t-1} + (1 - \beta^{-1})z_{ft} \tag{2.27}$$

To characterize the time-varying behavior of policymakers, let $s_t = M$ if the economy is in Regime M, and let $s_t = F$ if the economy is in Regime F. In Regime M, $\alpha(M) > 1$ and $\gamma(M) > 1$; in regime F, $0 \leq \alpha(F) < 1$ and $0 \leq \gamma(F) < 1$. As discussed, a LRE model that features either a Regime F or Regime M policy configuration is determinate. In Regime F (M), fiscal policy is active (passive) and monetary policy is passive (active). If the economy is not in Regime F or Regime M in a LRE model, then monetary and fiscal policy are both passive (active), in which case the model is indeterminate (explosive). Though the assumption that policy strictly switches between Regime M and Regime F configurations is highly restrictive, we assume this for expositional purposes, and will study a more general model of switching in the future.

The system given by (2.20)-(2.23), (2.25), and (2.27) yields a solution for $x_t = (\hat{y}_t, \hat{\pi}_t, \hat{i}_t, \hat{b}_t)'$. To utilize the forward method for solving this model, we must write the model in the form of (2.1). Before doing this, however, we add the following constraint to (2.21):

$$\begin{aligned} & \lambda E_t(\hat{b}_{t+1} - \hat{i}_{t+1} + \beta^{-1}\pi_{t+1} - (1-\beta^{-1})\rho_F z_{ft} \\ - \quad & (\beta^{-1} - \sum_{j=1}^{2} p_{ij}\gamma(j)(\beta^{-1}-1))\hat{b}_t - \sum_{j=1}^{2} p_{ij}\gamma(j)(\beta^{-1}-1)\hat{i}_t)) = 0 \end{aligned}$$

where λ is any nonzero scalar, and $s_t = i$. To derive this constraint, we forward the government budget constraint one-period and take expectations. The constraint forces expected inflation to be consistent with agents' expectations of future fiscal policy. Cho (2015) demonstrates the need for this constraint when using the forward method to solve "block-recursive" models that have an autonomous block and a dependent block. An autonomous block is subsystem of equations that have no behavioral dependence on the dependent block. For example, the Phillips Curve, IS curve, and interest rate rule constitute the autonomous block in our New Keynesian model because these equations do not explicitly depend on the government budget constraint and fiscal surplus rule, which constitute the dependent block. When we solve the model forward, expectations of inflation and output will not depend on, or be consistent with, current and expected future fiscal policy. In other words, agents will condition their expectations on information set, $\{\hat{\pi}_t, \hat{y}_t, \hat{i}_t, z_{mt}, s_t\}$, when we are interested in a class of equilibria corresponding to the following information set: $\{\hat{\pi}_t, \hat{y}_t, \hat{i}_t, \hat{b}_t, z_{mt}, z_{ft}, s_t\}$. If fiscal policy is always passive, then current and expected future output and inflation do not depend on fiscal policy, and the information adjustment is unnecessary. If, however, fiscal policy is active sufficiently often, then inflation and output–which are determined in the autonomous block–will not respond to stabilize government debt in the absence of the informational adjustment, and the conditions for determinacy will fail regardless of whether or not the model is determinate. Because we allow for active fiscal policy, we add this informational adjustment to the system of equations. After making this adjustment the system is written as:

$$A(i)x_t = BE_t x_{t+1} + C(i)x_{t-1} + Qu_t$$

where $u_t = (z_{mt}, z_{ft})'$, $s_t = i$and

$$A(i) = \begin{pmatrix} 1 & 0 & \sigma^{-1} & 0 \\ -\kappa & 1 & \lambda(-(1-\beta^{-1})\sum_{j=1}^{2} p_{ij}\gamma(j)) & \lambda(\beta^{-1} - (1-\beta^{-1})\sum_{j=1}^{2} p_{ij}\gamma(j)) \\ 0 & -\alpha(i) & 1 & 0 \\ 0 & \beta^{-1} & -1 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & \sigma^{-1} & 0 & 0 \\ 0 & \beta + \lambda(\beta^{-1}) & -\lambda & \lambda \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C(i) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma(i)(\beta^{-1}-1) & (\beta^{-1} - \gamma(i)(\beta^{-1}-1)) \end{pmatrix}$$

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1-\beta^{-1} \end{pmatrix}$$

Provided $A(s_t)$ is invertible for all s_t it is straightforward to show that $M(s_t) = A(s_t)^{-1}B$, $N(s_t) = A(s_t)^{-1}C(s_t)$ and $Q(s_t) = A(s_t)^{-1}$. We can now apply the forward method and assess the uniqueness, mean-square stability and E-stability of the resulting forward solution. Unsurprisingly, our numerical analysis supports the main analytical result of this paper. Specifically, consider the set of parameter values

$$\begin{array}{ccc} \widehat{\alpha(M)} \in [1,5] & \widehat{\alpha(F)} \in [0,1] & \widehat{\gamma(M)} \in [1,5] \\ \widehat{\gamma(F)} \in [0,1] & \kappa \in [0,1] & \beta \in [.975,1] \\ \sigma \in [0,5] & & \end{array}$$

The forward solution is found to be E-stable for all indicated combinations of parameter values whenever the sufficient conditions for determinacy are also satisfied. This result depends crucially on the agents information set, $I_t^1 = \{x_t, u_t, s_t\}$. When agents only observe $I_t^2 = \{x_{t-1},\, u_t,\, s_t\}$ at time t, there are cases in which determinacy does not imply E-stability. This appears to hinge on the fact that fiscal policy is periodically active; a determinate purely forward-looking New Keynesian model (i.e. a model with a passive fiscal authority) will always have E-stable forward solutions when agents only observe I_t^2. We leave this topic to future research.

Conclusion

We proved that the sufficient conditions for determinacy in Cho (2016) imply the E-stability of the forward solution in MS-DSGE models with lagged endogenous variables when agents condition their expectations of future endogenous variables on all current endogenous and exogenous variables. This result extends a well-known result in McCallum (2007), and is the first study to address the relationship between determinacy and E-stability in MS-DSGE models with lagged endogenous variables. Through applications to models of fiscal-monetary policy interactions, we also demonstrate that our result is not robust to changes in agents' information set.

In the future, we hope to extend the model of Eusepi and Preston (2013) to allow for switching in fiscal and monetary policy parameters. This extension allows us to explore new issues. First, we can analyze how the size and average maturity of government debt affects economies that face recurring changes in fiscal and monetary policy. In the presence of a permanent Regime F configuration, medium to long average maturities induce expectational instability. This is a significant result inasmuch as global data reveals substantial cross-country variation in the scale and composition of debt. The present study does not, however, account for switches in policy that may have occurred in the U.S. and abroad since the 1960s. We therefore see the addition of Markov-switching interest rate rule and surplus rule parameters as a natural extension. Second, this research allows us to better understand and characterize regions of the New Keynesian model's parameter space consistent with determinacy and E-stability in the presence of switching policy rules. Third, our treatment of this model may allow us to identify cases where the forward solution appears stable but violates the conditions for determinacy and indeterminacy in Cho (2016). This may help us better understand the extent to which Cho's conditions are not necessary. Finally, we want to study how switching in fiscal policy appears to impart expectational instability on an otherwise standard New Keynesian model when agents do not observe current endogenous varaiables. This is a subject of interest to us because the methods in Branch et al. (2013) and Reed (2015) show that determinacy implies E-stability when agents do not know the contemporaneous endogenous variables and fiscal policy is permanently passive. We take this topic up in Chapter 4.

CHAPTER III

THE POWER OF FORWARD GUIDANCE AND THE FISCAL THEORY OF THE PRICE LEVEL

Introduction

A growing literature offers strong theoretical support for the use of expansionary forward guidance on interest rates, particularly when interest rates are constrained by the zero lower bound (see, for example, Eggertsson and Woodford, (2003)). Despite the effectiveness of forward guidance in theory, the predictions of workhorse New Keynesian models do not accord well with empirical studies of the effects of forward guidance in the U.S. (e.g. Del Negro et al (2015), D'Amico and King (2015)). That is, while New Keynesian models predict large responses of inflation and output to forward guidance on short-term rates, the empirical evidence points to responses that are positive but modest. This shortcoming of the New Keynesian model is dubbed "The Forward Guidance Puzzle" (Del Negro et al. 2015), and it calls into question the ability of the standard New Keynesian model to predict the effects of anticipated monetary policy.

According to Del Negro et al. (2015), McKay et al. (2015), Carlstrom et al. (2012), Chung et al.(2014), Kiley (2014), the implausible responsiveness of output and inflation to forward guidance stems from three signature features of the New Keynesian model. First, consumption is excessively responsive to changes in interest rates. Second, the lack of a discount factor in the household's log-linearized Euler equation implies a strong response of consumption to long-run interest rates. Because forward guidance is designed to influence long-run rates, forward guidance naturally generates a large response in consumption through the Euler equation. Third, "front-loading" in the New Keynesian Phillips Curve renders inflation particularly sensitive to changes in current and future output. Together, the lack of discounting in the Euler equation and front-loading in the Phillips Curve generate a feedback loop that exacerbates the rise in inflation and output implied by forward guidance.

Several papers have addressed this puzzle by limiting the importance of these three features of the New Keynesian model. For instance, McKay et al (2015) mute the response of agents to forward guidance by introducing borrowing constraints that prevent agents from drawing down their savings over the forward guidance horizon. Gabaix (2016) introduces an explicit discount factor into the Euler equation and an additional discount factor into the Phillips Curve to model myopic agents. Del Negro et al (2015) show that a positive probability of death generates effective discounting in the Euler equation when they introduce a perpetual youth structure into the New Keynesian model. Chung et al. (2014) and Kiley (2014) introduce "sticky information" in the spirit of Mankiw and Reis (2002) to mitigate feedback effects from the Phillips Curve. Cole (2015) replaces rational expectations with a model of adaptive learning to demonstrate that bounded rationality lessens the effectiveness of forward guidance in specific policy experiments. Cochrane (2017) argues that equilibrium selection rules may help to explain the Forward Guidance Puzzle.

In contrast to previous attempts to explain the exaggerated response of inflation and output to forward guidance – which focus primarily on the specification of private sector behavior – this paper examines how the joint conduct of monetary and fiscal policy influences the effects of expansionary forward guidance in the New Keynesian model. Specifically, we show that the above mentioned exaggerated response of output and inflation to forward guidance may hinge on two assumptions (in addition to the three model features highlighted above): (1) the monetary authority employs an interest rate rule that satisfies the "Taylor Principle"; (2) fiscal policy is conducted in such a way that variation in fiscal surpluses acts to stabilize government debt, thereby rendering fiscal policy Ricardian. Our approach is most closely related to Cochrane (2017), which suggests that fiscal considerations may help select equilibria with smaller initial price jumps in response to anticipated policy announcements.[1] In contrast to Cochrane (2017), we explicitly characterize fiscal policy regimes and study how wealth effects arising in these regimes reduce the responses of inflation and output to forward guidance. Moreover, we model recurring fiscal regimes to capture how uncertainty about future fiscal policy impacts the effectiveness of forward guidance.

Our contribution borrows heavily from the Fiscal Theory of the Price Level literature, which models inflation as the outcome of both monetary and fiscal policy (see Leeper and Leith (2016) for a review of the Fiscal Theory of the Price Level). Work in this literature distinguishes between "passive" policymakers who are constrained to stabilize the government debt, and "active" policymakers who determine inflation. In the standard New Keynesian model we study here, monetary policy is active (passive) when interest rate responds respond strongly (weakly) to inflation, and fiscal policy is passive (active) when Ricardian equivalence is satisfied (violated). Careful consideration of the passive-active dichotomy reveals a number of channels through which the fiscal policy stance impacts the response of inflation and output to both fundamental and policy shocks.

This paper focuses on a specific channel through which active fiscal policy affects agents' perception of bond wealth. To illustrate this channel, we restrict attention to

[1] Cochrane (2017) uses a bond valuation equation, which we introduce in (2), to point out the idea that the large price adjustments predicted in the standard "forward-stable" model equilibrium must be supported by changes in the present value of expected future surpluses.

our model's intertemporal household budget constraint (assuming that all government debt is single-period debt):

$$E_t\left(\sum_{T=t}^{\infty} R_{t,T}P_TC_T\right) = E_t\left(\sum_{T=t}^{\infty}\left\{R_{t,T}[P_TY_T - P_T\tau_T]\right\}\right) + B_{t-1} \tag{3.1}$$

where $R_{t,T}$ is the stochastic discount factor from time t to T, C is consumption, τ is the government's real primary surplus, P_T is the price level at T, B_{t-1} is the government debt stock that matures at t, and Y is income. Under the assumption that $Y_T = C_T\ \forall T$, and after substituting for $R_{t,T} = \beta^{T-t}u'(Y_T)P_t/u'(Y_t)P_T$, this equation reduces to

$$\frac{B_{t-1}}{P_t} = E_t\left(\sum_{T=t}^{\infty}\beta^{T-t}\frac{u'(Y_T)}{u'(Y_t)}\tau_T\right) \tag{3.2}$$

Equation 3.2 is the sticky price version of bond valuation equation in Cochrane (2001), and it asserts that today's price level is determined by the real present value of expected future surpluses, $E_t\left(\sum_{T=t}^{\infty}\beta^{T-t}(u'(Y_T)/u'(Y_t))\tau_T\right)$, and the predetermined debt stock, B_{t-1}. From (3.1) it follows that any variation in B_{t-1} affects the household's consumption path, all else constant. When fiscal policy is passive, however, all else is not constant – any change in B_{t-1} induces an offsetting response in $\{\tau_T\}$ that leaves the households choice set intact. In other words, fiscal policy satisfies Ricardian equivalence. In an active fiscal policy regime, variations in bond wealth are not totally offset by changes in the stream of expected surpluses, and this implies that the change in bonds has wealth effects.

One source of the aforementioned variation in bond wealth is monetary policy. For example, a reduction in interest rates might force bonds to a lower equilibrium path as lower rates alleviate the burden of rolling over existing debt. In a model with active fiscal policy, households are not compensated for their lower bond holdings with tax cuts, which causes the household to feel nominally constrained through (3.1). Moreover, the fall in bonds places downward pressure on prices through (3.2). The resulting effect of this monetary expansion is an eventual fall in output and prices. Much of what follows in this paper depends on the fact that forward guidance on short-term interest rates appears in the model as a series of anticipated interest rate cuts.

We illustrate the role played by monetary-fiscal policy interactions in determining the effects of forward guidance by allowing fiscal (monetary) policy to be permanently or recurrently active (passive). Our results are threefold. First, we find that the presence of active fiscal policy allows for forward guidance to have wealth effects that dampen the response of output and inflation to forward guidance (potentially at the cost of implausible deflation). This result depends on the fact that agents view government debt as net wealth in a regime with active fiscal policy. Hence, an anticipated reduction in interest rates which places downward pressure on agents' nominal bond returns causes agents to feel more constrained today. This mutes agents' responses to lower long-run real interest rates and induces firms to lower prices.

Second, the presence of switching in fiscal and monetary policy has expectational spillover effects that may cause forward guidance to be less stimulative in the switching model's passive fiscal, active monetary policy regime. In such a setting, the *possibility* that fiscal policy may become active during the forward guidance horizon causes agents to become less optimistic about the effects of forward guidance in an economy where monetary and fiscal policy are *currently* active and passive, respectively. Interestingly, these spillover effects always attenuate the short-term effects of forward guidance, but can lead to more persistent responses of output and inflation, as we demonstrate in one specific case. Our Markov-switching approach helps to highlight the role that expectations play in generating a response of inflation and output to forward guidance.

Third, the presence of long-term government debt in a model with active fiscal, passive monetary policy introduces "revaluation effects" that mitigate the deflationary effects observed in the corresponding model with only short-term debt. We observe these effects because an anticipated reduction in short-term interest rates raises the market value of outstanding debt. Thus, while a reduction in interest rates lowers aggregate demand due to lower interest rate receipts, it can also raise aggregate demand by raising the price of the debt that households own. Such an effect cannot be observed in a model without long-term debt.

The paper is organized as follows: first, we develop the model; second, we explore the effects of forward in active fiscal, passive monetary policy regimes without switching; third, we extend these results to economies that experience switching in fiscal and monetary policy parameters; finally, we conclude.

Model

We use a basic New Keynesian model of the kind Woodford (1998) uses, and augment this model to allow for (1) a richer maturity structure of debt as in Woodford (2001), Eusepi and Preston (2013), and Leeper and Leith (2016); (2) Markov-switching in policy parameters as in Davig and Leeper (2011). This model features a representative household and firm, monopolistic competition in the production of intermediate goods, and price stickiness a la Calvo (1983) according to which $1 - \theta$ fraction of firms can change their prices each period. The model also allows the government to issue both bond portfolios, B_t^m, that have a geometrically decaying maturity structure, and short-term debt, B^s, which is held in net-zero supply. The government collects lump-sum taxes in accordance with an endogenous primary surplus rule, τ_t, and government purchases are assumed to equal 0, so that income, Y_t, equals C_t in equilibrium.

The model is linearized around the non-stochastic steady state with zero inflation. Let $\hat{z}_t \equiv ln(z_t) - ln(\bar{z})$ where $\bar{z}$ is the value of z in steady state. The behavior of households and firms then reduces to two-equations:[2]

$$\hat{y}_t = E_t\hat{y}_{t+1} - \sigma^{-1}(\hat{i}_t - E_t\hat{\pi}_{t+1}) + r_t^n \tag{3.3}$$

$$\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + \kappa\hat{y}_t + \mu_t \tag{3.4}$$

where y is the output gap, π is inflation, β is the household discount factor, σ^{-1} is the intertemporal elasticity of substitution and κ is defined in Appendix B. β, κ, and σ are positive by assumption, and β is also bounded above by 1. Moreover, r_t^n and μ_t evolve according to

$$r_t^n = \rho_n r_{t-1}^n + \epsilon_t^n \tag{3.5}$$

$$\mu_t = \rho_\mu \mu_{t-1} + \epsilon_t^\mu \tag{3.6}$$

Monetary policy is given by:

$$\hat{i}_t = \phi_y(s_t)\hat{y}_t + \phi_\pi(s_t)\hat{\pi}_t + \epsilon_t^{MP} + v_{1,t-1} \tag{3.7}$$

where ϵ^{MP} is an i.i.d monetary policy shock, and $v_{1,t}$ is a linear combination of L forward guidance shocks that obeys

[2] See Appendix B. for full-derivation of model equations

$$v_{1,t} = v_{2,t-1} + \epsilon^R_{1,t} \tag{3.8}$$

$$v_{2,t} = v_{3,t-1} + \epsilon^R_{2,t} \tag{3.9}$$

$$\vdots$$

$$v_{L,t} = \epsilon^R_{L,t} \tag{3.10}$$

such that $v_{1,t-1} = \sum_{l=1}^{L} \epsilon^R_{l,t-l}$, where $[\epsilon^R_{1,t}, \epsilon^R_{2,t}, ..., \epsilon^R_{L,t}]$ are the L forward guidance shocks announced at time t. This model of short-term interest rate guidance is borrowed from Laseen and Svensson (2011) and is widely used in the forward guidance literature. Intuitively, $\epsilon^R_{l,t}$ is a shock announce at time t that affects interest rates at time $t + l$. The general structure of forward guidance shocks given by (3.8)-(3.10) ensure that shocks announced at t are actually realized as intended at $t + l$. As shown in Appendix C and D, policymakers can use $(\epsilon^{MP}_t, \epsilon^R_{1,t}, \ldots, \epsilon^R_{L,t})$ to announce an interest rate peg between time t and $t + L$. To model recent instances of forward guidance we will peg i at or near the zero lower bound on i. Our specification allows for switching in policy parameters: s_t follows a S-state Markov chain, and the value of s_t determines ϕ_π and ϕ_y. Fiscal policy is characterized by the following linearized rule for primary surpluses:

$$\hat{\tau}_t = \gamma(s_t)(\hat{b}^m_{t-1} + \beta\rho\hat{P}^m_t) + z_{ft} \tag{3.11}$$

$$z_{ft} = \rho_F z_{f,t-1} + \epsilon_{ft} \tag{3.12}$$

where $\hat{b^m}_t$ is the percentage deviation of real bonds from steady state, z_{ft} is an exogenous fiscal policy shock, and ϵ_f is an exogenous mean-zero i.i.d innovation. γ is the fiscal authority's policy parameter and it follows the same Markov process as ϕ_y and ϕ_π. Fiscal policy must also satisfy the following budget constraint:

$$\hat{b}^m_{t-1} = \beta(1-\rho)\hat{P}^m_t + \beta\hat{b}^m_t + (1-\beta)\hat{\tau}_t + \hat{\pi}_t \tag{3.13}$$

where $\hat{P}^m_t$ is the price of the bond portfolio at time t and $\rho \in [0,1]$ captures the maturity structure of the government debt. While we relegate the derivation of this equation to Appendix B., the intuition behind the bond portfolio is fairly simple: the government issues $\hat{b}^m_t$ units of a nominal portfolio debt at time t that pays 1 unit of nominal income at time $t+1$, ρ units at time $t+2$, ρ^2 units at $t+3$ and so forth. This is the sense in which the maturity of debt is geometrically decaying. This structure allows us to introduce long-term debt into our model by using a single state variable that captures the average maturity of debt, ρ. The limiting cases of ρ illuminate how larger values of ρ correspond to longer average maturities: when $\rho = 0$, all debt is short term, and when $\rho = 1$, all debt is in the form of consols. As demonstrated in Appendix B, $\hat{P}^m_t$ satisfies

$$\hat{P}^m_t = -\hat{i}_t + \rho\beta E_t\hat{P}^m_{t+1} \tag{3.14}$$

The system given by (3.3)-(3.14) yields a solution for $x_t = (\hat{y}_t, \hat{\pi}_t, \hat{i}_t, \hat{b}_t, \hat{\tau}_t, \hat{P}^m_t)'$. We use Sims' (2002) method to solve the fixed regime model, and the forward method in Cho (2016) and Cho (2017) to solve the switching model.[3] A rational expectations

[3]The "block-recursive" structure of the model requires us to add a constraint to the switching model that renders agents' inflation expectations consistent with the government budget constraint.

equilibrium assumes the form:

$$x_t = \Omega(s_t)x_{t-1} + \Gamma(s_t)u_t$$

Parameters are selected so that the model under study is determinate. While there are no simple analytical conditions for determinacy in our switching model, Woodford (1998) gives simple conditions for determinacy in the case of non-switching (see Table 1):[4]

TABLE 1. Fixed Coefficent Model Determinacy Conditions

	$\phi_\pi > 1 - \frac{1-\beta}{\kappa}\phi_y$	$\phi_\pi < 1 - \frac{1-\beta}{\kappa}\phi_y$
$\gamma \in (1, \frac{\beta^{-1}+1}{\beta^{-1}-1})$	determinate	indeterminate
$\gamma \notin (1, \frac{\beta^{-1}+1}{\beta^{-1}-1})$	no stable solution	determinate

We say they the economy is in Regime M when $\phi_\pi > 1 - (1-\beta)\phi_y/\kappa$ and $\gamma \in (1, (\beta^{-1}+1)/(\beta^{-1}-1))$. and that the economy is in Regime F when $\phi_\pi < 1-(1-\beta)\phi_y/\kappa$ and $\gamma \notin (1, (\beta^{-1}+1)/(\beta^{-1}-1))$. In Regime M, fiscal policy is passive while monetary policy is active. This is the standard assumption in most New Keynesian research. In Regime F, fiscal policy is active while monetary policy is passive.

Fixed Coefficient Exercises

We now examine the effectiveness of forward guidance in the presence of fixed policy regimes (i.e. we constrain all policy parameters to be permanent). Our analysis involves three different model parameterizations: (1) a Regime M parameterization; (2) a Regime F parameterization with short-term debt ($\rho = 0$): (3) a Regime F parameterization with long-term debt. Table 6 in Appendix E contains the parameter

[4]We assume that $\phi_\pi(s_t) \geq 0$ for all s_t

values used in each of the three configurations, though our results are robust to different parameterizations.[5] Our analysis in this section also involves two distinct policy exercises that are commonly used in the literature. First, we examine the impulse responses of output and inflation to a single one unit k-period ahead forward guidance shock to the nominal interest rate. This exercise gives us useful intuition for the second policy experiment, which is the main result in this section. In that exercise, we examine the impulse responses of output and inflation to an announced 12-quarter interest rate peg that mimics aspects of the Federal Reserve's calendar-based forward guidance announcements in August 2011, January 2012, and September 2012 (see Del Negro et al (2015) for more details).

Exercise 1: Inspecting the Mechanism

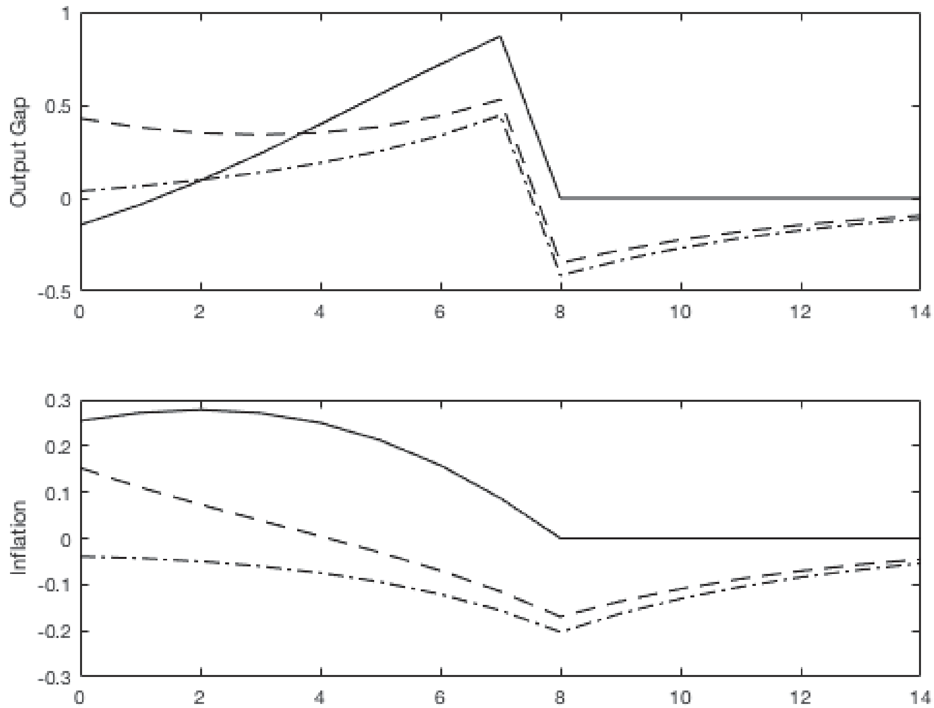
In order to better understand the mechanism driving our main results in the forward guidance experiments, we examine the effects of a one-time expansionary forward guidance shock to the short-term nominal interest rate under all three parameterizations. The exercise takes place as follows: at time t the central bank announces a negative one unit shock to i_{t+k} where $k \geq 0$. Agents respond at time t by adjusting hours worked, consumption, prices and bond holdings, and this generates paths for inflation and output that are plotted in Figure 1 for the cases where $k = 8$.[6]

Overall, output and inflation respond less favorably to forward guidance shocks in a Regime F economy. In a Regime M economy, the negative shock at the k-

[5] There is one exception: for small σ and as ϕ_π approaches 1 in Regime F, the Regime F impulse responses of output and inflation are strictly above the Regime M impulse responses before the realization of the shock in exercise 1 (after the shock, the Regime M impulse responses are above the Regime F responses). This applies only to our fixed regime results and we regard this as an unrealistic parameterization of the model. Our Markov-switching results are robust to reasonable parameterizations.

[6] Qualitatively similar results obtain for different choices of k

FIGURE 1. Impulse Responses to One-Unit Shock



the impulse responses of output and inflation to a one-unit anticipated shock to i_{t+8} at t. The solid line shows impulse responses in the Regime M model; the dashed line shows impulse responses in the Regime F model with long-term debt; the dashed-dotted line shows impulse responses in Regime F with only short-term debt.

horizon causes long run real rates to drop, which induces positive responses in output and inflation. These responses are magnified by the lack of a discount factor in the linearized Euler equation, which causes consumption and therefore output to be highly responsive to changes in long-run real rates. The response of inflation to the shock is driven, in part, by the front-loading in the Phillips curve and the presence of nominal rigidities: firms understand that demand will continue to rise until the shock is realized so they raise their prices the moment they have an opportunity to do so. These effects combine to cause a large stimulus.

In Regime F, however, output and inflation respond to the forward guidance shock through an additional channel: the decline in anticipated short-term interest rates reduces the return on bond holdings. Since agents in a Regime F economy treat their bond holdings as net wealth, expansionary forward guidance has negative wealth effects that counteract the stimulating effects of lower long-run real interest rates. In this framework, a k-period ahead shock initially lowers long-run real interest rates and raises consumption. At the time of the shock's realization, nominal wealth declines and puts downward pressure on prices through the bond valuation equation. After the shock is realized, agents reduce consumption to replenish bond holdings and this puts downward pressure on consumption and output. As with the Regime M case, firms respond by changing prices well in advance of the anticipated deflationary pressure. This might also be driven by the presence of price stickiness: firms recognize the probability that they won't be able to adjust prices at the time of the shock, so they adjust their prices as soon as they have an opportunity to do so. The overall effect of this price setting behavior is a large and persistent deflation. Hence, while the presence of these wealth effects generate less favorable and maybe more plausible responses of output to forward guidance, it comes at the cost of a deflation that is not obviously reconcilable with the data.

Figure 1 also reveals that the presence of long-term debt (i.e. $\rho > 0$) in Regime F leads to higher paths of output and inflation than in a Regime F economy without short-term debt (i.e. $\rho = 0$). This is because the presence of long-term debt introduces yet another channel through which forward guidance impacts output and inflation: the anticipated decline in short-term interest rates raises the price of outstanding debt, and therefore raises the market value of outstanding debt held by the household. This is a debt revaluation effect, and it leans against the aforementioned negative wealth effects.

One notable feature of the impulse response functions is that output responds more favorably to forward guidance on impact, i.e. at the time of the announcement, in the Regime F economies. We attribute this to one feature of the Regime M economy: monetary policy satisfies the Taylor Principle such that the increase in inflation observed on impact corresponds with higher real interest rates on impact. If we allow ϕ_π in all regimes to approach 1 from both directions, we observe similar responses in all economies on impact.

Exercise 2: The Fixed Regime Forward Guidance Experiment

Our second policy experiment in the fixed coefficient model assesses the effects of forward guidance on a specific path for interest rates. Using methods inspired by Del Negro et al. (2015), and Cole (2015), we study what happens when the central bank announces an interest rate target, $\bar{i}$, between time T and $T+L$.[7] We chose $L = 12$ to mimic the September 2012 FOMC statement that called for low interest rates through mid-2015. Additionally, $\bar{i} = 0$ is chosen as a target, but any interest rate target between 0 and 25 basis points may reasonably approximate the path implied by the September 2012 statement.[8] The economy is simulated for $T-1$ periods prior to announcement, and the simulations are repeated 10000 times. Figure 2 and 3 report the mean impulse responses of output, inflation and interest rates to the $L+1$ period anticipated interest rate peg. For simplicity's sake, I shut down shocks after time t so that $i_{t+l} = E_t i_{t+l}$ for $0 \leq l \leq L$. If shocks are present, monetary policymakers use

[7]See appendix C. for further details

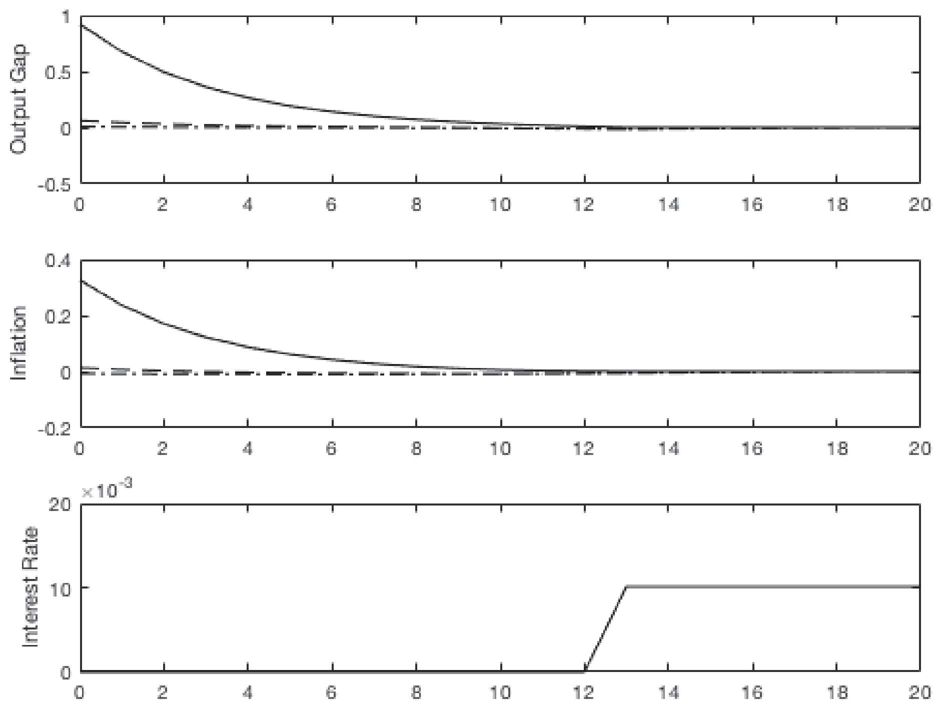
[8]The main qualitative results in this section are robust to any $\bar{i}$ below steady state, $i^* = \beta^{-1} - 1$

some combination of unanticipated and anticipated monetary policy shocks at $t+1$ to $t+L$ to maintain the peg and agents' expectation of the peg over the forward guidance horizon. As such, we regard this simplification as innocuous.

As with the previous exercises, Figures 2 and 3 demonstrate that output and inflation respond less favorably to expansionary forward guidance on interest rates under the assumption of active fiscal, passive monetary policy. In contrast to previous exercises, the forward guidance shocks do not induce a dramatic fall in output in the Regime F economies. This result may be driven by an important feature of the impulse responses in Figure 1: each expansionary forward guidance shock raises output before the shock is realized, and depresses output after the shock is realized (the latter effect is driven, in part, by a sharp drop in long-term real interest rates). Therefore, when the economy is hit by a sequence of such shocks, as in this section, the contractionary effects of realized forward guidance shocks are partially offset by the expansionary effects of unrealized shocks. In general equilibrium, this leads to a relatively flat trajectory for output (see Figure 3). Also in contrast to results from previous exercises, inflation responds much more positively to forward guidance on the interest rate path in the presence of long-term debt. We attribute this result to a particular strong revaluation effect, as forward guidance on $L+1$ future short-term interest rates has a huge impact on $\hat{P}_t^m$ (which is simply a weighted sum of expected future short-term interest rates) .

We emphasize that the strong responses of output and inflation in Regime M are a reflection of the forward guidance puzzle. Also note that Figure 3 uses a different vertical scale than Figure 2.

FIGURE 2. The 12-quarter Forward Guidance Horizon Experiment.

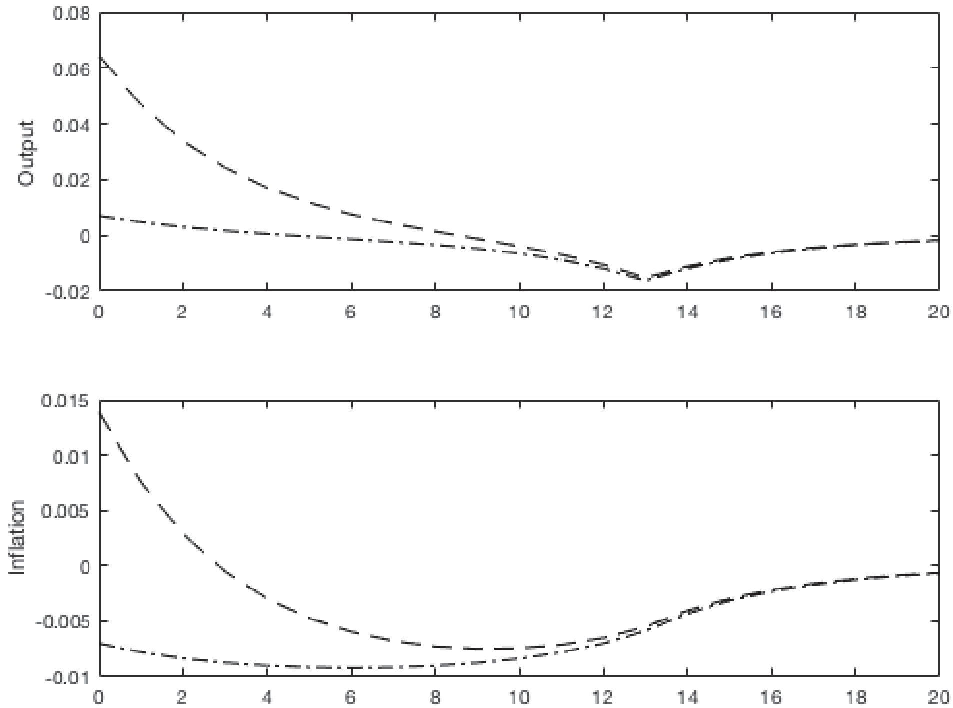


The solid line shows impulse responses in the Regime M model; the dashed line shows impulse responses in the Regime F model with long-term debt; the dashed-dotted line shows impulse responses in Regime F with only short-term debt.

Markov-Switching Forward Guidance Experiment

We now allow the policy stances of the monetary and fiscal authorities to periodically and recurrently change. Specifically, we assume that the economy switches between a Regime F configuration ($s_t = F$) and a Regime M configuration ($s_t = M$). This assumption is restrictive, but it allows us to get at one important mechanism: expectations of changing responses to forward guidance cause agents to behave differently today. These expectational spillovers shock the impulse responses of inflation and output in Regime M away from the paths implied by the corresponding

FIGURE 3. The 12-quarter Forward Guidance Horizon Experiment (Regime F).

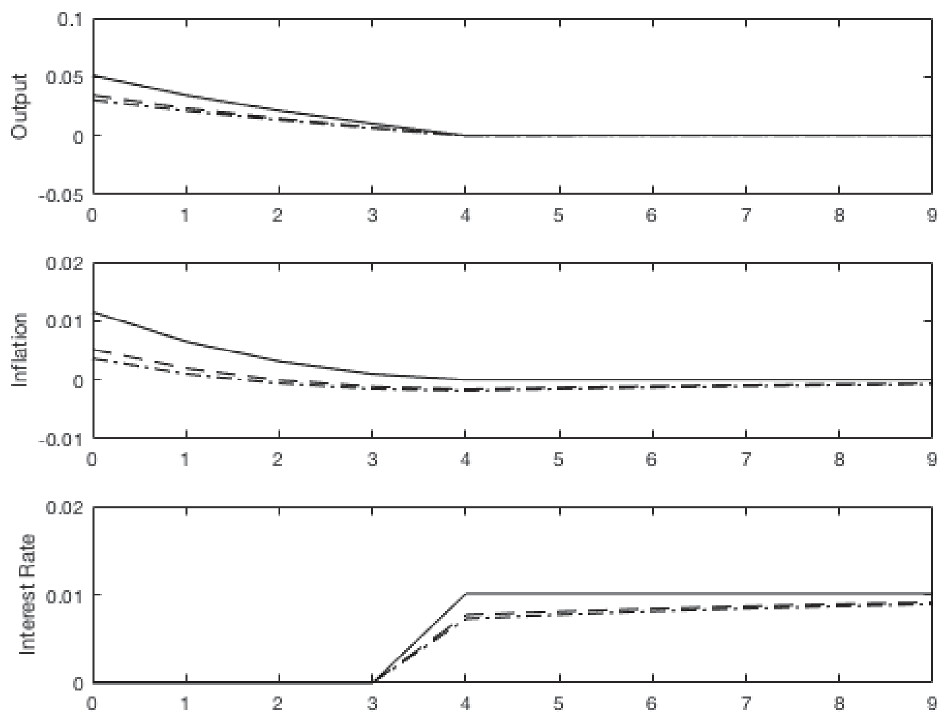


The 12-quarter Forward Guidance Horizon Experiment. The dashed line shows impulse responses in the Regime F model with long-term debt; the dashed-dotted line shows impulse responses in Regime F with only short-term debt.

fixed coefficient models, and may therefore help the impulse responses agree with the data.

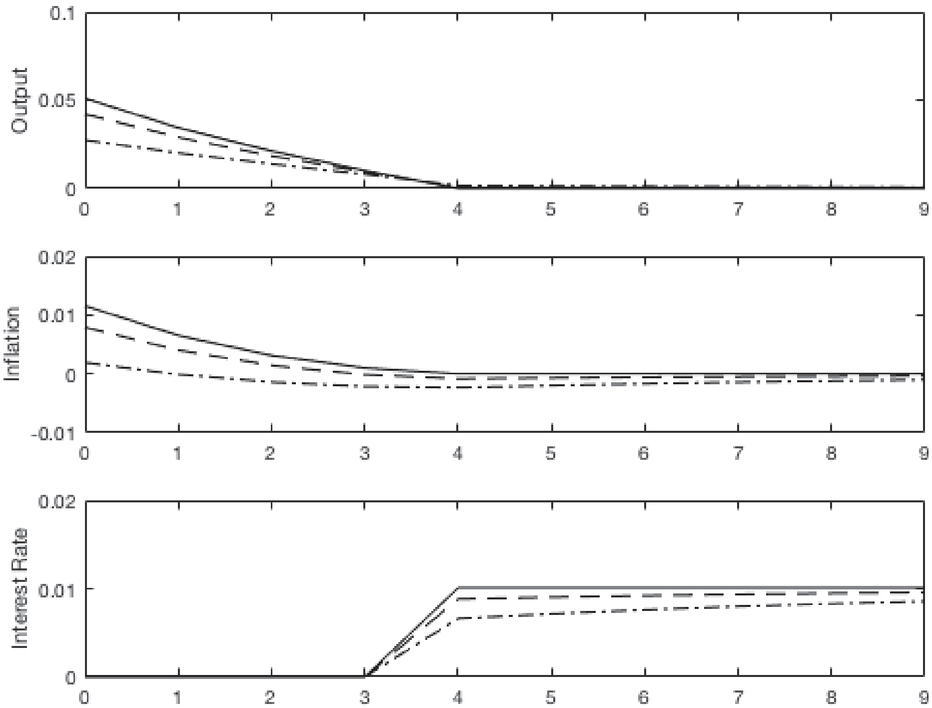
To illustrate this idea, we conduct a forward guidance experiment in the switching model. Specifically, we first assume that the economy is in Regime M when the central bank announces a sequence of shocks at time T such that $i_T = E_T i_{T+1} = ... = E_T i_{T+L} = \bar{i}$. We then assume that the economy remains in Regime M at $T + 1$, when another sequence of shocks is announced such that $i_{T+1} = E_{T+1} i_{T+2} = ... = E_{T+1} i_{T+L} = \bar{i}$. This process is repeated until $T + L$. This experiment shows how the switching economy responds to an announced $L+1$ period interest rate peg in Regime M. Figures 4-5 show the Regime M effects of this experiment when $L = 3$ using a parameterization inspired by a similar model in Ascari et al. (2017) (see Table 7 in Appendix E for the parameter values contained in Figures 4-6; see Appendix D a derivation for the policy experiment). We emphasize that agents do

FIGURE 4. Regime Switching Forward Guidance Experiment 1



The 3-quarter Forward Guidance Horizon Experiment, Parameterization 1. The solid line shows impulse responses in the fixed coefficient model Regime M; dashed line shows impulse responses in the switching model Regime M with long-term debt; the dashed-dotted line shows impulse responses in the switching model Regime M with only short-term debt.

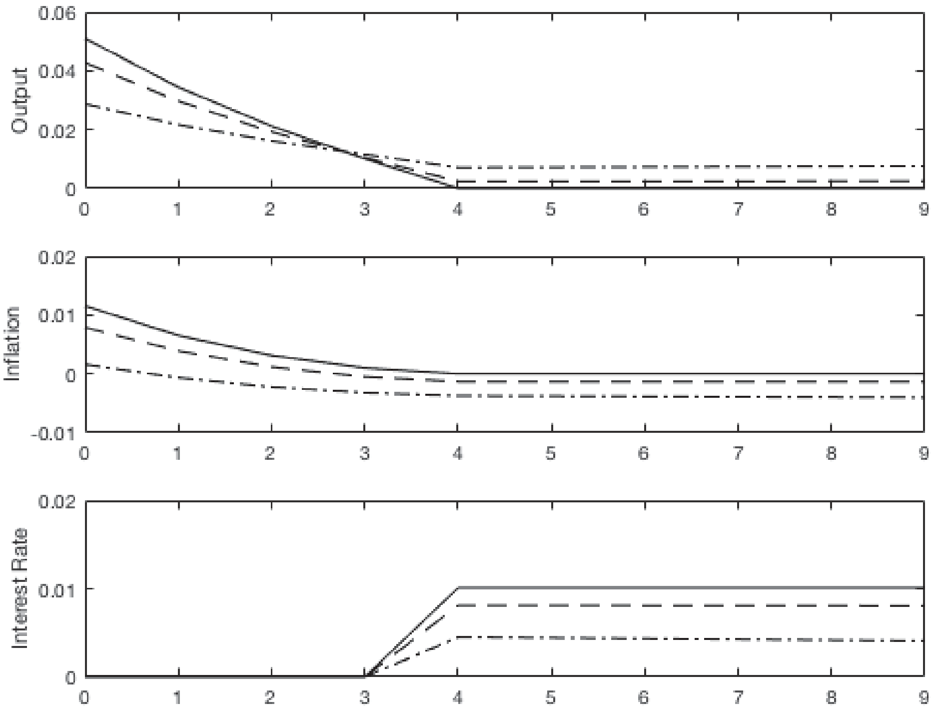
FIGURE 5. Regime Switching Forward Guidance Experiment 2



The 3-quarter Forward Guidance Horizon Experiment, Parameterization 2. The solid line shows impulse responses in the fixed coefficient model Regime M; dashed line shows impulse responses in the switching model Regime M with long-term debt; the dashed-dotted line shows impulse responses in the switching model Regime M with only short-term debt.

not expect the economy to remain in Regime M throughout the forward guidance horizon. Agents form rational expectations using the true transition probabilities (e.g. $E_T(s_{T+1} = M | s_T = M) = p_{MM}$ where p_{MM} is the probability of remaining in Regime M). We only hold M fixed to compare the Regime M impulse responses in the switching model, to the Regime M impulse responses in the fixed regime model. More generally, we could allow for regime-switching during our forward guidance experiment.

FIGURE 6. Regime Switching Forward Guidance Experiment 3



The 3-quarter Forward Guidance Horizon Experiment, Parameterization 3. The solid line shows impulse responses in the fixed coefficient model Regime M; dashed line shows impulse responses in the switching model Regime M with long-term debt; the dashed-dotted line shows impulse responses in the switching model Regime M with only short-term debt.

Relative to the fixed regime cases, expansionary forward guidance appears to be less stimulative in the switching model's Regime M. In Regime M, this is driven by the positive probability that the economy will switch to a state where the expansionary shock has negative wealth effects. Crucially, these spillover effects exist because policy is anticipated;[9] in an environment where all shocks are *i.i.d* unanticipated shocks, such

[9]A persistent unanticipated shock would deliver similar spillover effects. We leave this for future research.

spillover effects are not observed because there is zero probability that a given shock will affect the economy in a future regime.

While the qualitative results in Figures 4-5 are robust to different policy coefficients, structural parameters, transition probabilities, and forward guidance horizons, Figure 6 shows that the output and inflation impulse responses in the switching model can barely overshoot and undershoot the fixed regime responses after the interest rate peg is over when inflation reaction coefficients are high and the fiscal policy parameter in Regime M is relatively low.[10] We have two remarks about this particular result. First, a regime switch quickly eliminates the persistent output and inflation gaps. In the calibrated model, these switches occur every 20 periods on average. Second, lower inflation and higher output reaction coefficients in the interest rate rule help to raise i faster and close the gaps.

Conclusion

Standard New Keynesian models predict implausibly large and favorable responses of inflation and output to forward guidance on interest rates. This paper investigates the effects of forward guidance in a New Keynesian model with active fiscal policy and passive monetary policy. We find that the presence of active fiscal policy allows for forward guidance to have wealth effects that dampen the response of output and inflation to forward guidance, potentially at the cost of implausible deflation. In an active fiscal, passive money regime, the deflationary effects of forward guidance are mitigated by the presence of long-term debt. Moreover, the presence of switching in fiscal and monetary policy may have expectational spillover effects that cause forward guidance to be less stimulative in a passive fiscal, active money regime.

[10] See Appendix E. for parameter values

CHAPTER IV

MATURITY, DETERMINACY, AND E-STABILITY

Introduction

A vast literature examines determinacy and E-stability in New Keynesian models. This work may guide the design of monetary policy by offering tractable conditions for ensuring the existence of unique, learnable equilibria. For example, the Taylor Principle, by which central banks move real interest rates in the direction of inflation, is widely known to guarantee determinacy and E-stability in New Keynesian models when fiscal policy is "passive" in the sense described in Chapter 2 (Woodford (2003), Bullard and Mitra (2002)). Taking this point one step further, the simple Leeper (1991) conditions also apply to a broad class of New Keynesian models to help characterize determinacy when fiscal policy is not passive. Typically, those papers study linear approximations to economies that involve time-invariant fiscal and monetary policy regimes, despite ample econometric work that suggests recurring changes in the conduct of policy. Past work may assume time-invariant policy for at least one reason: tractability. The advent of new solution techniques and conditions for determinacy has made it possible to more fully examine determinacy and E-stability in a New Keynesian model with 2-state Markov-switching fiscal and monetary policy (see Cho (2016), Foerster et al (2016)). This paper uses those techniques to study determinacy (in the mean-square stability sense) and E-stability in a New Keynesian model with Markov-switching fiscal and monetary policy. We emphasize three results.

First, the maturity structure of government debt matters for determinacy and the existence of stable equilibria. We regard this as a significant result in part because the maturity structure of government debt is irrelevant for determinacy in the fixed regime analog of the model without a real risk premium on long-term debt (Eusepi and Preston (2011), Jin (2013)). Our results therefore suggest that time-variation in the fiscal policy stance on debt matters not only for equilibrium dynamics, but for the existence and uniqueness of rational expectations equilibria. More specifically, our results suggest that the maturity structure of debt impacts both the existence of sunspot equilibria and the number of stable fundamental (MSV) solutions. We demonstrate the former claim using a numerical search method from Cho (2016), and the latter claim using solution techniques from Foerster et al. (2016). We utilize these two approaches because they complement each other's strengths and weaknesses nicely. This result is significant for at least one other reason: a policy parameterization that yields determinacy in a model with only short-term debt may yield indeterminacy or no stable solutions in a model with more realistic debt maturity. As such, misspecified models that abstract away from rich maturity structures may offer suboptimal policy recommendations.

Second, determinacy generally implies E-stability in the 2-state switching model when agents do not observe contemporaneous endogenous variables and employ one-step-ahead decision rules. This finding extends our result in Chapter 2 and indicates that determinacy is mostly sufficient for the stability of an equilibrium under learning. Importantly, these results do not extend to cases where regimes lack persistence and fiscal policy is occasionally or permanently extremely active. While these regions of the parameter space are arguably unreasonable, we document them in this paper.

Finally, we derive the necessary and sufficient conditions for stability under infinite horizon learning, and numerically compare infinite horizon and one-step-ahead E-stability conditions. To the best of our knowledge, this is the first work to derive conditions for infinite horizon learning in a model with Markov-switching coefficients. Our preliminary analysis suggests that determinate equilibria are generally stable under both infinite-horizon and one-step-ahead learning.

The paper is organized as follows: first, we explore the relationship between debt maturity structure and the existence and uniqueness of rational expectations equilibria; second, we explore the relationship between determinacy and E-stability under one-step-ahead learning; third, we present the infinite-horizon learning results. Unless otherwise stated, our analysis involves the New Keynesian model developed in Appendix B. The following change of notation is used for convenience: $\gamma(\tilde{s_t}) = (1-\beta)\gamma(s_t)$

Maturity and Determinacy

We use New Keynesian model developed in B. to show that the maturity structure of debt (specifically, the average maturity of outstanding debt, ρ) matters for the existence and uniqueness of rational expectations equilibrium. As in Chapter 2, we cast our model in the form:

$$X_t = M(s_t)E_t(X_{t+1}) + N(s_t)X_{t-1} + Q(s_t)U_t$$

where X_t is $n \times 1$ vector of endogenous variables, U_t is $m \times 1$ vector of exogenous variables that follows

$$U_t = \rho U_{t-1} + \epsilon_t$$

A MSV, or fundamental, solution to the model assumes the form:

$$X_t = \Omega(s_t)x_{t-1} + \Gamma(s_t)U_t$$

Our analysis proceeds in a series of steps. First, we solve the models under study using the forward method in Cho (2016) and search for regions of the parameter space consistent with determinacy, indeterminacy and explosive MSV solutions across the parameter space. Since the conditions for determinacy in Cho (2016) are only sufficient, we must prove indeterminacy or the non-existence of stable MSV solutions whenever applicable. To prove indeterminacy, we either search for stable sunspot equilibria using a search routine developed in Cho (2016), or we solve for all the MSV solutions using methods from Foerster et al (2016) and examine their mean-square stability one by one. If multiple fundamentals solutions are stable, the model is indeterminate. To demonstrate explosiveness, we either appeal to a necessary condition for mean-square stability to rule out the existence of stable fundamental solutions, or we use the aforementioned routine in Foerster et al (2016) to demonstrate that all fundamentals solutions are explosive. We rely on such a complex scheme for checking indeterminacy and explosiveness because neither Cho (2016) nor Foerster et al (2016) offer necessary and sufficient conditions for determinacy.

Our approach starts with the methods in Cho (2016), which are explored extensively in Chapter 2. The results in that paper apply exclusively to the forward solution, whose equilibrium coefficients are acquired by taking the limit of the coefficients on lagged endogenous and exogenous variables in the forward solved model. Using the forward solution coefficients, Cho (2016) constructs two matrices with spectral radii that offer sufficient conditions for determinacy: when both spectral radii are inside the unit circle, the forward solution is the unique rational expectations equilibrium. This last condition is only sufficient for determinacy. If one or both spectral radii are outside of the unit circle, we must prove indeterminacy or the non-existence of stable MSV solutions.

When the forward solution is stable, but does not satisfy conditions for determinacy, we first search for sunspot equilibria of the form:

$$
\begin{aligned}
x_t &= \Omega^*(s_t)x_{t-1} + \Gamma^*(s_t)u_t + w_t && (4.1) \\
w_{t+1} &= \Lambda(s_t, s_{t+1})w_{t-1} + V(s_{t+1})V(s_{t+1})'\eta_{t+1} && (4.2)
\end{aligned}
$$

where η is mean-zero i.i.d., and $V(s_{t+1})$ is a matrix with orthonormal columns. $\Lambda(s_t, s_{t+1})$ and $V(s_{t+1})$ satisfy the following conditions for each $i, j \in \{1, 2, ..., S\}$:

$$
\begin{aligned}
\Lambda(i,j) &= V(j)\Phi(i,j)V(j)' && (4.3) \\
V(i) &= \sum_{j=1}^{S} p_{ij}F^*(i,j)V(j)\Phi(i,j) && (4.4) \\
r_\sigma(\bar{\Psi}_\Lambda) &= r_\sigma \begin{pmatrix} p_{11}\Lambda(1,1) & p_{21}\Lambda(2,1) & \cdots & p_{S1}\Lambda(S,1) \\ p_{12}\Lambda(1,2) & p_{22}\Lambda(2,2) & \cdots & p_{S2}\Lambda(S,2) \\ \vdots & & \ddots & \vdots \\ p_{1S}\Lambda(1,S) & p_{2S}\Lambda(2,S) & \cdots & p_{SS}\Lambda(S,S) \end{pmatrix} < 1 && (4.5)
\end{aligned}
$$

for some conformable matrix $\Phi(j)$, where $F^*(s_t) = \{I - E_t(M(s_t)\Omega^*(s_{t+1}))\}^{-1}M(s_t)$. The last condition is necessary and sufficient for mean-square stability. If that condition is satisfied, then equation (4.1) is mean-square stable as well. Moreover, we can show that the existence of a single mean-square stable w_t is sufficient for the existence of a continuum of mean-square stable processes. Indeterminacy follows.

Cho(2016) presents a simple method for detecting indeterminacy: minimize $r_\sigma(\bar{\Psi}_\Lambda)$ in (4.5) subject to constraints (4.2)-(4.4) with respect to $V(j)$ and $\Phi(i,j)$ using the fminsearch function in Matlab. In practice, this approach is inefficient for at least two reasons. First, our approach attempts to minimize a function with a relatively flat bottom. This means that it often takes hundreds of initial guesses to find a stable sunspot equilibrium.[1] Second, initial guesses for $V(j)$ and $\Phi(i, j)$ are arbitrary. This is justified by a lack of *a priori* reasons for picking particular values of those matrices as initial guesses.

Since we are working with a model that includes lagged endogenous variables, we know that the fundamental solution is not typically unique. Consequently, we might check to see if multiple fundamental solutions are stable when indeterminacy is suspected. As we demonstrate below, it is relatively straightforward to express the equilibrium coefficients as the solutions to a system of quadratic polynomials. Without switching, this system can be solved using the generalized Schur decomposition (Uhlig(1999), Klein(2000)). With switching coefficients, however, we cannot appeal to the cited results. Instead, we follow Foerster et al (2016) and use Gröbner bases. A Gröbner basis is a transformed system of polynomials with the same set of solutions as the original system of polynomials under study. While there are many ways to accomplish the aforementioned transformation, the Shape Lemma suggests that one very useful transformation generally exists. Here, we restate Foerster et al. (2016)'s presentation of the Shape Lemma

The Shape Lemma *There exists an open dense subset S of systems of n polynomial in n unknowns,* $\{x_1, ..., x_n\}$ *such that for every system in S, there exists a system with same roots that assumes the following form*

[1] Each attempt to minimize the spectral radius function takes several seconds at least. We find that for some parameterizations involving inflation reaction coefficients close to unity, over 5,000 initial guesses must be made to find a single stable sunspot equilibrium. The issues we discuss here are also addressed in Cho (2016)

$$x_1 = q_1(x_n)$$
$$x_2 = q_2(x_n)$$
$$\vdots$$
$$0 = q_n(x_n)$$

where each q_i is a univariate polynomial in n.

According to the Shape Lemma, we can solve a complicated system of quadratic polynomials by simply solving a series of univariate polynomials. We now apply these methods to the New Keynesian model we derive in Appendix B. To help deliver some key results concisely, we abstract from nominal rigidity and exogenous shocks.[2] Also, we will temporarily assume that there is only one policy Markov state ($s_t = 1 \; \forall t$) to illustrate the effect of ρ on the existence and uniqueness of model equilibrium. Together, these restrictions on the model yield the following system of equations that govern the laws of motion for inflation, π_t, bond portfolios, b_t, and the price of bond bortfolios, P_t:[3]

$$\begin{aligned}
\phi_\pi \pi_t &= E_t \pi_{t+1} \\
P_t &= -\phi_\pi \pi_t + \beta \rho E_t P_{t+1} \\
b_t &= \beta^{-1}(1-\tilde{\gamma}) b_{t-1} + (1-\rho+\rho\tilde{\gamma}) P_t + \beta^{-1}\pi_t
\end{aligned}$$

[2]Since we are only interested in studying the determinacy properties of our models, we can exclude shocks without affecting any of our results

[3]In the absence of nominal rigidity, the output gap, $y_t = 0 \; \forall t$

where the first equation combines $i_t = \phi_\pi \pi_t$ and the Fisher relation, and the second and third equations describe a no-arbitrage equation and the government budget constraint, respectively. Without exogenous shocks, a MSV solution to the above system of equations assumes the form $x_t = \Omega_x b_{t-1}$ for each $x \in \{\pi, b, P\}$ where the Ω_x solves:

$$\begin{aligned} \phi_\pi \Omega_\pi &= \Omega_\pi \Omega_b \\ \Omega_P &= -\phi_\pi \Omega_\pi(1) + \beta\rho\Omega_P \\ \beta^{-1}(1-\tilde{\gamma}) &= \Omega_b + (1-\rho+\rho\tilde{\gamma})\Omega_P + \beta^{-1}\Omega_\pi \end{aligned}$$

There are three solutions to this model. The stability of a solution is determined Ω_b which assumes one of three values depending on the solution at hand: ϕ_π, $\beta^{-1}(1 - \tilde{\gamma}(1))$, $1/\rho\beta$. The first solution is the active fiscal, passive monetary policy solution, in which inflation responds to debt, and the second solution is the active monetary, passive fiscal policy solution. The Leeper (1991) conditions for determinacy ensure that one and only one of those two solutions are inside the unit circle (which satisfies the condition for mean-square stability). Crucially, these roots do not depend on ρ, and the only root that does not on ρ is greater than one for all $\rho \in [0, 1]$. This is the sense in which the maturity structure of debt is irrelevant for determinacy in the fixed regime model. If, however, we allow for regime change (i.e. $s_t \in \{1, 2\} \ \forall \ t$ where s_t follows a 2-state Markov process), our simple model becomes:

$$\begin{aligned} \phi_\pi(s_t)\pi_t &= E_t \pi_{t+1} \\ P_t &= -\phi_\pi(s_t)\pi_t + \beta\rho E_t P_{t+1} \\ b_t &= \beta^{-1}(1-\tilde{\gamma}(s_t))b_{t-1} + (1-\rho+\rho\tilde{\gamma}(s_t))P_t + \beta^{-1}\pi_t \end{aligned}$$

A MSV solution to the above system of equations now assumes the form $x_t = \Omega_x(s_t)b_{t-1}$ for each $x \in \{\pi, b, P\}$ where the $\Omega_x(s_t)$ solve:

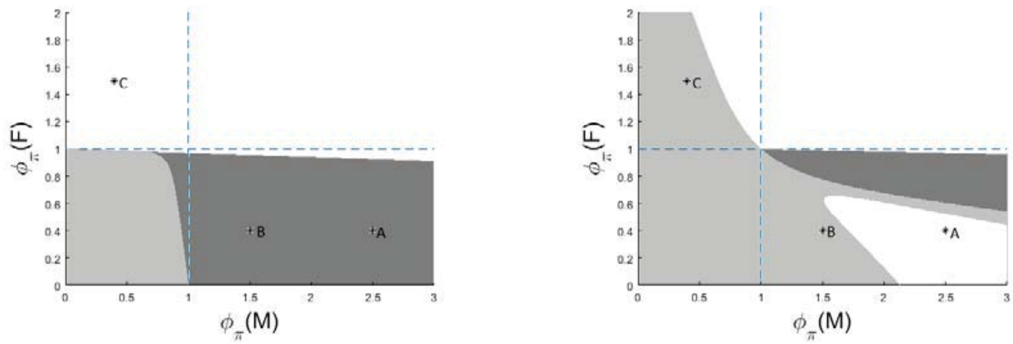
$$\begin{aligned}
\phi_\pi(1)\Omega_\pi(1) &= (p_{11}\Omega_\pi(1) + p_{12}\Omega_\pi(2))\Omega_b(1) \\
\phi_\pi(2)\Omega_\pi(2) &= (p_{21}\Omega_\pi(1) + p_{22}\Omega_\pi(2))\Omega_b(2) \\
\Omega_P(1) &= -\phi_\pi(1)\Omega_\pi(1) + \beta\rho(p_{11}\Omega_P(1) + p_{12}\Omega_P(2))\Omega_b(1) \\
\Omega_P(2) &= -\phi_\pi(2)\Omega_\pi(2) + \beta\rho(p_{21}\Omega_P(1) + p_{22}\Omega_P(2))\Omega_b(2) \\
\beta^{-1}(1-\tilde{\gamma}(1)) &= \Omega_b(1) + (1-\rho+\rho\tilde{\gamma}(1))\Omega_P(1) + \beta^{-1}\Omega_\pi(1) \\
\beta^{-1}(1-\tilde{\gamma}(2)) &= \Omega_b(2) + (1-\rho+\rho\tilde{\gamma}(2))\Omega_P(2) + \beta^{-1}\Omega_\pi(2)
\end{aligned}$$

As mentioned before, we find model MSV solutions using either the forward method outlined by Cho (2016) or the Gröbner basis method discussed in Foerster et al (2016). Since we are unable to find analytical solutions to the model, we set $p_{11} = p_{22} = .95$, $\tilde{\gamma}(1) = .05$, $\tilde{\gamma}(2) = -.05$, $\beta = .99$, though we can obtain similar results using vastly different parameterizations. Figure 7 shows determinacy regions in $(\phi_\pi(1), \phi_\pi(2))$ space for the model with short-term debt ($\rho = 0$) and the model with long-term debt ($\rho = .96$).[4]

Figure 7 makes it clear that variation in ρ can have substantial effects on the menu of policy options available to monetary policymakers. At point A, where $\phi_\pi(1) = 2.5$ and $\phi_\pi(2) = .4$, we use Cho (2016) to prove determinacy for the short-term model, and we use the Gröbner basis routine in Mathematica to show that there is no stable real-valued MSV solution when $\rho = .96$. At point B, where $\phi_\pi(1) = 1.5$ and $\phi_\pi(2) =$

[4] $\rho = .96$ roughly imitates the average duration of outstanding debt observed in recent U.S. data (25 quarters)

FIGURE 7. Determinacy and Maturity: Fisher Model



Left panel $\rho = 0$; Right panel $\rho = .96$; $p_{mm} = .95$; $p_{ff} = .94$; $\gamma(M) = .05$; $\gamma(F) = -.05$. The determinate region is dark gray; the indeterminate region is light gray; explosive region is white

.4, techniques from Cho (2016) still yield a unique stable MSV solution for the case where $\rho = 0$, but the Gröbner basis method now yields two stable MSV solutions when $\rho = .96$. At point C, we use the Gröbner basis routine to show that no stable MSV solutions exist when $\rho = 0$, but techniques from Cho (2016) reveal that we have sunspot indeterminacy when $\rho = .96$. These points demonstrate that the addition of long-term debt to the model can greatly complicate our determinacy analysis. In some cases, such as at point A, the addition of long-term debt reduces the number of equilibria, while at others, such as B and C, it can expand the set of stable MSV solutions and lead to indeterminacy.

When we back away from the assumption of flexible prices in the model, we allow for the output gap, y, to be nonzero. In this environment, equations of the form $y_t = \Omega_y(s_t)b_{t-1}$ give us the equilibrium dynamics of the output gap. Because nominal ridigity gives us non-zero output gaps, our equilibrium coefficients now must solve the following more difficult system of equation:

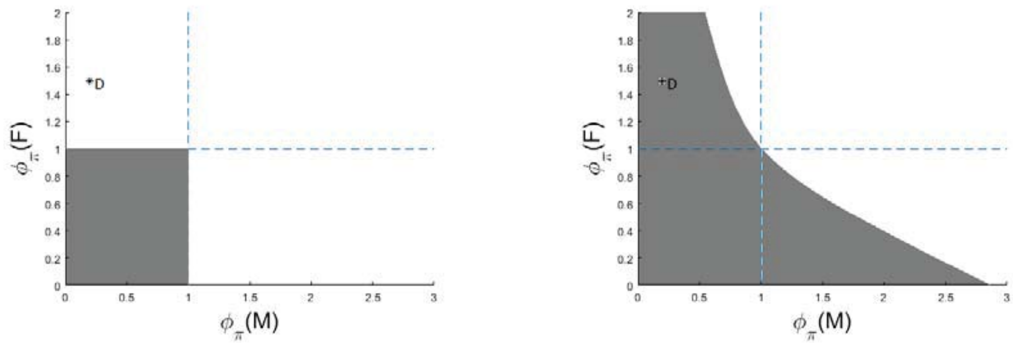
$$
\begin{aligned}
\Omega_y(1) &= \sum_j (p_{1j}(\Omega_y(j) + \sigma^{-1}\Omega_\pi(j)))\Omega_b(1) - \sigma^{-1}\phi_\pi(1)\Omega_\pi(1) \\
\Omega_y(2) &= \sum_j (p_{2j}(\Omega_y(j) + \sigma^{-1}\Omega_\pi(j)))\Omega_b(2) - \sigma^{-1}\phi_\pi(2)\Omega_\pi(2) \\
\Omega_\pi(1) &= \kappa\Omega_y(1) + \beta(p_{11}\Omega_\pi(1) + p_{12}\Omega_\pi(2))\Omega_b(1) \\
\Omega_\pi(2) &= \kappa\Omega_y(2) + \beta(p_{21}\Omega_\pi(1) + p_{22}\Omega_\pi(2))\Omega_b(2) \\
\Omega_P(1) &= -\phi_\pi(1)\Omega_\pi(1) + \beta\rho(p_{11}\Omega_P(1) + p_{12}\Omega_P(2))\Omega_b(1) \\
\Omega_P(2) &= -\phi_\pi(2)\Omega_\pi(2) + \beta\rho(p_{21}\Omega_P(1) + p_{22}\Omega_P(2))\Omega_b(2) \\
\beta^{-1}(1 - \tilde{\gamma}(1)) &= \Omega_b(1) + (1 - \rho + \rho\tilde{\gamma}(1))\Omega_P(1) + \beta^{-1}\Omega_\pi(1) \\
\beta^{-1}(1 - \tilde{\gamma}(2)) &= \Omega_b(2) + (1 - \rho + \rho\tilde{\gamma}(2))\Omega_P(2) + \beta^{-1}\Omega_\pi(2)
\end{aligned}
$$

Figure 8 uses this model to offer intuition on how longer maturity structures of debt held by households can generate an expanded determinacy region in the monetary policy parameter space. Consider point D, for example, which yields an unique equilibrium when $\rho = .96$ and no stable MSV solutions when $\rho = 0$. The reason point D generates explosive dynamics when $\rho = 0$ has to do with the extent to which monetary policy is active in Regime M.[5] The fiscal policy parameterization featured in this figure is too active to yield a stable Ricardian equilibrium. As such, increases in debt are viewed as net wealth, and if monetary policymakers respond using active monetary policy, explosive dynamics may result. For determinacy to obtain, monetary policymakers must implement a consistenly passive monetary policy rule.

As we lengthen the maturity structure, however, the aforementioned active monetary policy regime also revalues the outstanding debt stock in a manner that

[5] Here Regime M refers to the least active fiscal regime

FIGURE 8. Determinacy and Maturity: New Keynesian Model



Left panel $\rho = 0$; right panel $\rho = .96$; $p_{mm} = .95$; $p_{ff} = .94$; $\gamma(M) = .05$; $\gamma(F) = -.05$. The determinate region is dark gray; the indeterminate region is light gray; explosive region is white

tempers the explosive economic response. For example, if debt increases today, which without Ricardian equivalence raises inflation and therefore real interest rates under active monetary policy, agents will feel wealthier due to higher real rates of return on their net wealth, but they will also feel poorer because the interest rate increase reduced the price of oustanding debt. The former effect is destabilizing and present in both the long-term and short-term debt models, whereas the latter stabilizing effect is only prevalent in the long-term model. These revaluation effects at point D are evidently great enough to generate a MSV process with well-defined asymptotic variance.

As revealed by all of these figures, policy configurations that generate determinacy in the short-term debt model may yield indeterminacy in the corresponding long-term debt model, and vice versa. Consequently, policy recommendations stemming from models with only short-term debt, such as the models commonly employed in the New Keynesian literature, may yield suboptimal outcomes when fiscal policy is characterized by switching. Additionally, Figure 8 reveals the addition of long-term debt to the model may expand regions of the policy parameter space consistent with determinacy. Interestingly, fiscal policy does not need to switch to generate this expanded determinacy region. In other words, maturity allows central banks to choose from a wider menu of policy options that are consistent with determinacy and E-stability. This result is left for further study.

Determinacy and E-stability

This section builds on recent work by Ascari et al. (2016) and Cho and Moreno (2016), which characterizes conditions for determinacy in our two-state model with only short-term debt. Their analyses suggest an even more complicated role for monetary-fiscal policy coordination than in the fixed regime case where determinacy is completely described by the simple Leeper (1991) conditions. For example, their works separately confirm that switching between policy configurations that satisfy the Leeper (1991) determinacy conditions is not sufficient for determinacy in the switching model. Moreover, switching into explosive regime configurations or indeterminate regime configurations does not preclude determinacy in the switching model. Determinacy in the switching environment depends on the balance of policy. If, for example, fiscal policy is very active or monetary policy is very passive in one regime, then fiscal policy must be very passive or monetary policy must be very active in the other. Similarly, if monetary policy is predominantly active (passive) across regimes, then fiscal policy must be predominantly passive (active) across regimes as well.

We extend Ascari et al. (2016) and Cho and Moreno (2016) by studying the relationship between determinacy and E-stability in this model class. This section also extends our analysis in Chapter 1 by relaxing the assumption that agents observe contemporaneous endogenous variables when forming expectations of next period's endogenous variables. Instead, we assume that agents observe all contemporaneous exogenous variables, but only observe endogenous variables with a lag. That is, agents use the full history of endogenous variables up to $t-1$ when forming expectations at t.[6] First, we assume that agents observe all contemporaneous variables and study determinacy and E-stability numerically. We find that the forward solution is E-stable for all indicated combinations of parameter values whenever the sufficient conditions for determinacy are also satisfied:

$$\begin{array}{ccc} \phi(M) \in [1,5] & \phi(F) \in [0,1] & \tilde{\gamma}(M) \in (.01,.2] \\ \tilde{\gamma}(F) \in [-.1,01) & \kappa \in [0,1] & \beta \in [.975,1] \\ \sigma \in [0,5] & p_{mm} \in [.8,1) & p_{ff} \in [.8,1) \\ \rho \in [0,1] & & \end{array}$$

where p_{ff} and p_{mm} are the probabilities of remaining within regimes F and M, respectively. We emphasize two things about this set of parameter values. First, we present a very conservative set. It is certainly the case that parameter values outside of this set or consistent with determinacy and E-stability. Second, and more importantly, the works cited in the literature of Chapter 1 all use parameter values that fall in this set. This analysis therefore supports the conclusion that determinacy is sufficient for E-stability for reasonable parameterizations of the model.

To help illustrate our results, we reproduce figures from Cho and Moreno (2016) with the addition of E-stability regions (see Appendix J). As suggested earlier, determinacy implies E-stability for the parameter values considered therein. We also want to point out that for indeterminate regions of the parameter space, the forward solution is generally unstable when the conditions for determinacy in the

[6] Stability conditions are derived in E.

mean-stable sense fail. However, determinacy in the mean-stability sense is neither necessary nor sufficient for E-stability. This notwithstanding, the close relationship between mean-stability and learnability echo results in Branch, Davig and McGough (2013). That paper, which we describe in Chapter 2, studies a class of models without lagged endogenous variables. They find that a condition for determinacy, known as the Conditionally Linear Determinacy Condition (CLDC) implies E-stability. It is straightforward to show that the CLDC is equivalent to determinacy in the mean-stable sense.

Our results also suggest that determinacy is not sufficient for E-stability when agents do not observe contemporaneous endogenous variables. When the model features a persistent active fiscal policy regime with very large negative values of $\tilde{\gamma}$, as well a very short-lived regime (i.e. p_{11} to zero where Regime 1 is the short-lived regime), the relationship between determinacy and E-stability breaks down. This is true regardless of the fiscal policy stance in the short-lived regime, and is particularly relevant when inflation reaction coefficients are relatively high in both regimes. Our preliminary analysis suggests that most determinate equilibria become E-unstable as $\tilde{\gamma}(2)$ approaches negative infinity–this even applies to cases where all other parameters are insider the set offered above.

Infinite-Horizon Learning

We now present results in our model when agents employ infinite horizon decision rules. In order to accomplish this, we need to replace equations (3.3) and (3.4) in

Chapter 3 with the following equations:

$$\begin{aligned} y_t &= E_t \sum_{T \geq t} \beta^{T-t}[(1-\beta)y_{T+1} - \sigma^{-1}(i_T - \pi_{T+1}) + r_T^n] \\ \pi_t &= \kappa\pi_t + E_t \sum_{T \geq t} (\alpha\beta)^{T-t}[\kappa\alpha\beta y_{T+1} + (1-\alpha)\beta\pi_{T+1} + \mu_T] \end{aligned}$$

All other equations, including agents' perceived law of motion, are the same as before. If we assume that agents understand the aggregate probability laws associated to the evolution of inflation and output, as is assumed under rational expectations, these two equations reduce to (3.3) and (3.4) in Chapter 3. Under adaptive learning, however, it is not assumed that agents understand how other agents form expectations. As such, they may lack sufficient information to deduce the fact that their expectations are the aggregate expectations. The infinite horizon learning approach therefore avoids the strong assumption that learning agents know the relevant aggregate probability laws.

The actual laws of motion and stability conditions are derived in Appendix G.[7] To be the best of our knowledge, this is the first work to produce infinite horizon stability conditions in a macroeconomic model with Markov-switching coefficients. Generally speaking, we find that stability under one-step-ahead learning generally coincides with stability under infinite horizon learning for determinate models. This relationship between infinite horizon E-stability and one-step-ahead stability breaks down for intermediate average debt maturities, and when the central bank employs the following interest rate rule: $i_t = E_{t-1}\phi_\pi\pi_t$. These results echo results in Eusepi and Preston (2011, 2013), which show that intermediate average durations of outstanding debt, and the abovementioned policy rule are generally inconsistent with stability

[7]For brevity's sake, we present the actual laws of motion for an economy without persistent shock processes. We can relax this assumption and derive stability conditions for models with persistent processes as well.

under infinite horizon learning in an analogous model with fixed regimes. This suggests that we can dispense with algebraically burdensome infinite horizon learning models when studying models of this form.

Conclusion

To conclude, we explore determinacy and E-stability in a New Keynesian model with switching fiscal and monetary policy. Here we present three categories of results. First, the maturity structure of government debt matters for determinacy and the existence of stable equilibria in our switching model, which is not true in the analogous fixed coefficient model. We use two numerical solution techniques to show that maturity affects both the multiplicity of stable solutions, and the existence of sunspot equilibria. Second, determinacy generally implies E-stability when agents do not observe contemporaneous observable variables, but not for certain arguably unrealistic regions of the model parameter space. Third, we present conditions for stability under infinite-horizon learning in Markov-switching DSGE models and compare stability under infinite horizon and one-step-ahead learning. To the best of our knowledge, this is the first paper to derive these stability conditions in a model with switching coefficients.

CHAPTER V

PERFORMANCE OF SIMPLE INTEREST RATE RULES SUBJECT TO FISCAL POLICY

Introduction

This paper examines the performance and robustness of simple monetary policy rules in New Keynesian models with: (1) permanent or occasionally non-Ricardian fiscal policy; (2) long-term government debt. In these models, time-variation in the fiscal policy stance on debt is captured by Markov-switching in fiscal policy rule coefficients. Policy performance is measured in terms of a loss function that equals some weighted average of the variance of inflation and the variance of the output gap, and central banks are tasked with selecting implementable interest rate rules that minimize loss taking fiscal policy as given. While a primary goal of this project is to identify optimal interest rate rules in models with rational expectations, we endeavor to construct policies that are robust to parameter and model uncertainty, and that also perform well in models with constant gain learning.

Our contributions provide answers to three questions. First, should monetary policymakers respond to time-varying fiscal policy stances on the debt by implementing time-varying monetary policy? A growing body of work argues that fiscal policy stances on the debt are time-varying.[1] By offering a potential answer to this question, we may better understand the importance of precisely identifying the timing and magnitude of fiscal policy regime shifts.

[1]See Bianchi (2012), Bianchi and Ilut (2014), Davig and Leeper (2006, 2011), Gonzalez-Astudillo (2013), Kleim, Kriwoluzky, and Sarferaz (2015, 2016)

We find that the long-run or “global” responsiveness of fiscal policy to government debt determines whether the optimal interest rate rule is time-varying. When fiscal policy is “globally active” or “globally passive”, central banks typically lack reason to track the fiscal policy stance, and should instead implement time-invariant interest rate rules. In the case of globally active policy, we find that there are strikingly large regions of the parameter space for which time-invariant interest rate pegs are optimal. For “globally switching” policies that feature more balanced fiscal regimes, the optimal policy is both time-varying and parameter-dependent. Hence, we identify certain cases where monetary policymakers should track the timing and magnitude of fiscal policy regime changes.

Second, do the optimal policy rules under rational expectations perform well in models with adaptive learning? By answering this question, we hope to identify conditions under which the optimal rational expectations policy is robust to misspecifications about the true model of expectations formation. We find that interest rate pegs are also optimal in globally active models with least squares learning agents. Moreover, our learning agents in globally switching and passive models tend to prefer inflation reaction coefficients in passive fiscal regimes that are higher than the optimal rational expectations coefficients.

These last points are demonstrated in a constant gain learning model with observed states, and in a novel hidden Markov model of learning. To the best of our knowledge, this is the first paper to study a model with least squares learning agents who jointly estimate the equilibrium law of motion and state probabilities in a self-referential framework. In this environment, it is unclear whether optimal policies are robust to the exclusion of contemporaneous variables from agents’ information sets. This is because the non-linear structure of Markov-switching DSGE models prevents agents with lagged information sets from learning the MSV solution.[2] This calls into question the relevant class of equilibria in our policy analysis. Fortunately, in this environment, agents generally succeed in identifying policy states and learning equilibrium coefficients if they receive some contemporaneous signal of current policy. We emphasize that certain implications of the nowcasting problem just described, as well as the broader analysis of stability under learning in hidden Markov models, are left for future work.

Third, we ask: how does optimal monetary policy vary with the average maturity of debt? McClung (2012b) shows that the maturity structure of debt matters for determinacy in models with switching fiscal policy, which is not the case in analogous models with fixed fiscal regimes. We expand on this by showing how the menu of potential optimal policies available to central banks is particularly susceptible to the effects of maturity when fiscal policy is globally switching. Moreover, we show that uncertainty over fiscal policy variables can generate the kind of tradeoff between minimizing loss and maximizing probability of determinacy and E-stability that Evans and McGough (2007) discusses. A companion project to this paper explores whether the monetary authority ought to make balance sheet decisions that target the optimal debt maturity structure.

This paper most directly builds on the works of three optimal policy papers in the New Keynesian literature. First, this paper extends Schmitt-Grohe and Uribe (2007) which studies optimal simple monetary policy rules in fixed regime New Keynesian models with active and passive fiscal policy. We extend this paper by allowing for time-variation in fiscal policy stances. Second, this paper borrows

[2]Here, a MSV solution *only* depends on lagged endogenous variables, and contemporaneous exogenous shocks including the Markov state. We show that agents with lagged information sets can only learn equilibria that also depend on past Markov states.

heavily from Orphanides and Williams (2007), which studies the performance and robustness of simple monetary rules in simple monetary models with price-stickiness and learning agents. We build on their work by studying the robustness of optimal rational expectations monetary policy rules to misspecifications about private sector expectations in models with fiscal policy. Finally, we extend Chen et al (2015), which studies joint optimal monetary and fiscal policy in a model with switching policies. Their paper derives fully optimal joint policy rules through a Stackelberg game. In contrast, we look for optimal simple, implementable policy rules that are robust to misspecifications about private sector expectations. We also take the complementary view that fiscal policymakers do not engage in a sophisticated optimization routine to determine fiscal surpluses.

This paper also contributes to a growing literature on regime-switching policy in the context of the Fiscal Theory of the Price Level. For instance, Davig and Leeper (2011), Bianchi (2012, 2013), and Bianchi and Melosi (2014) estimate switching New Keynesian models such as the models in this paper and find evidence of fiscal and monetary policy switches in the U.S. Ascari et al. (2017) and Cho and Moreno (2016) attempt to generalize the determinacy conditions from Leeper (1991) to environments with switching coefficients. We extend their work by showing how maturity impacts determinacy in the presence of fiscal policy switching, and by more exhaustively studying how the across-regimes behavior of fiscal policymakers constrains the menu of monetary policies consistent with determinacy. Additionally this paper borrows heavily from, and attempts to contribute to, a learning literature involving models with monetary-fiscal policy interactions that includes Eusepi and Preston (2011, 2012, 2013) and Bianchi (2013) and Bianchi and Melosi (2014).

Finally, this paper contributes a hidden Markov model of least squares learning to the learning literature. While many hidden Markov models of learning are introduced by papers such as Bullard and Singh (2007) and Davig (2004), relatively few of them study agents who jointly estimate model parameters and state probabilities. Exceptions that do exist, such as Hansen and Sargent (2010), Hansen, Polson and Sargent (2010), and Johannes et al (2013) invariably involve Bayesian learners, and, almost invariably, [3] Bayesian learners in models without self-referential feedbacks.

Our approach differs from these other approaches in many ways. First, we study conditional least squares learners in a hidden Markov model, whereas other papers study Bayesian learning. Second, we concern ourselves with stability of rational expectations equilibrium in a model with hidden states. Because we can appeal to stochastic approximation papers that study convergence properties of our conditional least squares algorithm, future research may develop stability conditions that help extend the intuition of Evans and Honkapohja (2001) to our regime-switching models. Third, our agents estimate a Markov-switching VAR law of motion for endogenous variables.

The paper is organized as follows: first, we briefly introduce the model and estimation routine; second, we derive the optimal interest rate rules under rational expectations in models with short-term debt; third, we discuss the optimal rules under adaptive learning; fourth, we discuss optimal policy in the presence of long-term debt; finally, we conclude.

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[3] Hansen and Sargent (2010) is an exception, but their approach applies only to models that assume a very specific structure

Model and Method

We consider a class of log-linearized New Keynesian models that is augmented to include time-varying fiscal policy as in Davig and Leeper (2011), long-term maturity structure of debt as in Woodford (2001) and Eusepi and Preston (2013). In this class of models, private sector behavior is given by two equations of the form:

$$\begin{aligned}\tilde{y}_t &= E_t\tilde{y}_{t+1} - \sigma^{-1}(\hat{i}_t - E_t\hat{\pi}_{t+1}) + \sum_d u_t^d \\ \hat{\pi}_t &= \beta E_t\hat{\pi}_{t+1} + \kappa\tilde{y}_t + \sum_s u_t^s\end{aligned}$$

where all variables are expressed as percentage deviations from steady state, $\tilde{y}$ is the output gap, $\hat{\pi}$ is inflation, and $\hat{i}$ is the deviation of nominal interest rates from the nominal interest rate target. $\sum_d u_t^d$ and $\sum_s u_t^s$ are demand and supply shocks, respectively, that may include any number of exogenous processes acting on technology, preferences, market power, etc. To introduce fiscal policy into the model, we consider the log-linearized versions of the following equations:

$$\begin{aligned}P_t^m b_t + \tau_t &= \frac{b_{t-1}}{\pi_t}(1 + \rho P_t^m) + G_t \\ P_t^m &= \frac{1}{1+i_t}(1 + \rho E_t P_{t+1}^m)\end{aligned}$$

where b is real debt, G is real government spending, and τ is a surplus rule. $\hat{P}_t^m$ is the price of the bond portfolio at time t and $\rho \in [0,1]$ captures the maturity structure of the government debt. While we relegate the derivation of these equations to the appendix, the intuition behind the bond portfolio is fairly simple: the government issues b_t units of a nominal bond portfolio at time t that pays 1 unit of nominal income at time $t+1$, ρ units at time $t+2$, ρ^2 units at $t+3$ and so forth. In other words, government debt exhibits a geometrically decaying maturity structure. This structure allows us to introduce long-term debt into our model by using a single state variable that captures the average maturity of debt, ρ. The limiting cases of ρ illuminate how larger values of ρ correspond to longer average maturities: when $\rho = 0$, all debt is short term, and when $\rho = 1$, all debt is in the form of consols.

In this paper, we are primarily interested in how optimal monetary policy depends on τ, which is characterized by a rule of the form:

$$\begin{aligned} \tau_t &= \bar{\tau}\left(b_t(1+\rho P_t^m)\right)^{\gamma(s_t)} f_t \\ f_t &= f_{t-1}^{\rho_f} e^{\epsilon_t^f} \end{aligned}$$

where ϵ_t^f is some mean-zero i.i.d shock. τ adjusts some lump-sum component of the government's structural surplus in response to government liabilities, $b_t(1+\rho P_t^m)$. The responsiveness of this fiscal rule is determined by $\gamma(s_t)$, which is assumed to follow a two-state Markov process given by s_t. As we will discuss shortly, the value of γ determines which model variables need to stabilize government debt. Because the parameterization of this rule has general equilibrium implications for inflation and output, we allow the monetary policymaker to employ a log-linearized switching rule of the form:

$$\hat{i}_t = \rho_i \hat{i}_{t-1} + (1-\rho_i)(\phi_\pi(s_t)\hat{\pi}_t + \phi_y(s_t)\tilde{y}_t) + \epsilon_t^R$$

where s_t is the same process that drives variation in γ, $\hat{i}$ is the deviation of the nominal interest rate from its target. To impose structure on G_t and the private sector shocks in the model, we derive a model that is similar in spirit to the simple New Keynesian model in An and Schorfheide (2007). Specifically, government spending is given by

$$\begin{aligned}
G_t &= \xi_t Y_t \\
g_t &= \frac{1}{1-\xi_t} \\
ln(g_t) &= \rho_g ln(g_{t-1}) + \epsilon_t^g
\end{aligned}$$

According to this specification of government spending, a time varying fraction of output is consumed by the government. If we substitute this into the government budget constraint, it is straightforward to see that government debt depends directly on output. Therefore, government spending in our model introduces an output channel that may have implications for our results. In simple cases where we want to abstract away from this output channel, we simply set $\bar{G} = 0$ and $\epsilon_t^g = 0$ and model fiscal disturbances through f_t. To bring demand and supply shocks into the model, we assume that there are both markup shocks and shocks to household preferences. The model derivation is left for the Appendix H, but we can write our model in the form

$$\begin{aligned}
\tilde{y}_t &= E_t\tilde{y}_{t+1} - \sigma^{-1}(i_t - E_t\hat{\pi}_{t+1}) + \sigma^{-1}\rho_{\hat{z}}\hat{z}_t \\
\hat{\pi}_t &= \beta E_t\hat{\pi}_{t+1} + \kappa\tilde{y}_t + \hat{\mu}_t \\
\hat{i}_t &= \rho_i\hat{i}_{t-1} + (1-\rho_i)(\phi_\pi(s_t)\hat{\pi}_t + \phi_y(s_t)\hat{y}_t) + \epsilon_t^R \\
\hat{b}_t &= \beta^{-1}(\hat{b}_{t-1} - \pi_t) - (1-\rho)\hat{P}_{m,t} + \beta^{-1}\frac{\bar{G}}{\bar{b}}\hat{y}_t \\
&\quad -\beta^{-1}((1-\beta) + (1-\beta\rho)\frac{\bar{G}}{\bar{B}})\hat{\tau}_t + \beta^{-1}\hat{g}_t \\
\hat{\tau}_t &= \gamma(s_t)(\hat{b}_{t-1} + \beta\rho\hat{P}_{m,t}) + \hat{f}_t \\
\hat{P}_{m,t} &= -\hat{i}_t + \beta\rho E_t\hat{P}_{m,t+1} \\
\hat{f}_t &= \rho_{\hat{f}}\hat{f}_{t-1} + \epsilon_t^f
\end{aligned}$$

$$\hat{g}_t = \rho_g \hat{g}_{t-1} + \epsilon_t^g$$

$$\hat{\hat{z}}_t = \rho_{\hat{z}} \hat{\hat{z}}_{t-1} + \epsilon_t^z$$

$$\hat{\mu}_t = \rho_\mu \hat{\mu}_{t-1} + \epsilon_t^\mu$$

where z is the technology shock, μ is the cost-push shock, P is the transition probability matrix and $p_{ij} = Pr(s_t = j | s_{t-1} = i)$. All other variables are defined as before. In the baseline analysis, we calibrate our model so that the steady state government liabilities, $\bar{b}(1 + \rho\bar{P})$, equals the steady state level of output. We also set $\bar{G}/\bar{b}$ so that $\bar{G}/\bar{Y} = .2$ conditional of ρ. My main results do not seem to depend on these assumptions, except in rare special cases we discuss below. Having written the model, we are now in a position to define the fiscal policy stance on debt, and the monetary policy rule. To that end, we use the following two definitions.

<u>Definition 5</u>

A fiscal policy is defined by the following parameters: $\{p_{11}, p_{22}, \gamma(1), \gamma(2)\}$. Stated more thoroughly, a switching fiscal policy is fully characterized by *within-regime* responses to outstanding debt, given by $\{\gamma(1), \gamma(2)\}$ and by the transition probabilities $\{p_{11}, p_{22}\}$.

<u>Definition 6</u>

A monetary policy is defined by the parameters of the interest rate rule: $\{\phi_\pi(1), \phi_\pi(2), \phi_y(1), \phi_y(2), \rho_i\}$. In late sections, we may allow a monetary policy to be indexed by the average maturity of debt, ρ.

We subject our policy rule to a monetary policy shock to help account for fluctuations of i around its target value, or to capture any short-lived deviation of policy from the rule that might be caused by dissension between policymakers. In our simple model without debt, ϵ^R is isomorphic to a demand shock in the IS curve. As such, we do not need monetary policy to explore optimal monetary responses to demand shocks in a model with Ricardian dynamics. In our model, however, ϵ^R shows up in both the IS curve and the government budget constraint and this will have implications for optimal policy.

To help distinguish between policy regimes, we follow Leeper (1991) and describe an "active" policymaker as one who determines inflation without concern for the stability of debt, and a "passive" policymaker as one who directly acts to stabilize the evolution of debt. With respect to fiscal policy, $\gamma > 1$ characterizes a "passive" policy regime. When $\gamma > 1$, bonds evolve according to a stable autoregressive process so that changes in i and π are not needed to keep debt from exploding. Intuitively, $\gamma > 1$ means that surpluses adjust endogenously by an amount that is sufficient to pay down interest and principal on new debt issuance over an infinite horizon. In such an environment, forward looking agents recognize that any wealth effects stemming from debt issuance will be offset by future taxes and this renders policy Ricardian. Because fiscal policy stabilizes debt, the central bank is free to contain inflation as it pleases – ideally by employing an active monetary policy that satisfies the Taylor Principle.

When fiscal policy is active (i.e. $\gamma < 1$), surpluses do not rise by enough to offset any wealth effects coming from any new debt issuance and this causes consumption and inflation to rise in response to higher debt. Any rise in inflation that results from these wealth effects must be accommodated by central banks; if central banks raise interest rates by more than one-for-one in response to higher inflation, they will raise real debt service costs, leading to higher debt and therefore higher inflation in the future, and so on. Central banks therefore must respond weakly or passively to inflation so that inflation may erode the outstanding debt stock without generating additional debt service costs. Such monetary policy is said to be "passive" and is characterized by a violation of the Taylor Principle (e.g. $\phi_\pi < 1$).

Parameter values are chosen to so that the resulting model is determinate. While there are no simple analytical conditions for determinacy in our switching model, Leeper (1991)[4] gives simple conditions for determinacy in the case of non-switching (assuming $\bar{G} = 0$):[5]

TABLE 2. Leeper (1991) Determinacy Conditions

	$\phi_\pi > 1$	$\phi_\pi < 1$
$\gamma \in (1, \frac{\beta^{-1}+1}{\beta^{-1}-1})$	determinate	indeterminate
$\gamma \notin (1, \frac{\beta^{-1}+1}{\beta^{-1}-1})$	no stable solution	determinate

We say they the economy is in Regime M when $\phi_\pi > 1$ and $\gamma \in (1, \frac{\beta^{-1}+1}{\beta^{-1}-1})$. and that the economy is in Regime F when $\phi_\pi < 1$ and $\gamma \notin (1, \frac{\beta^{-1}+1}{\beta^{-1}-1})$. In Regime M, fiscal policy is passive while monetary policy is active. This is the standard assumption in most New Keynesian research. In Regime F, fiscal policy is active while monetary policy is passive. Our model features switching between Regime F and Regime M policy configurations. That is, our model features 2 states (i.e. $S = 2$) where each state is consistent with determinacy in the analogous fixed regime model[6]. We solve the model and check for determinacy in the mean-square-stable sense using techniques from Cho (2016).

[4]See Table 1 for Woodford (1998a) generalization of conditions

[5]We assume that $\phi_\pi(s_t) \geq 0$ for all s_t

[6]Despite the fact that each regime induces determinacy in a fixed regime model, the model with switching between determinate regimes is often explosive or yields indeterminacy

We now specify the optimization problem. The central bank chooses $\phi = (\phi_\pi(1), \phi_y(1), \phi_\pi(2), \phi_y(2), \rho_i, \rho) \forall s_t$ to minimize:

$$l(\phi) = var(\pi) + \lambda var(y)$$

The choice of ϕ that minimizes l is referred to as the optimal policy or "optimized" simple interest rate rule as in Orphanides and Williams (2007). The optimal policy is said to be time-invariant if $\phi_\pi(1) = \phi_\pi(2)$ and $\phi_y(1) = \phi_y(2)$, and is said to be time-varying otherwise. Though the average maturity of debt appears to be a fiscal policy tool, central banks are able to engage in large-scale asset purchase (LSAP) programs that "twist" the maturity structure of the debt held by the public. Consequently, we include this parameter in the central's bank's choice set in specific exercises. In addition to minimizing loss, the optimal policy should satisfy two criteria: (1) the optimal policy should implement a unique mean-square stable rational expectations equilibrium; (2) optimal inflation reaction coefficients must be non-negative.

As implied by our discussion of the Leeper (1991) conditions, the value of the fiscal policy parameter, γ, impacts the menu of policy options that central banks must choose from to contain inflation. When $\gamma < 1$, fiscal policy is active and monetary responses to inflation must be dovish; when γ is high and policy is passive, monetary responses must be aggressive. To help characterize how fiscal policy constrains central bankers in our model with time-varying policy stances, we employ a generalization of these conditions similar in spirit to conditions developed by Ascari et al (2017). Our taxonomy considers three types of fiscal policy stances: (1) "globally passive" policies that support a stable Ricardian equivalent equilibrium; (2) "globally active" policies that are more active than passive across regimes; (3) "globally switching" or "balanced" policies that are neither more active nor more passive across regimes. We note that both globally active and globally switching policies feature non-Ricardian dynamics; only globally passive policies are Ricardian. The following definitions help us characterize our three categories of switching fiscal policy and provide valuable intuition.

Definition 7

A fiscal policy is **globally passive** if $\phi_\pi(1) = \phi_\pi(2) = \alpha^P$ for all $\alpha^P > 1$ yields a determinate equilibrium.

A globally passive policy can be paired with any time-invariant interest rate rule that satisfies the Taylor Principle. In order for this to be true, fiscal policy must be Ricardian. Otherwise, we could choose a time-invariant active monetary policy that places debt on an explosive path. Because globally passive implies Ricardian equivalence and vice versa we can determine if a policy is globally passive using the following conditions passive (assuming $p_{11} + p_{22} > 1$):[7]

$$(p_{11} + p_{22} - 1)h_1^2 h_2^2 \;<\; 1 \tag{5.1}$$

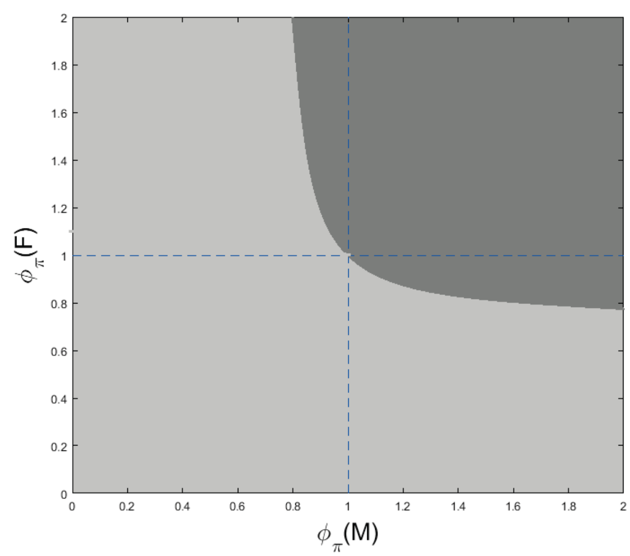
$$p_{11}h_1^2(1 - h_2^2) + p_{22}h_2^2(1 - h_1^2) + h_1^2 h_2^2 \;<\; 1 \tag{5.2}$$

where $h_i = \beta^{-1}(1 - (1 - \beta)\gamma(i))$ for $i = 1, 2$. These conditions, which are also presented in Ascari et al (2017), tell us when the budget constraint implies a mean-square stable autoregressive process for debt. If a fiscal policy satisfies these conditions, then debt evolves according to a mean-square stable autoregressive process without accommodation from the monetary authority and this allows monetary policymakers to determine inflation and output in the non-policy block of the New Keynesian model. Determinacy then requires that interest rates respond aggressively to inflation.

[7]We are interested in highly persistent regimes, which makes this a harmless assumption

Figure 9 shows regions of determinacy, indeterminacy and non-existence of stable solutions for a model with globally passive policy. As argued in Ascari et al. (2017), the determinacy region in Figure 9 presents something akin to the Long-Run Taylor Principle in Davig and Leeper (2007): fiscal policy can be very active for short amount of times, or modestly active with persistence, and the resulting equilibrium may still be Ricardian and determinate if policy is mostly passive overall. Note that a fixed passive fiscal policy regime is merely a special case of a globally passive policy.

FIGURE 9. Globally Passive Policy



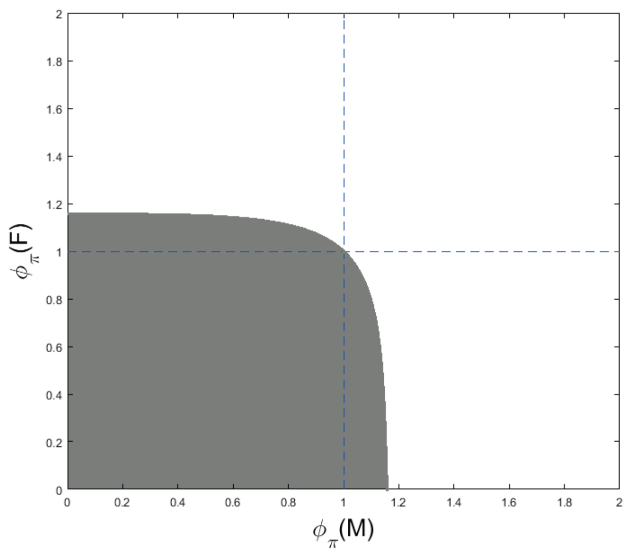
The determinacy region is dark gray; indeterminacy is light gray; no stable solutions is white

Definition 8

A fiscal policy is **globally active** if $\phi_\pi(1) = \phi_\pi(2) = \alpha^A$ for all $\alpha < 1$ yields a determinate equilibrium.

Stated equivalently, policy is globally active if a unique equilibrium exists when the monetary authority employs a time-invariant passive monetary policy. For a permanent passive monetary policy to be consistent with determinacy in our model, fiscal policy must be more active than passive overall. Figure 10 shows regions of determinacy, indeterminacy and explosiveness region for globally active fiscal policy. Note that a fixed active fiscal policy is merely a special case of a globally active policy.

FIGURE 10. Globally Active Policy



The determinacy region is dark gray; indeterminacy is light gray; no stable solutions is white

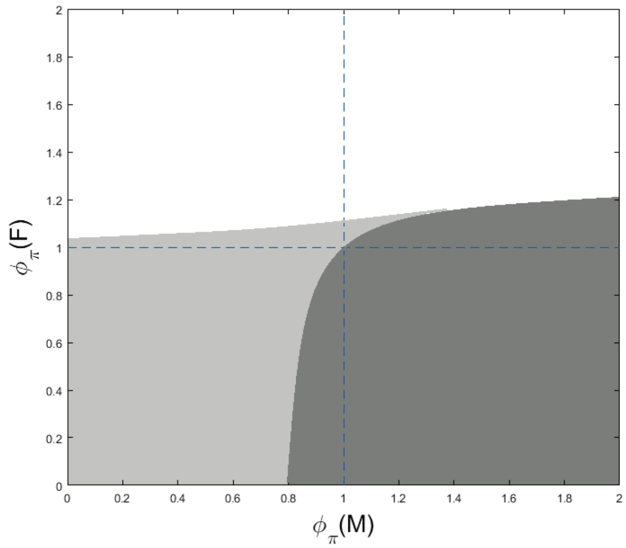
<u>Definition 9</u>

A fiscal policy is **globally switching** if there exists $\alpha^A < 1$ and $\alpha^P > 1$ such that neither $\phi_\pi(1) = \phi_\pi(2) = \alpha^A$ nor $\phi_\pi(1) = \phi_\pi(2) = \alpha^P$ yield a determinate equilibrium.

The set of globally switching fiscal policies is the complement of the set of globally active and passive policies. Intuitively, a globally switching policy is neither active enough in the long-run to support all passive monetary policies nor passive enough in the long-run to support all active monetary policies. These fiscal policies are balanced in the sense that they are not obviously more active or passive overall. For example, a globally switching policy may feature slow-changing, strongly active and strongly passive regimes, or fast-changing weakly active and weakly passive fiscal policy regimes. Table 3 offers very rough qualitative examples of how fiscal policies may be assigned to certain categories. Figure 11 shows determinacy regions for policies that feature highly persistent and/or strongly active and passive regimes. This figure suggests that determinacy requires monetary authorities to be hawkish during passive fiscal regimes and dovish during active fiscal regimes. Crucially, central banks cannot implement time-invariant policies such as permanent interest rate pegs because the overall fiscal policy stance is no longer mostly active or mostly passive. Figure 12 shows determinacy regions for policies that feature fast-changing and/or weakly active and passive regimes. In these scenarios, central bankers face a meager menu of policy options. Typically, determinacy regions for globally switching policies will resemble either Figure 11 or 12 depending on the strength of switching regime fiscal policy responses to debt and the persistence of regimes. Table 1 offers very rough qualitative examples of how fiscal policies may be assigned to certain categories.

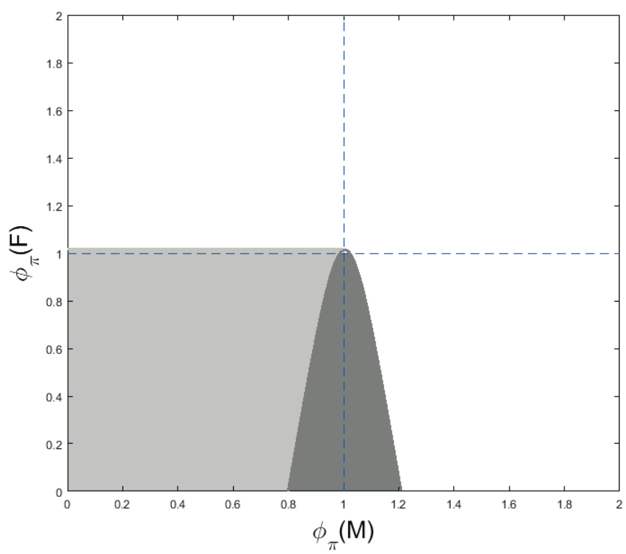
Before we present results we offer some final intuition about the fiscal policy taxonomy. Globally active and globally passive policies can be coupled with a wide range of time-invariant policies to deliver a determinate model, while globally switching policies must be paired with time-varying monetary policies for determinacy. One practical benefit of using time-invariant policies is that their implementation does not

FIGURE 11. Globally Switching Policy (Strong)



The determinacy region is dark gray; indeterminacy is light gray; no stable solutions is white

FIGURE 12. Globally Switching Policy (Weak)



The determinacy region is dark gray; indeterminacy is light gray; no stable solutions is white

TABLE 3. Overall Stance of Fiscal Policy

	low pers.; weak active	high pers.; weak active	low pers.; strong active	high pers.; strong active
low pers.; weak passive	**GS**	**GA**	**GA**	**GA**
high pers.; weak passive	**GP**	**GS**	**GS**	**GA**
low pers.; strong passive	**GP**	**GS**	**GS**	**GA**
high pers.; strong passive	**GP**	**GP**	**GP**	**GS**

Strength and persistence of fiscal regime and the overall stance of fiscal policy. "pers." = persistence; GA = "globally active"; GP = "globally passive"; GS = "globally switching"

require policymakers to actively track any changes in responsiveness of fiscal policy to debt. As we show next, time-invariant policies are going to perform well in models with globally active or passive policy.

Short-term Debt and Rational Expectations

In this section, I abstract away from long-term debt by assuming $\rho = 0$. We also assume that $\lambda = 0$ so that the central bank loss function equals that variance of inflation, but the optimal policies discussed in this section appear to perform well in applications with small λ such as the λ weights typically found in microfounded loss functions. Because we prioritize inflation-targeting, $\phi_y(1) = \phi_y(2) = 0$ reduces loss in our numerical search. Accordingly, we restrict our attention the interest rate rules of the form:

$$i_t = \rho_i i_{t-1} + (1 - \rho_i)\phi_\pi(s_t)\pi_t$$

We present our numerical results through a series of claims contained in this section.

Claim 1

> If fiscal policy is globally passive then for all parameterizations the optimal monetary policy response is to employ an interest rate rule with coefficients $\phi_\pi(s_t) = \bar{\phi}_\pi \quad \forall s_t$ and $\rho_i = \bar{\rho}_i$

Since determinacy requires that inflation and output be determined in the non-policy block, the optimal simple policy rule is identical to the optimal rule used in small-scale 3-equation models that consist of an IS curve, Phillips Curve and interest rate rule (see Woodford (2003)). Intuitively, a globally passive policy supports a mean-square stable autoregressive process for debt. Consequently, central banks do not need to accommodate fiscal policy and this allows monetary policymakers to determine inflation through the non-policy block.

When $\lambda = 0$, $\bar{\phi}_\pi \to \infty$ (see Woodford (2003)). Two features of this result should be emphasized. First, the optimal policy is time-invariant despite switching in the fiscal policy stance. Second, the monetarist equilibrium can be stable in models with persistent active fiscal policy. For example, the monetarist equilibrium is stable when $p_{11} = p_{22} = .95$, $\gamma(1) = 5$, $\gamma(2) = 0$, and $\bar{G} = 0$ despite the fact that fiscal surpluses are entirely exogenous half of the time. See Orphanides and Williams (2007) for a treatment of the optimal ρ_i when interest rates are determined in this environment

Claim 2

> If fiscal policy is globally active then for all reasonable parameterizations the monetary authorities should employ a permanent interest rate peg (i.e. $\phi_\pi(1) = \phi_\pi(2) = 0$) in order to minimize the variance of inflation.

While we cannot prove Claim 3 formally, our claim relies on the following numerical support: for all globally active policies in $p_{11} \in [.9, 1]$, $p_{22} \in [.9, 1]$, $\gamma(1) \in [-10, 10]$, and

$\gamma(2) \in [-10, 10]$, the interest rate peg is optimal. For the posterior mean calibration with added cost-push shocks and fiscal variables, we search over approximately 64,000 globally active policies and find that the interest rate peg is optimal for each one of them. We repeat this analysis for alternative reasonable calibrations (alternative shock covariance-variance structure, alternative persistence parameters for structural shocks, σ, κ) and find that this result is robust.

We add the word "reasonable" because non-Ricardian fiscal policy presents a tradeoff between stabilizing inflation in response to private sector shocks (i.e. demand and supply shocks), and stabilizing inflation in response to policy shocks. In conjunction with active fiscal policy, private sector shocks call for very high ρ_i (e.g. $\rho_i = .995$) and time-varying inflation reaction coefficients, while pegs perform best in response to monetary and fiscal policy shocks. As a result, the optimal monetary policy in a model with globally active fiscal policy depends on the net effect of these competing influences on inflation.

As it turns out, the private sector shock variances need to be very large relative to policy shock variances, or the private sector shocks need to be very persistent relative to policy shocks for the interest rate peg to be suboptimal. In particular, monetary policy shocks need to be very small relative to other shock variances. To illustrate this last point, we calibrate the model at the posterior mean, shut down each shock except for one private sector shock and ask: how large does the variance of the monetary policy shock need to be for the interest rate peg to be optimal?

When we set $\bar{G} = 0$, so that output no longer impacts debt through the budget constraint and set $\rho_u = .99$ where ρ_u is the supply shock persistence term, we need for the variance of the monetary policy shock to be greater than .014% of the variance of the i.i.d innovation to the supply shock to get an optimal interest rate peg. For $\rho_u =$

.9, the monetary policy shock needs to be greater than .0017% of the variance of the same innovation to the supply shock. When we increase $\bar{G}$ to .2, the interest rate peg is optimal even when cost-push is the only shock in the model.

When we set $\bar{G} = 0$, so that output no longer impacts debt through the budget constraint and set $\rho_z = .99$ where ρ_z is the demand shock persistence term, the variance of the i.i.d. innovation to the demand shock must be less than 5 times the variance of the monetary policy shock for the peg to be optimal. This suggests that demand shocks are a bigger threat to the optimal interest rate peg. However, when $\rho_z = .9$, the monetary policy shock only needs to be greater than .025% of the variance of the i.i.d innovation to the demand shock for the peg to be optimal. Of course, these exercises exclude fiscal policy shocks, and those shocks help to select the peg. For example, if we shut down monetary policy shocks and set all remaining shock parameters to their posterior mean values, we can set $\rho_z = .99$ and still have an optimal peg.

Because intuition supports the inclusion of policy shocks in our model, and because it is highly unlikely that an estimated model will reject the inclusion of policy shocks, we regard cases where the peg is suboptimal as special cases involving potentially unreasonable parameterizations of the model. We also note that pegs are quite often *nearly* optimal in that loss is often close to 0% higher under the peg when compared to the optimum. However, we have found cases where loss is as much as 3% higher under the peg.

Non-Ricardian equilibria in our model allows us to generate some intuition concerning the optimality of interest rate pegs. Suppose some shock (e.g. a fiscal shock) raises the outstanding debt stock today. Since agents perceive government debt as net wealth, this will raise consumption and inflation. This is one sense in which debt determines inflation under a globally active policy. The amount of inflation generated in general equilibrium depends on monetary policy, however. As such, monetary policy determines how inflation feeds back to stabilize debt. If an interest rate peg is in place, a large inflation will occur today, which pushes debt in the direction of its steady state value. On the other hand, if the central bank allows interest rates to respond positively, then debt service costs will increase today, which creates higher debt tomorrow and so on. The higher expected path of debt raises time t inflation expectations, so that inflation is both higher today and propagated into the future. In a similar thought experiment, Leeper and Leith (2016) show that the present value of inflation will be higher under the responsive interest rate than under the peg in their small-scale New Keynesian model. Once they solve for the equilibrium path of inflation, it's straightforward to show that the sharp, sudden responses of inflation under the peg are consistent with less volatility in inflation. The complexity of our non-linear model makes it very difficult to repeat a similar experiment in this paper. However, Claim 3 strongly suggests that their results generalize to models with time-varying fiscal stances – even models with recurring passive fiscal policy regimes.

To understand why pegs perform so well in globally active models, it's important to recall the fact that debt both determines inflation and is stabilized by inflation in any non-Ricardian equilibrium. This means that any shock to government debt (i.e. any shock appearing in the budget constraint) will have an affect on inflation and output. To fix things, consider a shock which raises debt. Since agents perceive government debt as net wealth, this will raise consumption and inflation. This is one sense in which debt determines inflation under a globally active policy. The amount of inflation generated in general equilibrium depends on monetary policy, however. As such, monetary policy determines how inflation feeds back to stabilize debt. If an interest rate peg is in place, a large inflation will occur today, which pushes debt in the direction of its steady state value. On the other hand, if the central bank allows interest rates to respond positively, then debt service costs will increase today, which creates higher debt tomorrow and so on. The higher expected path of debt raises time t inflation expectations, so that inflation is both higher today and propagated into the future. In a similar thought experiment, Leeper and Leith (2016) show that the present value of inflation will be higher under the responsive interest rate than under the peg in their small-scale New Keynesian model. Once they solve for the equilibrium path of inflation, it's straightforward to show that the sharp, sudden responses of inflation under the peg are consistent with less volatility in inflation. The complexity of our non-linear model makes it very difficult to repeat a similar experiment in this paper. However, Claim 3 strongly suggests that their results generalize to models with time-varying fiscal stances – even models with recurring passive fiscal policy regimes.

While pegs are broadly consistent with stable inflation in our non-Ricardian model, $\phi_\pi(1) = \phi_\pi(2) = 0$ does not guarantee determinacy for all fiscal policies that violate the abovementioned conditions (i.e. mean-square stable common-factor sunspots may exist). In particular, indeterminacy obtains if fiscal policy is too passive in one regime. For example, if $p_{11} = p_{22} = .95$, $\gamma(1) = 2$, $\gamma(2) = -5$ then fiscal policy is sufficiently active for the interest rate peg to deliver determinacy. If, however, $\gamma(1) = 2$ is replaced by $\gamma(1) = 5$, then policy is too passive in regime 1 for the interest rate peg to deliver determinacy. These are globally switching equilibria and determinacy requires that they be paired with special optimal equilibria.

Claim 3

Optimal GS monetary policies are time-varying and parameter dependent

For example, the optimized inflation reaction coefficients for the policy given by $p_{11} = p_{22} = .95$, $\gamma(1) = 5$, $\gamma(2) = -5$ and for the policy given by $p_{11} = p_{22} = .95$, $\gamma(1) = 2$, $\gamma(2) = 0$, are $(\phi_\pi(1), \phi_\pi(2), \rho_i) = (3.3, 0, .99)$ and $(\phi_\pi(1), \phi_\pi(2), \rho_i) = (2.97, .73, .99)$, respectively. These particular optimized coefficients come from the expected posterior loss exercise we introduce in the next paragraph. Since the optimized policy favors large swings in inflation responses, policy inertia is undesired (i.e. $\rho_i = 0$ is optimal).

While results in the globally active and globally passive settings hinge only on fiscal policy parameters (for reasonable parameterizations of shock processes), the optimal policy in globally switching models depends on any model parameter that impacts determinacy conditions. This means that we need to choose parameter values in order to draw conclusions about optimal policy in the globally switching models. To help inform our selection of model parameter values, we estimate the following truncated model using Bayesian techniques:[8]

$$
\begin{aligned}
\hat{y}_t &= E_t\hat{y}_{t+1} - \sigma^{-1}(i_t - E_t\hat{\pi}_{t+1}) + (1-\rho_g)\hat{g}_t + \sigma^{-1}\rho_{\hat{z}}\hat{z}_t \\
\hat{\pi}_t &= \beta E_t\hat{\pi}_{t+1} + \kappa(\hat{y}_t - \hat{g}_t) + \hat{\mu}_t \\
\hat{i}_t &= \rho_i\hat{i}_{t-1} + (1-\rho_i)(\phi_\pi\hat{\pi}_t + \phi_y\hat{y}_t) + \epsilon_t^R \\
\hat{g}_t &= \rho_g\hat{g}_{t-1} + \epsilon_t^g \\
\hat{z}_t &= \rho_{\hat{z}}\hat{z}_{t-1} + \epsilon_t^z \\
\hat{\mu}_t &= \rho_\mu\hat{\mu}_{t-1} + \epsilon_t^\mu \\
\tilde{y}_t &= \hat{y}_t - \hat{g}_t
\end{aligned}
$$

where the final equations reflects the fact the natural rate of output equals $\hat{g}_t$ in our model, thus allowing us to glean information about government shocks from estimates

[8]See Appendix I. for tables containing information about our prior and posterior distributions

of the IS and Phillips Curves. Using the separated partial means test to test the convergence of our estimates, we believe that the best results emerge when we place dogmatic priors over ρ_u and σ_μ and estimate only the non-policy block with the interest rate. Since this exercise intends to consider counterfactural policies, estimates of the underlying fiscal policy stance are unnecessary. However, estimates pertaining to the shock processes and other private sector coefficients help to discipline our analysis towards parameter regions that agree better with the data. In extensions of the present work, we intend to estimate a fuller DSGE model.

After sampling from the posterior distribution, we follow Cogley et al. (2011) and compute the expected posterior loss associated with each policy parameterization. A Monte Carlo average of the following expected posterior loss function is computed:

$$\int l(\phi)P(\tilde{\theta}|Y)d\tilde{\theta}$$

where $\tilde{\theta} = (\kappa, \sigma, \rho_g, \rho_z, \rho_\mu, \sigma_g, \sigma_z, \sigma_\mu, \sigma_r)$. For the previously mentioned case where $\gamma(1) = 5$ and $\gamma(2) = -5$, $p_{11} = p_{22} = .95$, the optimal policy is given by $\phi_\pi(1) = 3.33$, $\phi_\pi(2) = 0$, $\rho_i = 0.99$. Intuitively, monetary policy should be active in the passive fiscal regime, and very passive in the active fiscal regime.

To sum up, the optimal policy response depends on whether fiscal policy is globally active, globally passive or globally switching. If fiscal policy is globally passive, then optimal policy is time-invariant and calls for large inflation reaction coefficients. If fiscal policy is not globally passive, then interest rate pegs deliver the fundamental solutions that minimize loss. If, however, fiscal policy is globally switching then interest rate pegs lead to indeterminacy. In those settings, and those settings alone, the optimal policy is time-varying.

TABLE 4. Optimal Inflation Reaction Coefficients under RE when $\rho = 0$

Type	$(\gamma(1),\ \gamma(2)$	**optimal** $(\phi_\pi(1),\ \phi_\pi(2))$
GP	(5, 0)	(∞, ∞)
GS	$(5, -5)$	(3.33, 0)
GS	(2, 0)	(2.73, .72)
GA	$(2, -5)$	(0, 0)

Adaptive Learning

In this section we relax the assumption that agents form rational expectations, and study policy performance in a model where agents attempt to learn the equilibrium law of motion for the model's endogenous variables, and form forecasts of future variables according to an estimated perceived law of motion. Relative to rational expectations models, models with learning agents feature instabilities that arise from agents' forecast errors. Specifically, agents' forecast errors affect the model's data generating process, thereby changing future data points and future estimates of the model's coefficients. This self-referential feature of our model fundamentally changes the way in which policy interacts with expectations to contain inflation and output. As such, the inclusion of adaptive learning in our analysis provides an important robustness check. Our main conclusion is that the optimized simple policy rules studied under rational expectations are robust to misspecifications of the underlying model of expectations employed by agents. That is, the optimized policy rules under rational expectations are optimal or nearly optimal in models with adaptive learning agents, with exceptions in the case of globally switching policy.

We present our results in two sections.First, westudy a learning model in which agents observe the model's endogenous variables (with a reasonable lag), exogenous driving processes and the underlying Markov state that drives variation in fiscal and monetary policy rules. When agents observe the underlying Markov state, they can easily update parameter estimates using a *within-state* recursive least squares algorithm that resembles the least squares algorithm developed and discussed in Evans and Honkapohja (2001). While this learning specification provides a natural first step away from the rather strong assumption that agents form rational expectations, it still assumes that agents easily observe something an applied econometrician would not: the underlying state of policy. We therefore develop a model of learning that backs away from this assumption.

In our hidden Markov model of adaptive learning, agents estimate the same perceived law of motion, but do not observe the underlying Markov state (i.e. agents find themselves in a hidden Markov model). Because agents do not observe the stance of fiscal and monetary policy, they cannot use the recursive least squares algorithm employed in the learning model with observed states. Instead, we allow agents to use the recursive MLE algorithm and the recursive conditional least squares algorithm developed in Krishnamurthy and Yin (2002) and LeGland and Mevel (1997) to update parameter estimates after observing the model's endogenous variables and exogenous driving processes. We emphasize two main results from this section. First, this is, to the best of our knowledge, the first paper to study least squares learning agents who estimate a Markov-switching autoregressive equilibrium law of motion in a self-referential model with hidden Markov states.[9] That is, previous research does not jointly estimate perceived laws of motion and the Markov state probabilities. We therefore regard this section as a springboard for future research on the use of hidden markov models of learning. Second, the exogeneity[10] of policy rule coefficients makes it possible for agents to infer the underlying state with some reasonable accuracy. Hence,

[9]See literature review for details on related papers

[10]By "exogeneity" we mean that the coefficients in the equilibrium law of motion for policy variables τ and i do not depend on agents' beliefs

model dynamics in a model with learning agents and hidden states are very similar to model dynamics in the learning model with observed states. We conclude that it is potentially reasonable to assume that agents observe policy switches, but it remains to be seen whether with assumption is strong in models where Markov-switching affects non-policy variables such as trend growth.

Observed Markov States

We now develop a model of learning in which agents observe the underlying Markov state (i.e. they observe the underlying policy stance). The model dynamics are still given by an actual law of motion, which can be constructed from the log-linearized equilibrium conditions given previously:

$$x_t = A(s_t)E_t x_{t+1} + B(s_t)x_{t-1} + C(s_t)z_t \tag{5.3}$$

where $x = (\pi\ y\ i\ b\ \tau\ P)'$ and $z = (g\ \hat{z}\ \epsilon^R\ \mu\ \epsilon^f)$. Under rational expectations, agents know the full structure given by (5.3) and can solve for the rational expectations equilibrium. Under adaptive learning, however, agents do not know (5.3) and are therefore incapable of computing the true mathematical expectations of tomorrow's variables. Despite the fact that agents are not fully rational, we still endow agents with sophisticated beliefs about the law of motion governing inflation, output, etc., in equilibrium. Specifically, we give agents the following perceived law of motion (PLM):

$$x_t = a(s_t) + b(s_t)x_{t-1} + c(s_t)z_t \tag{5.4}$$

Notice that this perceived law of motion has the same functional form as the rational expectations equilibrium law of motion, which implies that agents may conceivably learn the rational expectations equilibrium law of motion if their estimates of $a(s_t)$, $b(s_t)$, and $c(s_t)$ converge to their rational expectations values (i.e. if $a(s_t) \rightarrow 0_{n\times 1}$, $b(s_t) \rightarrow \Omega(s_t)$ and $c(s_t) \rightarrow \Gamma(s_t)$ for all s_t). If a rational expectations equilibrium can be learned, it is said to be "stable under learning" or "expectationally stable" ("E-stable") (see McClung (2016, 2017b) for more about E-stability in this class of models). E-stable rational expectations equilibria are easier to rationalize in the sense that adaptive learning supports a coordination story for their realization, and in the sense that they are robust to the unreasonably strong assumptions that undergird rational expectations. Our task in this section is to study the volatility of inflation and output when agents beliefs about the structure of the economy are close to the unique rational expectations equilibrium implemented by the monetary policy rule.

To make our model of learning fully operational, we must specify agents' information set, their estimation strategy, and the full process through which expectations interact with predetermined variables to pin down the endogenous variable values. We begin by specifying agents' time t information set, I_t, which includes all past observations of x, and all past and current observations of z and s. Formally: $I_t = \{y_{t-1}, y_{t-2}, \ldots, y_0; z_t, z_{t-1}, \ldots, z_0; s_t, s_{t-1}, \ldots, s_0\}$. We could exclude z_t from the information set (i.e. only include past values of z) and obtain similar results. Using observations in I_t, agents will update their estimates of the coefficients in (5.4) using the following *within-regime* learning algorithm:

$$\Phi(s_t)_{st} = \Phi(s_t)_{st-1} + \psi_{s_t} R(s_t)_{st}^{-1} u_t (x_t - \Phi(s_t)'_{st-1} u_t) \tag{5.5}$$

$$R(s_t)_{st} = R(s_t)_{st-1} + \psi_{s_t} (u_t u_t' - R(s_t)_{st-1}) \tag{5.6}$$

where $\Phi(s_t)_{st} = (a_t(s_t), b_t(s_t), c_t(s_t))'$ are the time-t estimates of regime s_t coefficients, $u_t = (1, x'_{t-1}, z'_t)'$, and st is the number of realizations of state s_t up until and including time t. Alternatively, we might use a learning algorithm that estimates a dummy variable regression where elements in u are interacted with dummy variables that take on values of 1 or 0 depending on the underlying Markov state. The last feature of the algorithm we need to define is the gain parameter, ψ_{s_t}. Intuitively, ψ_{s_t} attaches a weight to each new observation and therefore determines the extent to which new information impacts parameter estimates. If we give each observation equal weight by setting $\psi = 1/t_{s_t}$, where t_{s_t} is the number of realizations of s_t up until time t, then our learning algorithm becomes the conditional recursive least squares estimator of Φ. Clearly, as $t \to \infty$ the estimates converge to some value, which may be the rational expectations equilibrium coefficients depending on initial beliefs and the E-stability of the equilibrium under study. Alternatively, we might allow agents to give more weight to recent observations by using a constant gain parameter,$\psi = \bar{\psi}$, where $\bar{\psi}$ is some scalar. In constant gain learning algorithms, beliefs will never converge, but may converge to some distribution centered on the rational expectations equilibrium. These algorithms are considered appropriate in settings where agents may expected structural changes in the model, or in settings where agents simply value recent data more than older data.

Having specified the learning algorithm, we now outline the sequence of events that lead to an equilibrium at time t:

1. Agents observe z_t and s_t and add those to their information sets.

2. Using I_t and time $t-1$ estimates $a_{t-1}(s_t)$, $b_{t-1}(s_t)$, $c_{t-1}(s_t)$. Agents form forecasts, $\hat{E}_t x_{t+1}$:

$$\begin{aligned}\hat{E}_t x_{t+1} &= (p_{s_t 1} a(1)_{t-1} + p_{s_t 2} a(2)_{t-1}) + \\ &\quad (p_{s_t 1} b(1)_{t-1} + p_{s_t 2} b(2)_{t-1})(a(s_t)_{t-1} + b(s_t)_{t-1} x_{t-1} + c(s_t)_{t-1} z_t) + \\ &\quad (p_{s_t 1} c(1)_{t-1} + p_{s_t 2} c(2)_{t-1}) \rho z_t\end{aligned}$$

3. x_t is generated from the actual law of motion, (5.3), which gives us time t endogenous variables as a function of beliefs and predetermined variables

4. Agents observe x_t and add it to their information sets

5. Agents use (5.5)-(5.6) to update their estimates

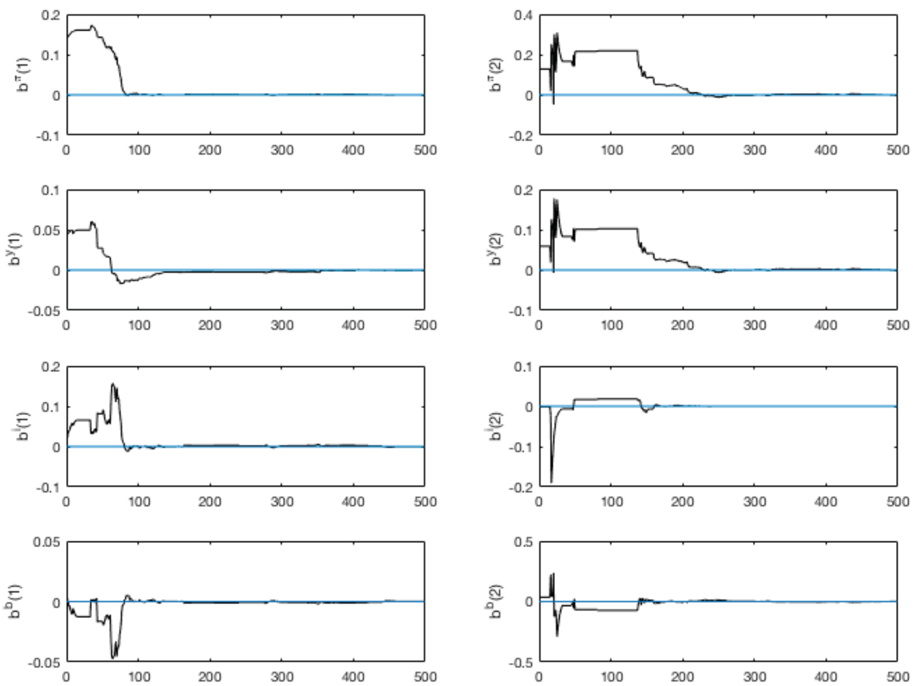
6. Forward t to $t+1$ and repeat steps 1-5.

Before studying policy performance in this environment, we first use a decreasing gain parameter see whether agents can learn the rational expectations equilibrium corresponding to each of the parameterizations we consider. Initial beliefs about $a(s_t)$, $c(s_t)$ for $s_t = 1, 2$ are set to zero, while initial beliefs about $b(s_t)$ are perturbed around $\Omega(s_t)$.[11] For all parameterizations we consider here, beliefs eventually converge to their rational expectations equilibrium values. Figure 13 illustrates the convergence of beliefs for the posterior mean calibration with $\gamma(1) = 5$, $\gamma(2) = -5$, $\phi_\pi(1) = 3$, $\phi_\pi(2) = 0$. In this figure, as well Figure 16 we plot the difference of actual beliefs and rational

[11] We set initial beliefs about the VAR coefficients away from zero (but still far from their REE values) to help improve the rate of convergence of beliefs to the REE. We also want to mention that beliefs may not converge to the rational expectations equilibrium for all initial values; E-stability is a local stability concept that only applies to beliefs that are in some neighborhood of their potential convergence points.

expectations equilibrium beliefs over time (i.e. a value of 0 means that beliefs equal the rational expectations equilibrium).

FIGURE 13. Coefficient Estimate Errors and Observed State Learning



The left-hand column features the VAR-coefficients on independent variable lagged debt in regime 1; right-hand column features the VAR-coefficients on independent variable lagged debt in regime 2. Notice that beliefs are held fixed when they correspond to an inactive state (e.g. notice the flat, "mesas" in the state 2 coefficients between t=50 and t=150).

To help better understand the impact that learning has on model dynamics, we study policy performance in a model with constant gain learning and a gain parameter equal to .01.[12] In such a model, we cannot compute the unconditional variance of

[12]The learning algorithm is also augmented with a ridge correction mechanism as in Slobydan and Wouters (2012), and projection facility that prevents estimates from updating if the updated

TABLE 5. Optimal Coefficients under Adaptive Learning

Type	$(\gamma(1),\ \gamma(2))$	**optimal RE coeff.** $(\phi_\pi(1),\ \phi_\pi(2),\ \rho_i)$	**optimal AL coeff.** $\phi_\pi(1),\ \phi_\pi(2),\ \rho_i)$	**Projection Facility (per 100,000)**
GS	(5, −5)	(3.3, 0, .99)	(3.68, 0, .99)	110
GA	(2, −5)	(0, 0, 0)	(0, 0, 0)	86

The larger inflation reaction coefficients under learning echoes a result from Orphanides and Williams (2007). 4 is the largest inflation reaction coefficient used in this particular numerical search. Despite the small gain parameter and infrequent use of the projection facility, the model is frequently unstable for $\psi > .02$

inflation and output. We therefore approximate the variance of inflation and output by simulating the model for 100,000 periods and computing sample variances.[13] Because these simulations are more computationally intensive, we do not compute expected posterior losses. Instead, we set non-policy parameters equal to their posterior mean (with added cost-push shock), and make inferences based on this model calibration. Otherwise, the procedure for measuring performance is the same as the procedure used in the RE model: we search over monetary policy parameters and find the set of interest rate rule coefficients that minimizes the variance of inflation and output. Table 5 presents our main findings.

Unobserved Markov States (Hidden Markov Model)

The learning model with observed states provides valuable evidence that the optimized simple rules under rational expectations are robust to misspecifications of private sector expectations. However, that model makes one potentially unreasonable

parameters imply a Markov-switching VAR that is not mean-square-stable. Intuitively, the projection facility formalizes the notion that agents reject unstable models. We invoke the projection facility and ridge correction mechanism in far less than 1% of simulated periods

[13]Each simulation uses the same 100,000 realizations of shocks. We do this to help mitigate the potential for large outlier shocks to bias our sample variances.

assumption: agents observe the state of fiscal and monetary policy. In practice, applied econometricians do not observe the stance of fiscal and monetary policy. Instead, econometricians use techniques developed in papers such as Hamilton (1989) to identify the probable state of the economy at any point in time. Because a lot of adaptive learning research begins with the premise that our models' agents should be no more informed and rational than the econometricians among us, we endeavor in this section to remove s_t from the information set, I_t, and study the model-implied dynamics of inflation and output. We refer to this new model as the hidden Markov model of learning. Before deriving the hidden Markov model of learning, we emphasize that self-referential feedback in this model not only poses the risk of destabilizing agents' beliefs about model coefficients; forecast errors act on both future coefficient estimates and agents' inferences about the underlying state. One may therefore expect additional expectations-induced volatility in this model.

As it turns out, the structure of our model makes it possible for agents to infer the underlying state with reasonable accuracy so that the removal of Markov states from agents' information set only raises the volatility of inflation and output slightly. This last point is partly explained by an argument made in Bianchi (2013) which states that fully rational agents can perfectly infer today's state if they observe contemporaneous and past x, z. Their argument relies on the fact that rational agents know all of the S within-regime systems of equations (i.e. $x_t = \Omega(s_t)x_{t-1} + \Gamma(s_t)z_t$) that may determine x_t. All agents in their model need to do to perfectly infer the state is compute each of the S equations until they find the correct system of equations. Their argument does not apply to our framework; if agents hold incorrect beliefs about the economy – as they always do in a model of constant gain learning, or before beliefs converge – they may make horrible inferences about the state of the economy. Despite this limitation, the equilibrium coefficients of the policy rules are exogenous to beliefs, which makes it easy for agents to learn the rational expectations equilibrium law of motion for fiscal surpluses and infer from it the underlying state of the economy with reasonable but far from perfect accuracy. We emphasize that other equilibrium coefficients do depend on agents' beliefs, so that our model is still self-referential.

As before, agents beliefs about the law of motion for endogenous variables is given by the PLM in (5.4). In what follows, we consider two information structures. First, we assume that $I_t = \{y_{t-1}, y_{t-2}, \ldots, y_0; z_{t-1}, \ldots, z_0\}$. After examining the potential convergence points of beliefs, and pointing out the exogeneity of the surplus law of motion, we then add surpluses, τ_t, to I_t and demonstrate that agents' beliefs can converge to the rational expectations equilibrium. Under both information structures, agents do not observe s_t, which implies that they cannot use (5.5)-(5.6) to update their beliefs. To get around the difficulty presented by the hidden Markov process, we rely on techniques from Krishnamurthy and Yin (2002) and LeGland and Mevel (1997), which present "online" or recursive algorithms for learning the coefficients of an exogenous Markov-switching autoregression. Specifically, we use the recursive maximum likelihood estimator (RMLE) from both papers, and the recursive conditional least squares estimator (RCLS) from LeGland and Mevel (1997). While newer alternatives to these algorithms exist outside of the stochastic approximation literature, we rely on these papers because they present convergence results that may prove useful in extensions of the current analysis.

The algorithms described in both papers make inferences about the coefficients, $\Phi(s_t)$, and the Markov process, s_t, using two related recursive processes. First, agents make inferences about s_t using a prediction filter of the form introduced by Hamilton (1989). To develop this filter we first define *within-regime* conditional densities for x,

$f_{s_t} = f(x_t|x_{t-1}, x_{t-2}, ..., z_t, z_{t-1}, ..., s_t; \Phi(s_t)_{t-1})$. In a model with normally distributed $i.i.d$ innovations to our exogenous driving process, f_{s_t} assumes the following form:

$$f_{s_t} = (2\pi)^{-t/2}|\Sigma|^{-.5}exp\{-.5(x_t - \mu(s_t)_{t-1})'\Sigma^{-1}(x_t - \mu(s_t)_{t-1})\}$$

where $\mu(s_t)_{t-1} = a_{t-1}(s_t) + b_{t-1}(s_t)x_{t-1} + c_{t-1}(s_t)z_t$ and Σ is the covariance-variance matrix for the $i.i.d$ innovations to z. To make future calculations easier, we define the following matrices:

$$f_t = (f_{1t}, f_{2t} \ldots, f_{St})'$$

$$F_t = diag(f_{1t}, f_{2t} \ldots, f_{St})$$

Let $\hat{p}_{i,t|t-1} = Pr(s_t = i|I_t)$, and $\hat{p}_{t|t-1} = (\hat{p}_{1t|t-1}, \hat{p}_{2t|t-1}, \ldots, \hat{p}_{St|t-1})'$. $\hat{p}_t$ follows the recursion:

$$\hat{p}_{t+1|t} = \frac{P'F_t\hat{p}_{t|t-1}}{f_t'\hat{p}_{t|t-1}} \tag{5.7}$$

where it is assumed that agents know the true transition probabilities in P. The prediction filter in the last equation completely describes how agents recursively compute their predictions for today's state. Because inferences about s_t are made prior to time t, agents can, at best, infer s_{t-1} perfectly. As we show below, this feature of our model makes it impossible for agents' beliefs to converge to the rational expectations equilibrium studied in previous sections, and is the primary reason why we argue for the addition of τ_t to I_t. The second recursive process in the algorithms presented by Krishnamurthy and Yin (2002) and LeGland and Mevel (1997) updates the parameter estimates, $\Phi(s_t)$, according to:

$$\Phi_t = \Phi_{t-1} + \gamma S(x_t, I_t; \Phi_{t-1}) + \epsilon_t M_t$$

where Φ_t is a $k \times 1$ vector[14] that contains the elements of $\Phi(s_t)$ for all s_t, γ is the gain parameter and M_t is a correction term (i.e. we use a projection facility in our implementation of their algorithms). Let Φ_t^l denote the l-th element of Φ_t. The function $S(x_t, I_t; \Phi_{t-1})$ is the only thing that varies across the two algorithms we use in the paper. For the RMLE algorithm, $S(x_t, I_t, \Phi_{t-1})$ is given by the following equations:

$$S(x_t, I_t, \Phi_{t-1}) \quad = \quad (S^1(x_t, I_t, \Phi_{t-1}), \ldots, S^k(x_t, I_t, \Phi_{t-1}))'$$

where

$$S^l(x_t, I_t, \Phi_{t-1}) = \frac{f_t' \omega_t^l}{f_t' \hat{p}_{t|t-1}} + \frac{(\partial f_t' / \partial \Phi_t^l) \hat{p}_{t|t-1}}{f_t' \hat{p}_{t|t-1}} \tag{5.8}$$

for all $l \in \{1, ..., k\}$ and $\omega_t^l = \frac{\partial \hat{p}_{t|t-1}}{\partial \Phi_t^l}$. We update ω_t^l recursively as follows:

$$\omega_{t+1}^l = R_{1t} \omega_t^l + R_{2t} \tag{5.9}$$

where

$$\begin{aligned} R_{1t} &= P'(I - \frac{F_t \hat{p}_{t|t-1} \mathbf{1}_s'}{f_t' \hat{p}_{t|t-1}}) \frac{F_t}{f_t' \hat{p}_{t|t-1}} \\ R_{2t} &= P'(I - \frac{F_t \hat{p}_{t|t-1} \mathbf{1}_s'}{f_t' \hat{p}_{t|t-1}}) \frac{(\partial F_t)/(\partial \Phi_t^l) \hat{p}_{t|t-1}}{f_t' \hat{p}_{t|t-1}} \end{aligned}$$

[14]In our model with $S = 2$, n endogenous variables and m endogenous variables, $k = 2(n(n+1) + nm)$

Equation (5.7), (5.8) and (5.9), plus initial conditions, give us the RMLE algorithm. To derive the RCLS we only need to change our definition of $S^l(x_t, I_{t-1}, \Phi_{t-1})$ as follows:

$$S^l(x_t, I_t, \Phi_{t-1}) = (\phi_{\Phi_{t-1}}(x_t - \phi'_{\Phi_{t-1}}\hat{p}_{t|t-1}))'\omega_t^l + (\frac{\partial \phi_{\Phi_{t-1}}}{\partial \Phi_{t-1}^l}(x_t - \phi'_{\Phi_{t-1}}\hat{p}_{t|t-1}))'\hat{p}_{t|t-1} \quad (5.10)$$

where $\phi_{\Phi_{t-1}}$ is a matrix that collects the conditional mean for each state (i.e. $\mu(s)_{t-1}$ for each $s \in \{1, \ldots, S\}$). Before outlining the events leading to a temporary equilibrium, we emphasize that this algorithm is very similar to the algorithm presented in (5.5)-(5.6). Specifically, if agents observe the state so that ω becomes a vector of zeros (since $(I - \frac{F_t\hat{p}_{t|t-1}\mathbf{1}'_s}{f'_t\hat{p}_{t|t-1}} \to 0_S)$, and they replace $\hat{p}_{t|t-1}$ with $\hat{p}_{t|t} = (1\ 0)'$ or $\hat{p}_{t|t} = (0\ 1)'$ to reflect this knowledge, then this algorithm becomes the recursive estimator used in the model with observed states with $R(s_t)_{s_t} = I$. We can now outline the sequence of events that lead to an equilibrium at time t:

1. Agents update information sets.

2. Using I_t and time $t-1$ estimates $a_{t-1}(s_t)$, $b_{t-1}(s_t)$, $c_{t-1}(s_t)$. Agents form forecasts, $\hat{E}_t x_{t+1}$:

$$\begin{aligned}
\hat{E}_t x_{t+1} \quad = \quad & (\hat{p}_{1t|t-1}p_{11} + \hat{p}_{2t|t-1}p_{21})a(1) + (\hat{p}_{1t|t-1}p_{12} + \hat{p}_{2t|t-1}p_{22})a(2) + \\
& \hat{p}_{1t|t-1}p_{11}b(1)(a(1) + b(1)x_{t-1} + c(1)z_t) + \\
& \hat{p}_{1t|t-1}p_{12}b(2)(a(1) + b(1)x_{t-1} + c(1)z_t) + \\
& \hat{p}_{2t|t-1}p_{11}b(1)(a(2) + b(2)x_{t-1} + c(2)z_t) + \\
& \hat{p}_{2t|t-1}p_{11}b(1)(a(2) + b(2)x_{t-1} + c(2)z_t) + \\
& ((\hat{p}_{1t|t-1}p_{11} + \hat{p}_{2t|t-1}p_{21})c(1) + (\hat{p}_{1t|t-1}p_{12} + \hat{p}_{2t|t-1}p_{22})c(2))\rho z_t
\end{aligned}$$

3. x_t is generated from the actual law of motion, (5.3), which gives us time t endogenous variables as a function of beliefs and predetermined variables

4. Agents observe x_t and add it to their information sets

5. Agents use (5.7), (5.8), (5.9), or (5.7), (5.9), and (5.10) to update their coefficient estimates and prediction filter

6. Forward t to $t+1$ and repeat steps 1-5.

Before presenting results, it is important to note that our hidden states information structure prevents agents from learning the rational expectations equilibrium studied in all previous sections. This is because agents only form $\hat{p}_t$ using $t-1$ information. Hence, if agents perfectly infer s_{t-1} – which is the best they can do – they still hold the following beliefs about s_t: $\hat{p}_{t|t-1} = (p_{s_{t-1}1}, p_{s_{t-1}2})' < (1,1)'$. In this best case scenario, agents' beliefs about the VAR coefficients, $b(s_t)$, will not converge to a solution of (5.3). If, instead, agents allow their beliefs about PLM coefficients to depend on both s_t and s_{t-1} then this information structure may allow agents to learn solutions to the following fixed point condition:

$$b(s_t, s_{t-1}) = A(s_t) \sum_{j=1}^{2} \sum_{h=1}^{2} p_{s_{t-1}j} p_{jh} b(h,j) b(j, s_{t-1}) + B(s_t) \tag{5.11}$$

These solutions, which we refer to as history-dependent equilibria, do solve (5.3). However, they do not satisfy the following fixed point condition:

$$b(s_t) = A(s_t)(p_{s_t 1} b(1) + p_{s_t 2} b(2)) b(s_t) + B(s_t) \tag{5.12}$$

which is a necessary condition for solutions of the form, $b(s_t)$. While beliefs are no longer consistent with the rational expectations equilibria we examined up until now, we nonetheless find that beliefs can converge[15]. Hence, while beliefs never converge to the rational expectations equilibrium, they may nonetheless be stable over time and converge to values that may be relatively close to the original rational expectations equilibrium.

To identify potential convergence points consistent with (5.11), we use the Groebner basis approach from Foerster et al (2016). We then explore issues of uniqueness and E-stability pertaining to this class of equilibria. Initial evidence suggests that policy parameters widely associated with determinacy in the preceding analysis may admit multiple mean-square stable history dependent equilibria that satisfy the fixed point condition in (5.11). Moreover, these equilibria do not appear to be stable under learning. Since this class of equilibria is arguably relevant in settings where agents cannot observe contemporaneous variables, we intend to further explore these issues of uniqueness and expectational stability in future work.

Figure 14 plots $\hat{p}_1$ over time. In our calibration $p_{11} = .95$ so that oscillation in their beliefs between .05 and .95 implies that they're inferring s_{t-1} almost perfectly. To better understand how agents so successfully infer the underlying state of the economy, despite initial incorrect beliefs about the structure of the economy, we redefine $x = (\tilde{x}, \tau)'$ where $\tilde{x} = (y, \pi, i, b, P)'$ and point out that the actual law of motion for x (after beliefs are substituted in) may be written as:

$$\begin{pmatrix} \tilde{x}_t \\ \tau_t \end{pmatrix} = \begin{pmatrix} \tilde{\Omega}(s_t; \Phi_{t-1}) \\ \Omega_\tau(s_t) \end{pmatrix} \begin{pmatrix} \tilde{x}_{t-1} \\ \tau_{t-1} \end{pmatrix} + \begin{pmatrix} \tilde{\Gamma}(s_t; \Phi_{t-1}) \\ \mathbf{e}_6' \end{pmatrix} z_t \tag{5.13}$$

where $\Omega_\tau(s_t) = (0\ 0\ 0\ \gamma(s_t)\ 0\ 0)$, and $\mathbf{e}_6' = (0\ 0\ 0\ 0\ 0\ 1)$. Clearly, the evolution of τ_t is only endogenous to beliefs through b_{t-1}; the coefficients governing the evolution of

[15]Even for constant gain parameters beliefs appear to converge to a distribution around a fixed point

τ are exogenous, which suggests that agents will quickly learn the law of motion for τ and then make accurate inferences for $\hat{p}_t$ that rely on the marginal density:

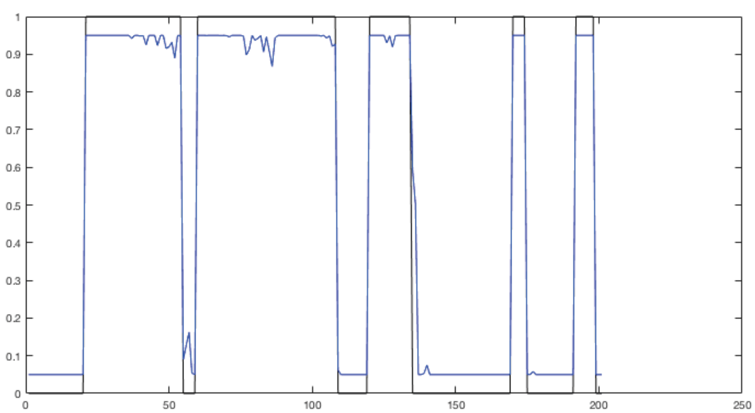
$$f_{s_t}^{\tau} = f(\tau_t | x_{t-1}, \Phi_{t-1}) \tag{5.14}$$

The marginal density in (5.14) is so essential for correct inference of s_{t-1} that we can redefine our prediction filter using only the marginal densities for surpluses and get results that are nearly identical to the results displayed in Figure 14). The fact that surpluses are determined at the beginning of t (i.e. all shocks and b_{t-1} have been realized by beginning of t, so that τ_t is fixed before agents form expectations), begs an important question about timing: should agents be able to observe τ_t at the beginning of t? That is, should I_t include τ_t? If agents observe τ_t at t, they may be able to perfectly infer s_t. This allows for agents to learn solutions of the fixed point problem given by (5.12) to coincide, i.e. so that agents may actually learn the rational expectations equilibrium under study. To support this idea numerically, we first redefine the prediction filter:

$$\begin{aligned}
f_t^{\tau} &= (f_{1t}^{\tau}, f_{2t}^{\tau} \ldots, f_{St}^{\tau})' \\
F_t^{\tau} &= diag(f_{1t}^{\tau}, f_{2t}^{\tau} \ldots, f_{St}^{\tau}) \\
\hat{p}_{t|t}^{\tau} &= \frac{F_t^{\tau} \hat{p}_{t|t-1}^{\tau}}{f_t'^{\tau} \hat{p}_{t|t-1}^{\tau}} \\
\hat{p}_{t+1|t}^{\tau} &= P' \hat{p}_{t|t}^{\tau}
\end{aligned}$$

Now agents use $\hat{p}_{t|t}^{\tau}$ instead of $\hat{p}_{t|t-1}$ when forming expectations at time t. As shown in Figure 15 agents can now infer the current state very effectively, which allows them to learn the rational expectations equilibrium under study in the previous section, as demonstrated by Figure 16. In Figure 16, we initialize beliefs away from the rational expectations equilibrium[16], set $\psi = t^{-2/3}$ (as in LeGland and Mevel (1997)) and estimate the model using the RCLS algorithm. We also use a projection facility that prevents agents from accepting a mean-square-unstable PLM, but this facility is invoked in far less than .1% of periods simulated. Compared to Figure 13, the rate of convergence is slow under RCLS, but this may be driven the errors in the prediction filter (Figure 15) and the large decreasing gain parameter $t^{-2/3}$. We find that the optimal policy results in the observed states learning section generalize to the hidden Markov model of learning.

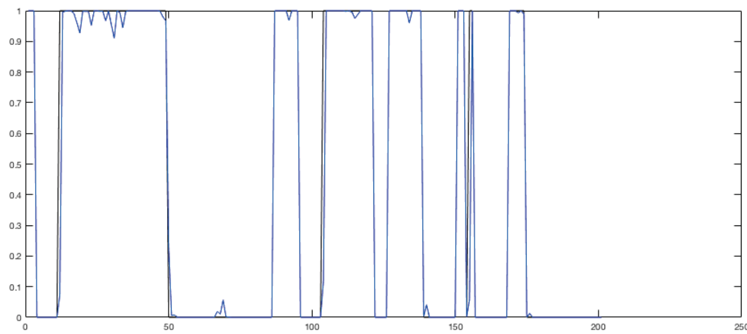
FIGURE 14. Estimating the Policy State (Lagged Information)



Blue line is $\hat{p}_{1,t|t-1}$; black line equals 1 if $s_t = 1$ and 0 otherwise

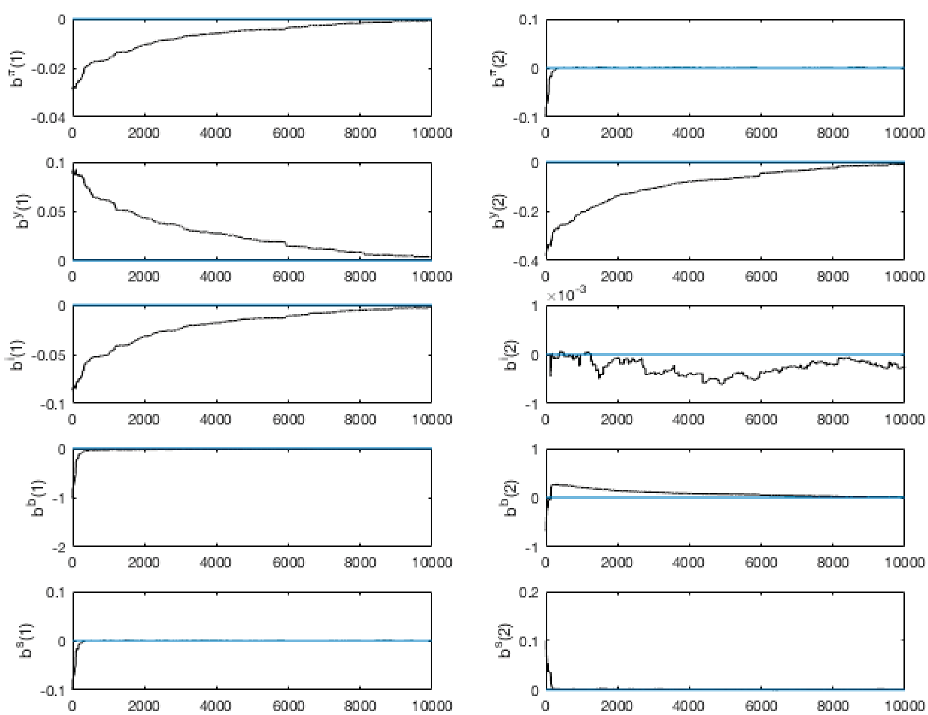
[16]As seen in the third subplot in the second column of Figure 16, initial beliefs about the dependence of i on b in regime F are unintentionally close to 0. Our results do not depend on this initial belief.

FIGURE 15. Estimating the Policy State (Contemporaneous Information)



Blue line is $\hat{p}_{1,t|t}$; black line equals 1 if $s_t = 1$ and 0 otherwise

FIGURE 16. Coefficient Estimate Errors in Hidden Markov Model Learning



The left-hand column features the VAR-coefficients on independent variable lagged debt in regime 1; right-hand column features the VAR-coefficients on independent variable lagged debt in regime 2.

Long-term Debt

In this section, we relax the assumption that $\rho = 0$ and introduce long-term debt into our model. While this innovation helps to bring our model closer to reality, it also creates a debt revaluation channel through which monetary and fiscal policy interact to affect agents' perceptions of bond wealth in non-Ricardian economies. This debt revaluation works as follows: if interest rates are reduced (increased), then the price of outstanding debt, given by $\hat{P^m}$, increases (decreases) and this positively (negatively) affects agents' perception of their own net wealth. This revaluation channel can often lean against the wealth effects created by movements in debt service costs, a tendency demonstrated in a host of papers including McClung (2017a).

In addition to the creation of a revaluation channel, the introduction of long-term debt can alter the menu of monetary policies that induce a determinate equilibrium when fiscal policy switches and is non-Ricardian (see McClung (2017b)). The fact that maturity matters for determinacy in our simple switching DSGE model is a novel result insofar as the average maturity of debt does not matter for determinacy in the corresponding fixed regime model (see Jin (2013), for example). To illustrate the impact that maturity has on determinacy consider figures 17 and 18.

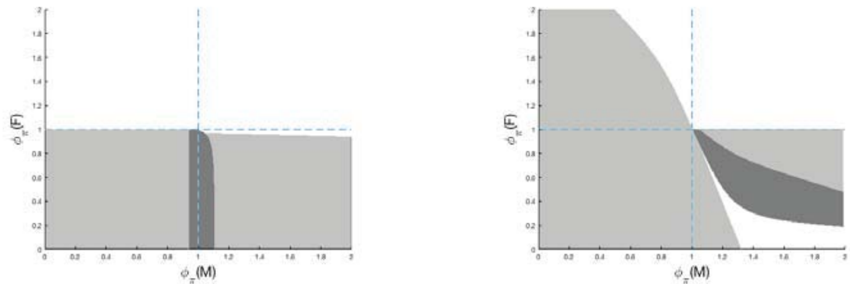
The fact that maturity matters for determinacy complicates the policymaker's problem in at least two ways. First, the policymaker now has an incentive to identify the steady state average maturity given by ρ when fiscal policy is non-Ricardian. Without knowledge of ρ, the policymaker cannot properly identify the menu of policies that induce a unique equilibrium, which may prevent them from finding the optimized policy. Second, the policymaker now has an incentive to consider balance sheet decisions that affect the value of ρ when solving their optimization problem. In our model, the relevant measure of government debt is government debt held by the household, not purely the debt issued by the fiscal authority itself. As such, central banks can impact the maturity structure of debt held by households by engaging in Operation Twist-style policies in which households and the monetary authority exchange short-term debt for long-term debt. Figure 17 illustrates a case where monetary policymakers may realize an incentive to lengthen the maturity debt held by households, while Figure 18 illustrates the opposite case. We hope to use ρ as a proxy for these debt operations by adding ρ to the central bank's choice set.

The search for an optimal ρ is further complicated by the fact that we face uncertainty over the true value of ρ.[17] We might address this uncertainty by assigning a prior distribution to ρ, adding ρ to θ then estimating the model using Bayesian techniques. Using simple priors over ρ we can generate a tradeoff between expected posterior loss and the probability that a given policy implements a unique mean-square stable and E-stable equilibrium. For one simple prior over ρ, the policy that maximizes the probability of determinacy and E-stability (at .985) when $p_{11} = p_{22} = .95$, $\gamma(1) = 5$, $\gamma(2) = -1$, involves $\phi_\pi(1) = 1.2$, $\phi_\pi(2) = .9$ and an expected posterior loss of 4.15. If we replace $\phi_\pi(1) = 1.2$, $\phi_\pi(2) = .9$ with $\phi_\pi(1) = 1.3$, $\phi_\pi(2) = .8$, we reduce the probability of determinacy and E-stability to .917, but we also reduce expected posterior loss to 2.57. The addition of uncertainty over ρ therefore introduces a tradeoff between minimizing loss and maximizing the probability of determinacy and E-stability, a tradeoff first recognized by Evans and McGough (2007). Uncertainty over all other fiscal policy parameters will almost surely present a similar tradeoff.

We believe that parameter uncertainty and the addition of an extra dimension in ρ to our policy problem generates complications that are beyond the scope of the

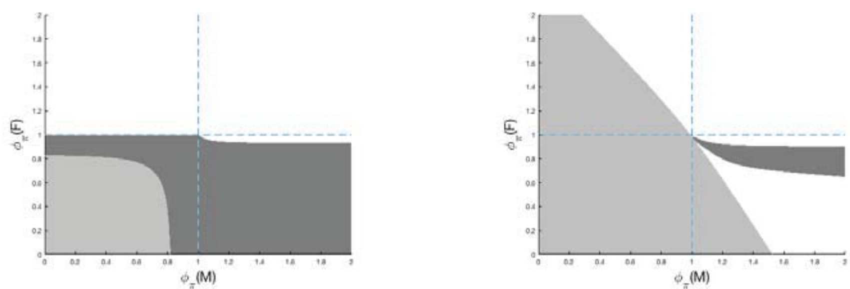
[17] The average maturity of debt, equal to $(1 - \beta\rho)^{-1}$ in our model has been estimated using U.S. data. However, because the actual maturity structure of U.S. debt does not decay geometrically, it is not clear whether or not such estimates should be used to select ρ

FIGURE 17. Lengthening Maturity Expands Policy Menus



Left panel $\rho = 0$, right panel $\rho = .96$. $p_{mm} = .98$, $p_{ff} = .95$, $\gamma(M) = .02$, $\gamma(F) = -.01$. The determine region is dark gray; the indeterminate region is light gray; explosive region is white

FIGURE 18. Shortening Maturity Expands Policy Menus



Left panel $\rho = 0$, right panel $\rho = .96$. $p_{mm} = .98$, $p_{ff} = .95$, $\gamma(M) = .02$, $\gamma(F) = -.01$. The determine region is dark gray; the indeterminate region is light gray; explosive region is white

present analysis. However, we hope to fully explore issues pertaining to long-term debt in an estimated DSGE framework in the near future.

Conclusion

This paper examines the performance and robustness of simple monetary policy rules in models with learning agents subject to: (1) permanent or occasionally non-Ricardian fiscal policy; and/or (2) the presence of long-term government debt.

My analysis indicates that the “global” response of the fiscal policymaker to debt determines the optimal monetary policy response. When fiscal policy is globally passive or globally active the optimal monetary policy rule features time-invariant coefficients with high inflation reaction coefficients in globally passive models and interest rate pegs in globally active models. In cases where fiscal policy features balanced or strong switching between active and fiscal policy stances, the optimal monetary policy rule features switching coefficients. These results are robust to adaptive learning, including a novel hidden Markov model of learning we introduce in the paper. For this reason, we should want to better understand how the presence of long-term debt affects the optimal monetary policy in a model with switching fiscal policy stances.

CHAPTER VI

CONCLUSION

Concluding Summary

The work which comprises this dissertation demonstrates the extent to which conventional and unconventional policy outcomes, as well as the existence, uniqueness and expectational stability of rational expectations solutions, depend on the expectational effects of time-varying policy. These findings suggest that uncertainty over future fiscal policy may curb the effectiveness of monetary policy, or otherwise constrain the actions of central bankers. Additionally, this work examines the relationship between determinacy and expectational stability in a general class of Markov-switching DSGE models.

Chapter 2 of my dissertation generalizes McCallum (2007) and is the first to address the relationship between determinacy and E-stability in Markov-switching Dynamic Stochastic General Equilibrium (MS-DSGE) models with lagged endogenous variables. I prove that the sufficient conditions for determinacy in Cho (2016) imply the E-stability of the forward solution in MS-DSGE models with lagged endogenous variables when agents condition their expectations of future endogenous variables on current endogenous and exogenous variables. The class of models studied in this paper is very general, and nests a wide array of models that are frequently studied in modern macroeconomics.

In Chapter 3, I study the impact of expansionary forward guidance in a simple New Keynesian model with recurring or permanent active fiscal policy. This work addresses and offers a potential solution to the simple New Keynesian model's prediction that expansionary forward guidance can generate an implausibly large stimulus. I find that the introduction of permanent or recurring active fiscal policy dampens the response of output and inflation to forward guidance in the New Keynesian model. Moreover, the presence of regime-switching policy introduces expectational effects that cause forward guidance to be less stimulative in our regime-switching model's active money, passive fiscal policy regime. Finally, the introduction of long-term debt affects the magnitude of the stimulus resulting from forward guidance in models with active fiscal policy.

In Chapter 4, I explore determinacy and E-stability in a New Keynesian model with switching fiscal and monetary policy. Here I present three categories of results. First, the maturity structure of government debt matters for determinacy and the existence of stable equilibria in our switching model, which is not true in the analogous fixed coefficient model. I use two numerical solution techniques to show that maturity affects both the multiplicity of stable solutions, and the existence of sunspot equilibria. Second, determinacy generally implies E-stability when agents do not observe contemporaneous observable variables, but not for certain arguably unrealistic regions of the model parameter space. Third, this chapter presents conditions for stability under infinite-horizon learning in Markov-switching DSGE models and compares stability under infinite horizon and one-step-ahead learning. To the best of my knowledge, this is the first paper to derive these stability conditions in a model with switching coefficients.

Finally, Chapter 5 examines the performance and robustness of simple monetary policy rules in models with learning agents subject to: (1) permanent or occasionally active fiscal policy; and/or (2) the presence of long-term government debt. My analysis indicates that the "global" response of the fiscal policymaker to debt determines the optimal monetary policy response. When fiscal policy is globally passive or globally active the optimal monetary policy rule typically features time-invariant coefficients with high inflation reaction coefficients in globally passive models and interest rate pegs in globally active models. In cases where fiscal policy features balanced or strong switching between active and fiscal policy stances, the optimal monetary policy rule features switching coefficients. These results extend to models with adaptive learning, including a hidden Markov model of learning never seen before in the literature.

Extensions and Future Work

Current work in progress examines the implications of debt maturity structure for central bank balance sheet decisions and the performance of simple interest rate rules subject to time-varying fiscal policy. Because the maturity structure of debt has major implications for the menu of interest rate rules available to monetary policymakers, this project may answer two questions. First, how might optimal simple interest rate rules depend on both the maturity structure of debt and fiscal policy regimes? Second, what central bank balance sheet decisions help to contain inflation and output in an economy with regime switching fiscal policy?

Other work in progress explores issues of uniqueness and expectational stability in a general class of Markov-switching DSGE models with lagged information structures. This line of research is motivated by Chapter 5, which shows that the beliefs of learning agents who do not observe contemporaneous variables, including underlying Markov states, cannot converge to the class of rational expectations equilibria considered in most Markov-switching DSGE analyses. Because learning applications commonly exclude contemporaneous variables from information sets, this result suggests that a separate class of equilibria that depend on past Markov states may be relevant in specific policy applications. This project attempts to learn various properties of this class of equilibria.

Additionally, I endeavor to further study the expectational stability of rational expectations equilibria and convergence of beliefs in hidden Markov models of adaptive learning. Papers in the stochastic approximation literature have studied the properties of recursive algorithms that estimate parameters of Markov-switching autoregressive processes with hidden states. I would like to apply convergence results in that literature to my work on learning in regime-switching DSGE models.

Future work should also extend themes in the dissertation to larger, more realistic models with more sophisticated policy rules. While simple fiscal policy rules help us understand the link between policy interactions and general equilibrium outcomes, the performance of monetary policy should be studied in fuller models that include rich debt maturity structures, capital, etc., and that allow for more sophisticated policy rules and more than two policy states. These projects could involve additional Bayesian model estimation of regime switching DSGE models. Such exercises help to resolve parameter and model uncertainty, therefore offering greater insight into the robustness of optimal policy

APPENDIX A

DERIVATION OF FIRST E-STABILITY CONDITION

To solve for $DT_B(\bar{B})$, we linearize $T_B(B)$ at the forward solution and vectorize the resulting equation. We then use the following identification rule: if $vec(dT_B) = Avec(dB)$ then $A = DT_B(B)$, where $dB = (dB(1)\ dB(2) \cdots dB(S))$ and dT_B is the linearized system of equations. Using the rule: $d(F(X)^{-1}) = -F(X)^{-1}(dF)F(X)^{-1}$, we obtain the following linearization of $T_B(B)$:

$$
\begin{aligned}
dT_B &= \begin{pmatrix} (\Xi(1,B)^{-1}M(1)(\sum_{j=1}^{S} p_{1j}dB(j))\Xi(1,B)^{-1}N(1))' \\ (\Xi(2,B)^{-1}M(2)(\sum_{j=1}^{S} p_{2j}dB(j))\Xi(2,B)^{-1}N(2))' \\ \vdots \\ (\Xi(S,B)^{-1}M(S)(\sum_{j=1}^{S} p_{Sj}dB(j))\Xi(S,B)^{-1}N(S))' \end{pmatrix}' \\
&= \Xi(1,B)^{-1}M(1)p_{11}(dB) \begin{pmatrix} \Xi(1,B)^{-1}N(1) & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & & & 0_n \end{pmatrix} \\
&+ \Xi(1,B)^{-1}M(1)p_{12}(dB) \begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ \Xi(1,B)^{-1}N(1) & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & & & 0_n \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \cdots \\
&+ \quad \Xi(1,B)^{-1}M(1)p_{1S}(dB)\begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ \Xi(1,B)^{-1}N(1) & & & 0_n \end{pmatrix} \\
&+ \quad \Xi(2,B)^{-1}M(2)p_{21}(dB)\begin{pmatrix} 0_n & \Xi(2,B)^{-1}N(2) & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & & & 0_n \end{pmatrix} \\
&+ \quad \Xi(2,B)^{-1}M(2)p_{22}(dB)\begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ 0_n & \Xi(2,B)^{-1}N(2) & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & & & 0_n \end{pmatrix} \\
&+ \quad \cdots \\
&+ \quad \Xi(2,B)^{-1}M(2)p_{2S}(dB)\begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & \Xi(2,B)^{-1}N(2) & & 0_n \end{pmatrix} \\
&+ \quad \cdots \\
&+ \quad \Xi(S,B)^{-1}M(S)p_{SS}(dB)\begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & 0_n & & \Xi(S,B)^{-1}N(S) \end{pmatrix}
\end{aligned}
$$

Using the rule $vec(ABC) = C' \otimes Avec(B)$, and the identification rule, we obtain:

$$
\begin{aligned}
DT_B(B) &= \begin{pmatrix} (\Xi(1,B)^{-1}N(1))' & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & 0_n & \cdots & 0_n \end{pmatrix} \otimes \Xi(1,B)^{-1}M(1)p_{11} \\
&+ \begin{pmatrix} 0_n & (\Xi(1,B)^{-1}N(1))' & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & 0_n & \cdots & 0_n \end{pmatrix} \otimes \Xi(1,B)^{-1}M(1)p_{12} \\
&+ \cdots \\
&+ \begin{pmatrix} 0_n & 0_n & \cdots & (\Xi(1,B)^{-1}N(1))' \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & 0_n & \cdots & 0_n \end{pmatrix} \otimes \Xi(1,B)^{-1}M(1)p_{1S} \\
&+ \begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ (\Xi(2,B)^{-1}N(2))' & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & 0_n & \cdots & 0_n \end{pmatrix} \otimes \Xi(2,B)^{-1}M(2)p_{21} \\
&+ \begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ 0_n & (\Xi(2,B)^{-1}N(2))' & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & 0_n & \cdots & 0_n \end{pmatrix} \otimes \Xi(2,B)^{-1}M(2)p_{22}
\end{aligned}
$$

$$+ \quad \cdots$$

$$+ \begin{pmatrix} 0_n & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & & \ddots & \\ 0_n & 0_n & \cdots & (\Xi(S,B)^{-1}N(S))' \end{pmatrix} \otimes \Xi(S,B)^{-1}M(S)p_{SS}$$

At the forward solution, $(B(1)\ B(2)\cdots B(S)) = (\Omega^*(1)\ \Omega^*(2)\cdots\Omega^*(S))$. Moreover, it is straightforward to show that $\Xi(i,\bar{B})^{-1}N(i) = \{I - M(i)(\sum_{j=1}^{S} p_{ij}\Omega^*(j))\}^{-1}N(i) = \{I - M(i)(E_t(\Omega^*(s_{t+1}))\}^{-1}N(i) = \Omega^*(i)$ and $\Xi(i,\bar{B})^{-1}M(i) = \{I - M(i)(\sum_{j=1}^{S} p_{ij}\Omega^*(j))\}^{-1}M(i) = \{I - M(i)(E_t(\Omega^*(s_{t+1}))\}^{-1}M(i) = F^*(i)$ where E_t denotes rational expectations here. After substituting these equilibrium expressions into the Jacobian, we obtain the following Jacobian evaluated at the forward solution:

$$DT_B(\bar{B}) = \begin{pmatrix} p_{11}\Omega^*(1)' \otimes F^*(1) & p_{12}\Omega^*(1)' \otimes F^*(1) & \cdots & p_{1S}\Omega^*(1)' \otimes F^*(1) \\ p_{21}\Omega^*(2)' \otimes F^*(2) & p_{22}\Omega^*(2)' \otimes F^*(2) & \cdots & p_{2S}\Omega^*(2)' \otimes F^*(2) \\ \vdots & & \ddots & \vdots \\ p_{S1}\Omega^*(S)' \otimes F^*(S) & p_{S2}\Omega^*(S)' \otimes F^*(S) & \cdots & p_{SS}\Omega^*(S)' \otimes F^*(S) \end{pmatrix}$$

APPENDIX B

NEW KEYNESIAN MODEL DERIVATION

Throughout the dissertation we study an economy that is populated by a large number of infinite-lived identical household-firms indexed by $j \in [0,1]$. Each household-firm is a monopolistically competitive producer of a unique product variety indexed by $i \in [0,1]$, where $i = j$ denotes the product of household-firm j. Household-firm j engages in a decision-making process to maximize the following objective:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{(C_t^j)^{1-\sigma}}{1-\sigma} - \omega(y_t(j)) \right)$$

subject to

$$\int_0^1 p_t(i) c_t^j(i) di + E_t(R_{t,t+1} B_t^j) \leq W_t^j + p_t(j) y_t(j) + P_t(z_t - \tau_t) \tag{B.1}$$

$$C_t^j = \left(\int_0^1 c_t^j(i)^{1-\frac{1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}}$$

where $c^j(i)$ is household-firm j's consumption of good i; W_t^j denotes the nominal value of the bond portfolio that the household holds at the beginning of t and W_0 is given; $R_{t,T}$ is the stochastic discount factor between time t and T; $y(j)$ is the quantity of product variety j produced by the household-firm; z is a lump-sum transfer from the government; τ is a lump-sum tax; ω is a strictly convex function; $p_t(j)$ and P_t are the price of product variety j and the price level, respectively. To preclude arbitrage opportunities, we assume that all asset prices are determined by stochastic discount factors. This implies, for example, that $Q_{t,t+1} = \frac{1}{1+i_t} = E_t(R_{t,t+1})$ where $Q_{t,t+1}$ is the price of a single-period government bond at time t, and i_t is the nominal interest rate on a riskless one-period bond. Furthermore, market completeness is assumed. The sequence of flow constraints implied by (B.12) yields the following intertemporal constraint:

$$\sum_{t=0}^{\infty} E_0\{R_{0,t}\int_0^1 p_t(i)c_t^j(i)di\} \leq \sum_{t=0}^{\infty} E_0\{R_{o,t}\big(p_t(j)y_t(j)+P_t(z_t-\tau_t)\big)\}+W_t^j$$

Since each household-firm is identical and markets are complete, we assume that each household-firm has the same initial wealth level. This induces agents to engage in a process of perfect risk-sharing that generates identical equilibrium paths for household consumption and so forth. As a result, we can drop the j subscript and treat household-firm j as the representative household and firm.

The household-firm chooses (1) how to allocate its expenditures among the product varieties; (2) how much to consume or save in each period; (3) how much to produce in each period. We study these three decision processes in turn. In making these decisions, the representative household-firm acts as a "price-taker" by taking the actions of other household-firms as given (i.e. the household-firm takes P_t and Y_t as given). Additionally, the household-firm faces a price rigidity when solving its producer problem, and we discuss this in greater detail below.

In this environment, a rational expectations equilibrium is a collection of stochastic processes such that each household-firm chooses sequences of consumption, asset portfolios, and prices that maximizes its objective given $\{P_t, Y_t, z_t, \tau_t\}$ and a specification for fiscal and monetary policy; such that net demand of assets by private household-firms equals the supply of government debt. By studying the aforementioned three decision-making processes of the representative household-firm, we uncover conditions that characterize such an equilibrium.

We present the first of these problems–the problem of maximizing C_t subject to a given level of expenditure–in the form of a Lagrangean:

$$L = (\int_0^1 c_t(i)^{1-\frac{1}{\epsilon}} di)^{\frac{\epsilon}{1-\epsilon}} - \mu(\int_0^1 p_t(i)c_t(i)di - X_t)$$

where X_t is the minimum level of expenditure. Differentiating with respect to $c_t(z)$ yields the following optimality condition:

$$\frac{\epsilon}{\epsilon - 1}(\int_0^1 c_t(i)^{1-\frac{1}{\epsilon}} di)^{\frac{\epsilon}{\epsilon-1}-1}(1 - \frac{1}{\epsilon})c_t(z)^{-\frac{1}{\epsilon}} - \mu p_t(z) = 0$$

which can be combined with the first-order condition for any other product variety (e.g. product variety i) to obtain:

$$(\frac{c_t(z)}{c_t(i)})^{-\frac{1}{\epsilon}} = \frac{P_t(i)}{P_t(z)}$$

Now, we can substitute this into the expenditure function and solve for $c_t(i)$:

$$\begin{aligned} X_t &= \int_0^1 p_t(i)c_t(i)di = \int_0^1 p_t(i)(\frac{p_t(i)}{p_t(z)})^{-\epsilon} c_t(z)di \\ &= \frac{c_t(z)}{p_t(z)^{-\epsilon}} \int_0^1 p_t(i)^{1-\epsilon} di \end{aligned}$$

Because $P_t = (\int_0^1 p_t(i)^{1-\epsilon} di)^{1/(1-\epsilon)}$, this last equation implies:

$$c_t(z) = \frac{X_t}{P_t}(\frac{p_t(z)}{P_t})^{-\epsilon} \tag{B.2}$$

We can then substitute the analogous equation for $c_t(i)$ this into the definition for C_t:

$$C_t = \left(\int_0^1 c_t(i)^{1-\frac{1}{\epsilon}} dj\right)^{\frac{\epsilon}{\epsilon-1}} = \frac{X_t}{P_t^{1-\epsilon}}\left(\int_0^1 p_t(j)^{1-\epsilon} dj\right)^{\frac{-\epsilon}{1-\epsilon}} = \frac{X_t}{P_t^{1-\epsilon}} P_t^{-\epsilon} \tag{B.3}$$

$$\therefore P_t C_t = X_t = \int_0^1 p_t(i) c_t(i) di \tag{B.4}$$

Equations (B.2)-(B.4) therefore imply the following demand schedule for good i:

$$c_t(i) = C_t\left(\frac{p_t(i)}{P_t}\right)^{-\epsilon}$$

Henceforth, let government purchases equal 0 in every period. This assumption delivers the following market-clear condition:

$$Y_t = C_t \tag{B.5}$$

Accordingly, the demand schedule for product variety i may be expressed as:

$$y_t(i) = Y_t\left(\frac{p_t(i)}{P_t}\right)^{-\epsilon} \tag{B.6}$$

The optimal consumption-savings plan of the representative household must satisfy: (1) $Y_t = C_t$ for all t; (2) the intertemporal household budget constraint (with equality) in each period; (3) a consumption Euler-equation that can be derived through a variational argument:

$$\beta E_t\left\{\frac{u'(C_{t+1})}{u'(C_t)}\frac{P_t}{P_{t+1}}\right\} = \frac{1}{1+i_t} \tag{B.7}$$

Condition (B.1) along with complete risk-sharing imply [1]:

$$\sum_{T=t}^{\infty} E_t R_{t,T} P_t C_T = \sum_{T=t}^{\infty} E_t\big\{R_{t,T}[P_T Y_T + P_T(z_T - \tau_T)]\big\} + W_t \tag{B.8}$$

where $W_t = B^m_{t-1}(1 + \rho P^m_t)$ is nominal outstanding government debt at beginning of t. Combining with the government flow constraint,

$$B^m_{t-1}(1 + \rho P^m_t) = P_t(\tau_t - z_t) + P^m_t B^m_t$$

yields the transversality condition:

$$\lim_{t\to\infty} E_t[R_{t,T} W_T] = 0$$

We now turn to the pricing-production decision of the household-firm. Because price determines quantity through the demand schedule, we assume that the household-firm chooses price when solving for the optimal production schedule. The firm is constrained by a price friction of the form developed in Calvo (1983). Each period, $1 - \theta$ fraction of firms are randomly allowed to reset prices, while the remaining θ fraction continue to charge last period's price. This means that a firm expects its price to persist for $1/(1-\theta)$ periods into the future each time it resets prices. As a result, it is natural for the household-firm to treat θ as a discount factor and choose a single price that maximizes the discounted sum of future profits:

$$\sum_{k=0}^{\infty} \theta^k \big\{\Lambda_t E_t[R_{t,t+k}\mathcal{P} y_{t+k}(\mathcal{P})] - \beta^k E_t[\omega(y_{t+k}((P)))]\big\}$$

[1] Under full insurance, identical households with identical initial wealth levels choose identical optimal consumption paths. Moreover, $\int_0^1 B^j(t)dj = B_t$ follows from the assumption that net demand for assets by private households equals supply of government debt

where Λ_t is the marginal utility of household income at t. As in Woodford (1998a), we treat Λ_t as a constant, and proceed to the first-order condition:

$$\sum_{k=0}^{\infty} \theta^k E_t \{ \Lambda [R_{t,t+k} (\frac{\mathcal{P}}{P_{t+k}})^{-\epsilon} Y_{t+k} (1-\epsilon) - \beta^k \omega'(y_{t+k}((P)))(-\epsilon) \frac{\mathcal{P}^{-\epsilon-1}}{P_{t+k}^{-\epsilon}} Y_{t+k} \} = 0$$

Multiply both sides of the first-order condition by $\frac{\mathcal{P}}{\Lambda(1-\epsilon)}$ to obtain

$$\begin{aligned} \sum_{k=0}^{\infty} \theta^k E_t \{ R_{t,t+k} (\frac{\mathcal{P}}{P_{t+k}})^{-\epsilon} \mathcal{P} Y_{t+k} - \beta^k \Lambda^{-1} \omega'(y_{t+k}((P)))(\frac{\epsilon}{\epsilon - 1}) \frac{\mathcal{P}^{-\epsilon}}{P_{t+k}^{-\epsilon}} Y_{t+k} \} &= 0 \\ \sum_{k=0}^{\infty} \theta^k E_t \{ [R_{t,t+k} (\frac{\mathcal{P}}{P_{t+k}})^{-\epsilon} Y_{t+k} \big(\mathcal{P} - \frac{\beta^k}{R_{t,t+k}\Lambda} \omega'(y_{t+k}(\mathcal{P}))(\frac{\epsilon}{\epsilon-1}) \big) \} &= 0 \end{aligned}$$

To further simplify the first-order condition, consider the following two equations:

$$\begin{aligned} \beta^k \frac{u'(Y_{t+k})}{y'(Y_t)} \frac{P_t}{P_{t+k}} &= R_{t,t+k} \\ \Lambda_t &= u'(Y_t)/P_t \end{aligned}$$

The first equation is a necessary and sufficient condition for household optimization, while the second equation is an expression for the marginal utility of income. We substitute these equations into the first-order condition to yield:

$$\sum_{k=0}^{\infty} \theta^k E_t \{ [R_{t,t+k} (\frac{\mathcal{P}}{P_{t+k}})^{-\epsilon} Y_{t+k} \big(\mathcal{P} - S_{t+k,t} (\frac{\epsilon}{\epsilon - 1}) \big) \} = 0 \tag{B.9}$$

$$\frac{\omega'(y_{t+k}(\mathcal{P}))}{u'(Y_{t+k})} P_{t+k} = S_{t+k,t} \tag{B.10}$$

$S_{t+k,t}$ captures the household's expected marginal costs at time $t+k$. A sufficient but not necessary condition for optimality is that $\mathcal{P} = \frac{\epsilon}{\epsilon-1} S_{t+k,t}$ for all $t+k$. In this case, the optimal price is always a mark-up of $\frac{\epsilon}{\epsilon-1}$ over marginal costs. Since $\mathcal{P}$ is the same for all firms who change price at t it is straightforward to show that

$$P_t = [\theta P_{t-1}^{1-\epsilon} + (1-\theta)\mathcal{P}]^{\frac{1}{1-\epsilon}} \tag{B.11}$$

We are now in a position to characterize the non-policy aggregate demand (AD) and aggregate supply (AS) blocks of the model. The non-policy AD block is given by equations (B.5)-(B.8), and the AS block is given by equations (B.9)-(B.11). The AD equations give us consumption demand, bond holdings and rates of return subject to monetary and fiscal policy and a path for prices. The AS equation gives us a path for the price index and optimal prices subject to AD. To complete the model, we discuss simple fiscal and monetary policy arrangements. First, the monetary authority uses an interest rate rule of the form:

$$i_t \quad = \quad \Phi(\pi_t, s_t, \epsilon_t^{MP}, V_{1,t-1})$$

where π_t is inflation, ϵ_m is an exogenous mean-one i.i.d. shock, $v_{1,t-1}$ is the log of ${V_{1,t-1}}^2$, and s_t follows the 2-state Markov process described in Section 5. The fiscal authority only issues a bond portfolio, B_t^m, with a maturity that declines at a rate $\rho \in [0,1]$. Under this maturity structure, the quantity of government debt issued at $t-1$ that matures at $t+j$ is:

$$B_{t-1}(t+j) = B_{t-1}^m \rho^j$$

[2]Outside of section 3, we set $V_{1,t-1}$ $\forall t$. The addition of $V_{1,t-1}$ allows us to model forward guidance

The evolution of the government's bond portfolio satisfies that following budget constraint:

$$B^m_{t-1}(1 - \sum_{j \geq 0} Q_t(t+j)\rho^{j-1}) = P_t(\tau_t - z_t) + B^m_t \sum_{j \geq 0} Q_t(t+j)\rho^j$$

where $Q_t(t+j)$ is the price of debt that matures at time $t+j$ and is sold at t.. To simplify the government budget constraint, we define the price of the bond portfolio, P^m_t, as:

$$P^m_t = E_t \sum_{j \geq 0} Q_t(t+j)\rho^j$$

Furthermore, we can show that bond prices follow a recursive formulation:

$$P^m_t = Q_t(t+1)(1 + \rho E_t P^m_{t+1}) \tag{B.12}$$

which allows us to rewrite the government budget constraint as

$$B^m_{t-1}(1 + \rho P^m_t) = P_t(\tau_t - z_t) + P^m_t B^m_t$$

given B^m_{-1}. The government also implements a rule that adjusts real primary surpluses in response to the market value of real debt. If we let $S_t = \tau_t - z_t$ denote the real primary surplus, then we may characterize this rule as:

$$\begin{aligned} S_t &= S^* + \Gamma(s_t)(\frac{B^m_{t-1}}{P_{t-1}} - b^{m*}) + Z_{ft} \\ Z_{ft} &= \left(Z^{\rho_F}_{f,t-1}\right)\epsilon_f(t) \end{aligned}$$

where Z_{ft} is an exogenous fiscal shock process and ϵ_F is an exogenous mean-one i.i.d fiscal policy shock.

We add these policy equations to the non-policy AD block to completely characterize aggregate demand. To better analyze the equilibrium dynamics of the model, we linearize the equations of the AD and AS blocks. The linearized AD equations appear in equations (3.3), (3.5), (3.7)-(3.14) in Chapter 3. To arrive at equation (3.4) in section 3 which is the linearized AS curve, we linearize equations (B.9)-(B.11):

$$\hat{\mathcal{P}}_t = (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k E_t\{\hat{s}_{t+k,t} + \sum_{s=t+1}^{t+k}\hat{\pi}_s\} \tag{B.13}$$

$$\hat{s}_{t+k,t} = (\omega^{-1} + \sigma^{-1}\hat{y}_t - \theta\omega^{-1}[\hat{\mathcal{P}}_t - \sum_{s=t+1}^{t+k}\pi_s]) \tag{B.14}$$

$$\hat{\pi}_t = \frac{1-\theta}{\theta}\hat{\mathcal{P}}_t \tag{B.15}$$

where $\hat{\mathcal{P}}_t$ is the percentage deviation of optimal price over the price index from its steady state value of 1, $\hat{S}_{t+k,t}$ is the percentage deviation of marginal costs over the price index from its steady state value of 1 over the markup and $\omega = \frac{\omega'(Y^*)}{\omega''(Y^*)Y^*}$. To arrive at the linearized AS curve, we substitute equation (B.14) into equation (B.13) and quasi-difference to obtain:

$$\hat{\mathcal{P}}_t = \frac{\kappa\theta}{1-\theta}\sum_{k=0}^{\infty}(\theta\beta)^k E_t\hat{y}_{t+k} + \sum_{k=1}^{\infty}E_t\hat{\pi}_t \tag{B.16}$$

where

$$\kappa \equiv \frac{(1-\theta)(1-\theta\beta)}{\theta}\frac{\omega+\sigma}{\sigma(\omega+\theta)}$$

Substituting equation (B.16) into equation (B.15) yields the linearized AS equation in (3.4).

APPENDIX C

FIXED REGIME FORWARD GUIDANCE EXPERIMENT

This appendix shows how to implement an anticipated interest rate peg at time T (i.e. $i_T = E_T i_{T+1} = ... = E_T i_{T+L} = \bar{i}$) using the forward guidance shocks introduced in section 2. First, it helps to rewrite the equilibrium relationships as

$$Y_t = GY_{t-1} + \bar{\Psi}\bar{\epsilon}_t + \tilde{\Psi}\tilde{\epsilon}_t$$

where $Y_t = (\hat{y}_t, \hat{\pi}_t, \hat{i}_t, r_t^n, \mu_t, v_{1,t}, v_{2,t}, \ldots, v_{L,t}, \hat{\tau}_t, \hat{b}_t^m, \hat{P}_t^m)'$, $\bar{\epsilon}_t = (\epsilon_t^n, \epsilon_t^\mu, \epsilon_t^f)'$, and $\tilde{\epsilon}_t = (\epsilon_t^{MP}, \epsilon_{1,t}^R, \ldots, \epsilon_{L,t}^R)'$. It follows that the equilibrium process for $\hat{i}$ is given by an equation of the form:

$$\hat{i}_t = G_i Y_{t-1} + \bar{\Psi}_i\bar{\epsilon}_t + \tilde{\Psi}_i\tilde{\epsilon}_t$$

Assume $\bar{\epsilon}_T = 0$, for simplicity (we can relax this). Then:

$$\begin{aligned}
i_T &= G_i Y_{T-1} + \tilde{\Psi}_i\tilde{\epsilon}_T \\
E_T i_{T+1} &= G_i^2 Y_{T-1} + (G\tilde{\Psi})_i\tilde{\epsilon}_T \\
&\vdots \\
E_T i_{T+L} &= G_i^{L+1} Y_{T-1} + (G^L\tilde{\Psi})_i\tilde{\epsilon}_T
\end{aligned}$$

where G_i^k and $(G^k\tilde{\Psi})_i$ denote the rows of G^k and $(G^k\tilde{\Psi})$ that correspond to the nominal interest rate for $k = 1, ..., L+1$. If we set $i_s = \bar{i}$ for all $s \in \{T, T+1, \ldots, T+L\}$, then we have a system of $L+1$ equations in $L+1$ unknowns which are the elements of

$\tilde{\epsilon}_T$. The solution of this system is given by:

$$\begin{pmatrix} \tilde{\Psi}_i \\ (G\tilde{\Psi})_i \\ \vdots \\ (G^L\tilde{\Psi})_i \end{pmatrix}^{-1} \left(\bar{i}1_{L+1\times 1} - \begin{pmatrix} G_i \\ G_i^2 \\ \vdots \\ G_i^{L+1} \end{pmatrix} Y_{T-1} \right) = \tilde{\epsilon_T}$$

where $1_{L+1\times 1}$ is a $L+1\times 1$ vector of ones. We implement the interest rate peg by announcing $\tilde{\epsilon}_T$ at T. If we also suppose that $\bar{\epsilon}_s = 0$ for all $s \in \{T, T+1, \ldots, T+L\}$ and $\tilde{\epsilon}_s = 0$ for all $s \in \{T+1, \ldots, T+L\}$ then $\tilde{\epsilon}_T$ will also successfully implement the interest rate *ex post* (i.e. $i_T = i_{T+1} = ... = i_{T+L} = \bar{i}$). If we relax this last assumption (e.g. if $\bar{\epsilon}_s \neq 0$ for some some $s \in \{T+1, ..., T+L\}$), then the central bank will have to announce shocks after T to defend the peg. Regardless of whether shocks are present, the central bank can always use these shocks to defend an L-horizon peg.

APPENDIX D

REGIME SWITCHING FORWARD GUIDANCE EXPERIMENT

First, it helps to rewrite the equilibrium relationships as

$$Y_t = G(s_t)Y_{t-1} + \bar{\Psi}(s_t)\bar{\epsilon}_t + \tilde{\Psi}(s_t)\tilde{\epsilon}_t$$

where $Y_t = (\hat{y}_t, \hat{\pi}_t, \hat{i}_t, r_t^n, \mu_t, v_{1,t}, v_{2,t}, \ldots, v_{L,t}, \tau_t, \hat{b}_t^m, \hat{P}_t^m)'$, $\bar{\epsilon}_t = (\epsilon_t^n, \epsilon_t^\mu, \epsilon_t^f)'$, and $\tilde{\epsilon}_t = (\epsilon_t^{MP}, \epsilon_{1,t}^R, \ldots, \epsilon_{3,t}^R)'$. It follows that the equilibrium process for $\hat{i}$ is given by an equation of the form:

$$\hat{i}_t = G(s_t)_i Y_{t-1} + \bar{\Psi}(s_t)_i\bar{\epsilon}_t + \tilde{\Psi}(s_t)_i\tilde{\epsilon}_t$$

We now suppose that $\bar{\epsilon}_s = 0$ for all $s \in \{T, T+1, \ldots, T+L\}$ and $\tilde{\epsilon}_s = 0$ for all $s \in \{T+1, \ldots, T+L\}$. As in Appendix A.2. we can relax this assumption and allow the central bank to defend the peg using shocks after T. Next, we define the following matrices:

$$\begin{aligned}
K_{s_t}^1 &= (p_{s_t1}G(1)_i + p_{s_t2}G(2)_i) \\
K_{s_t}^2 &= (p_{s_t1}p_{11}G(1)_iG(1) + p_{s_t1}p_{12}G(2)_iG(1) \\
&+ p_{s_t2}p_{21}G(1)_iG(2) + p_{s_t2}p_{22}G(2)_iG(2)) \\
K_{s_t}^3 &= (p_{s_t1}p_{11}^2G(1)_i(G(1))^2 + p_{s_t1}p_{11}p_{12}G(2)_i(G(1))^2 + p_{s_t1}p_{12}p_{21}G(1)_iG(2)G(1) \\
&+ p_{s_t1}p_{12}p_{22}G(2)_iG(2)G(1) + p_{s_t2}p_{21}p_{11}G(1)_iG(1)G(2) \\
&+ p_{s_t2}p_{21}p_{12}G(2)_iG(1)G(2) + p_{s_t2}p_{22}p_{21}G(1)_i(G(2))^2 + p_{s_t2}p_{22}^2G(2)_i(G(2))^2)
\end{aligned}$$

where $G(s_t)_i$ is the row of $G(s_t)$ corresponding to the nominal interest rate, i, for $s_t \in \{1,2\}$ and $p_{s_t k}$ is the probability of transition from state s_t to state k for $k \in \{1,2\}$. Under these assumptions:

$$\begin{aligned}
i_T &= G(s_T)_i Y_{T-1} + \tilde{\Psi}(s_T)_i \tilde{\epsilon}_T \\
E_T i_{T+1} &= K^1_{s_T}(G(s_T)Y_{T-1} + \tilde{\Psi}(s_T)\tilde{\epsilon}_T) \\
E_T i_{T+2} &= K^2_{s_T}(G(s_T)Y_{T-1} + \tilde{\Psi}(s_T)\tilde{\epsilon}_T) \\
E_T i_{T+3} &= K^3_{s_T}(G(s_T)Y_{T-1} + \tilde{\Psi}(s_T)\tilde{\epsilon}_T)
\end{aligned}$$

If we set $i_s = \bar{i}$ for all $s \in \{T, T+1, T+3\}$, then we have a system of 4 equations in 4 unknowns which are the elements of $\tilde{\epsilon}_T$. The solution to this system is the set of shocks that sets the interest rate peg. The solution is given by:

$$\begin{pmatrix} \tilde{\Psi}(s_T)_i \\ K^1_{s_T}\tilde{\Psi}(s_T) \\ K^2_{s_T}\tilde{\Psi}(s_T) \\ K^3_{s_T}\tilde{\Psi}(s_T) \end{pmatrix}^{-1} \left(\bar{i} 1_{4\times 1} - \begin{pmatrix} G(s_T)_i \\ K^1_{s_T} G(s_T) \\ K^2_{s_T} G(s_T) \\ K^3_{s_T} G(s_T) \end{pmatrix} Y_{T-1} \right) = \tilde{\epsilon_T}$$

where $1_{L+1\times 1}$ is a 4×1 vector of ones.

Since $E_T i_{T+1} \neq i_{T+1}$ is typically true in the presence of switching coefficients and non-absorbing states, the central bank will typically have to announce a new sequence of shocks at $T+1$ to defend the interest rate peg (i.e. the central bank issues new monetary shocks to ensure that $i_{T+1} = E_{T+1} i_{T+2} = E_{T+1} i_{T+3} = \bar{i}$). Define $\tilde{\epsilon_{T+1}} = (\epsilon^{MP}_{T+1}, \epsilon^R_{1,T+1}, \epsilon^R_{2,T+1}, 0)'$. Then at time $T+1$:

$$
\begin{aligned}
i_{T+1} &= G(s_{T+1})_i Y_T + \tilde{\Psi}(s_{T+1})_i \tilde{\epsilon}_{T+1} \\
E_{T+1} i_{T+2} &= K^1_{s_{T+1}}(G(s_{T+1})Y_T + \tilde{\Psi}(s_{T+1})\tilde{\epsilon}_{T+1}) \\
E_{T+1} i_{T+3} &= K^2_{s_{T+1}}(G(s_{T+1})Y_T + \tilde{\Psi}(s_{T+1})\tilde{\epsilon}_{T+1})
\end{aligned}
$$

If we set $i_s = \bar{i}$ for all $s \in \{T+1, T+3\}$, then we have a system of 3 equations in 3 unknowns, which we solve for $\epsilon^{MP}_{T+1}, \epsilon^{R}_{1,T+1}, \epsilon^{R}_{2,T+1}$ as before. This process repeats itself in $T+2$ where we use equations for i_{T+2} and i_{T+3} to solve for the pair $(\epsilon^{MP}_{T+2}, \epsilon^{R}_{1,T+2})$ that sets $i_{T+2} = E_{T+2} i_{T+3} = \bar{i}$. Then again at $T+3$ we use the equilibrium equation for i_{T+3} to solve for the ϵ^{MP}_{T+3} that sets $i_{T+3} = \bar{i}$

APPENDIX E

TABLES

TABLE 6. Fixed Coefficient Model Parameterization

	Description	**Regime M**	**Regime F (short-term)**	**Regime F (long-term)**
σ	CRRA parameter	1	1	1
β	Discount Factor	.99	.99	.99
κ	Phillips Curve Slope	.1	.1	.1
ϕ_π	Feedback Inflation	1.5	0	0
ϕ_y	Feedback Output	0	0	0
ρ_n	AR(1) natural rate	.5	.5	.5
ρ_μ	AR(1) cost-push	.5	.5	.5
ρ	Average Debt Maturity	0	0	.93
γ	Feedback Debt	2	.1	.1

TABLE 7. Regime-Switching Model Parameterizations

	$\gamma(M)$	$\gamma(F)$	$\phi_\pi(M)$	$\phi_\pi(F)$	p_{MM}	p_{FF}
Figure 4	20	-5	1.5	0	.95	.95
Figure 5	20	-5	1.5	.8	.95	.95
Figure 6	5	-5	1.5	.8	.95	.95

Section 4 parameterizations same as Section 3 parameterizations except for the above values.

APPENDIX F

LAGGED INFORMATION ONE-STEP-AHEAD E-STABILITY

We consider the class of models developed in section 2.3.1. Suppose agents observe the current state, s_t, and know the elements of P, but do not know X_t. Agents have the following perceived law of motion:

$$X_t = A(s_t) + B(s_t)X_{t-1} + C(s_t)U_t$$

where $A(i)$ is $n \times 1$, $B(i)$ is $n \times n$ and $C(i)$ is $n \times m$. In this section, we solve for agents' state-contingent expectations and derive the state-contingent T-map. For now, we assume $S = 2$, but this proof can be extended to any finite Markov-Chain. If $s_t = i$ then:

$$\begin{aligned}
E_t(X_{t+1}) &= E(X_{t+1}|s_t = i) \\
&= p_{i1}A(1) + p_{i2}A(2) + (p_{i1}B(1) + p_{i2}B(2))X_t + (p_{i1}C(1) + p_{i2}C(2))\rho U_t \\
&= p_{i1}A(1) + p_{i2}A(2) + (p_{i1}B(1) + p_{i2}B(2))(A(i) + B(i)X_{t-1} \\
&\quad +C(i)U_t) + (p_{i1}C(1) + p_{i2}C(2))\rho U_t \\
&= (p_{i1}A(1) + p_{i2}A(2) + (p_{i1}B(1) + p_{i2}B(2))(A(i))) \\
&\quad +((p_{i1}B(1) + p_{i2}B(2))B(i))X_{t-1} + \\
&\quad ((p_{i1}C(1) + p_{i2}C(2))\rho + (p_{i1}B(1) + p_{i2}B(2))C(i))U_t
\end{aligned}$$

Substituting $E_t(X_{t+1})$ into (2.1) in Chapter 2 yields the actual data generating process:

$$
\begin{aligned}
X_t &= M(i)(p_{i1}A(1)+p_{i2}A(2)+(p_{i1}B(1)+p_{i2}B(2))(A(i))) \\
&+ (M(i)((p_{i1}B(1)+p_{i2}B(2))B(i))+N(i))X_{t-1} \\
&+ (M(i)((p_{i1}C(1)+p_{i2}C(2))\rho+(p_{i1}B(1)+p_{i2}B(2))C(i))+Q(i))U_t
\end{aligned}
$$

This delivers the state-contingent T-map:

$$
T\begin{pmatrix} A(1) \\ A(2) \\ B(1) \\ B(2) \\ C(1) \\ C(2) \end{pmatrix} = \begin{pmatrix} M(1)((p_{11}+p_{11}B(1)+p_{12}B(2))A(1)+p_{12}A(2)) \\ M(2)((p_{22}+p_{21}B(1)+p_{22}B(2))A(2)+p_{21}A(1)) \\ M(1)(p_{11}B(1)^2+p_{12}B(2)B(1))+N(1) \\ M(2)(p_{22}B(2)^2+p_{21}B(1)B(2))+N(2) \\ M(1)((p_{11}C(1)\rho+(p_{11}B(1)+p_{12}B(2))C(1))+p_{12}C(2))\rho+Q(1) \\ M(2)((p_{22}C(2)\rho+(p_{21}B(1)+p_{22}B(2))C(2))+p_{21}C(1))\rho+Q(2) \end{pmatrix}
$$

The block of the T-map associated to $B = (B(1)' \; B(2)')'$ decouples from the other blocks. This block is given by:

$$
T_B(B) = \begin{pmatrix} M(1)(p_{11}B(1)^2+p_{12}B(2)B(1))+N(1) \\ M(2)(p_{22}B(2)^2+p_{21}B(1)B(2))+N(2) \end{pmatrix} = \begin{pmatrix} T_B^1(B) \\ T_B^2(B) \end{pmatrix}
$$

To assess E-stability, we begin by stabilizing the following differential equation:

$$
\frac{dB}{dt} = T_B(B) - B
$$

Let $DT_B(\bar{B})$ denote the Jacobian of T_b evaluated at the MSV solution, $\bar{B} = (\Omega^*(1)'$ $\Omega^*(2)')'$. Since T_B is continuously differentiable, Proposition 5.6 in Evans and Honkapohja (2001) tells us that $\bar{B}$ is asymptotically stable if the eigenvalues of $DT_B(\bar{B})$ have real parts less than one. Alternatively, we can analyze the stability of (3) by analyzing the stability of the following differential equation:

$$\frac{d\tilde{B}}{dt} = T_{\tilde{B}}(\tilde{B}) - \tilde{B}$$

where $\tilde{B} = (B(1)\ B(2))$ and

$$T_{\tilde{B}}(\tilde{B}) = \begin{pmatrix} T_B^1(B) & T_B^2(B) \end{pmatrix}$$

Now let $DT_{\tilde{B}}(\bar{\Omega}^*)$ denote the Jacobian of T_b evaluated at the MSV solution, $\bar{\Omega}^* = (\Omega^*(1)\ \Omega^*(2))$. Since $T_{\tilde{B}}$ is continuously differentiable, $\bar{\Omega}^*$ is asymptotically stable if the eigenvalues of $DT_{\tilde{B}}(\bar{\Omega}^*)$ have real parts less than one. Finally, since $DT_B(\bar{B})$ is similar to $DT_{\tilde{B}}(\bar{\Omega}^*)$, the asymptotic stability of the $\bar{\tilde{B}}$ in (4) implies the asymptotic stability of $\bar{B}$ in (3). To solve for $DT_{\tilde{B}}(\bar{\Omega}^*)$, we linearize $T_{\tilde{B}}(\tilde{B})$ at the REE and vectorize the resulting equation. We then use the following identification rule: if $vec(dT_{\tilde{B}}) = Avec(d\tilde{B})$ then $A = DT_{\tilde{B}}(\bar{\Omega}^*)$, where $d\tilde{B} = (dB(1)\ dB(2))$ and $dT_{\tilde{B}}$ is the linearized system of equations.

$$
\begin{aligned}
dT_{\tilde{B}} &= \begin{pmatrix} (p_{11}M(1)((dB(1))B(1) + B(1)dB(1)))' \\ (p_{22}M(2)((dB(2))B(2) + B(2)dB(2)))' \end{pmatrix} \\
&+ \begin{pmatrix} (p_{12}M(1)(B(2)(dB(1)) + (dB(2))B(1)))' \\ (p_{21}M(2)(B(1)(dB(2)) + (dB(1))B(2)))' \end{pmatrix}' \\
&= (p_{11}M(1)B(1) + p_{12}M(1)B(2))d\tilde{B} \begin{pmatrix} I_n & 0_n \\ 0_n & 0_n \end{pmatrix} + p_{11}M(1)d\tilde{B} \begin{pmatrix} B(1) & 0_n \\ 0_n & 0_n \end{pmatrix} \\
&+ p_{12}M(1)d\tilde{B} \begin{pmatrix} 0_n & 0_n \\ B(1) & 0_n \end{pmatrix} + (p_{22}M(2)B(2) + p_{21}M(2)B(1))d\tilde{B} \begin{pmatrix} 0_n & 0_n \\ 0_n & I_n \end{pmatrix} \\
&+ p_{22}M(2)d\tilde{B} \begin{pmatrix} 0_n & 0_n \\ 0_n & B(2) \end{pmatrix} + p_{21}M(2)d\tilde{B} \begin{pmatrix} 0_n & B(2) \\ 0_n & 0_n \end{pmatrix}
\end{aligned}
$$

Using the rule $vec(ABC) = C' \otimes Avec(B)$, and the identification rule, we obtain:

$$
\begin{aligned}
DT_{\tilde{B}}(\tilde{B}) &= \begin{pmatrix} I_n & 0_n \\ 0_n & 0_n \end{pmatrix} \otimes (p_{11}M(1)B(1) + p_{12}M(1)B(2)) \\
&+ \begin{pmatrix} B(1)' & 0_n \\ 0_n & 0_n \end{pmatrix} \otimes (p_{11}M(1)) + \begin{pmatrix} 0_n & B(1)' \\ 0_n & 0_n \end{pmatrix} \otimes (p_{12}M(1)) \\
&+ \begin{pmatrix} 0_n & 0_n \\ 0_n & I_n \end{pmatrix} \otimes (p_{22}M(2)B(2) + p_{21}M(2)B(1)) \\
&+ \begin{pmatrix} 0_n & 0_n \\ 0_n & B(2)' \end{pmatrix} \otimes (p_{22}M(2)) + \begin{pmatrix} 0_n & 0_n \\ B(2)' & 0_n \end{pmatrix} \otimes (p_{21}M(2))
\end{aligned}
$$

At the REE, $(B(1)\ B(2)) = (\Omega^*(1)\ \Omega^*(2)) = \bar{\Omega^*}$. E-stability requires the real parts of $DT_{\tilde{B}}(\bar{\Omega^*})$ to be less than one. We now turn to the equation for $A = (A(1)'\ A(2)')'$:

$$T_A(A) = \begin{pmatrix} M(1)(p_{11} + p_{11}B(1) + p_{12}B(2)) & p_{12}M(1) \\ p_{21}M(1) & M(2)(p_{22} + p_{21}B(1) + p_{22}B(2)) \end{pmatrix} A$$

Using the same methods as before, we obtain the following Jacobian evaluated at the REE where $\bar{A} = (0'_{n\times 1}\ 0'_{n\times 1})'$:

$$\begin{aligned} DT_A(\bar{A}, \bar{B}) &= \begin{pmatrix} M(1)(p_{11}\Omega^*(1) + p_{12}\Omega^*(2)) & 0_n \\ 0_n & M(2)(p_{21}\Omega^*(1) + p_{22}\Omega^*(2)) \end{pmatrix} \\ &+ \begin{pmatrix} p_{11}M(1) & p_{12}M(1) \\ p_{21}M(2) & p_{22}M(2) \end{pmatrix} \end{aligned}$$

E-stability requires the real parts of Ψ_M to be less than one. Finally, we consider the equation for $C = (C(1)'\ C(2)')'$:

$$\begin{aligned} T_C(C) &= \begin{pmatrix} p_{11}M(1)B(1) + p_{12}M(1)B(2) & 0_n \\ 0_n & p_{21}M(2)B(1) + p_{22}M(2)B(2) \end{pmatrix} C \\ &+ \begin{pmatrix} p_{11}M(1) & p_{12}M(1) \\ p_{21}M(2) & p_{22}M(2) \end{pmatrix} C\rho \end{aligned}$$

Using the same methods as before, we obtain the following Jacobian evaluated at the REE where $\bar{C} = (\Gamma(1)', \Gamma(2)')'$:

$$
\begin{aligned}
DT_C(\bar{B}, \bar{C}) &= \rho \otimes \begin{pmatrix} p_{11}M(1) & p_{12}M(1) \\ p_{21}M(1) & p_{22}M(2) \end{pmatrix} \\
&+ I_m \otimes \begin{pmatrix} M(1)(p_{11}\Omega^*(1) + p_{12}\Omega^*(2)) & 0_n \\ 0_n & M(2)(p_{21}\Omega^*(1) + p_{22}\Omega^*(2)) \end{pmatrix}
\end{aligned}
$$

The REE solution $\bar{A}, \bar{B}, \bar{C}$ is E-stable if:

i. all the eigenvalues of $DT_A(\bar{A}, \bar{B})$ have real parts less than 1,

ii. all the eigenvalues of $DT_{\tilde{B}}(\bar{\Omega}^*)$ have real parts less than 1, and,

iii. all the eigenvalues of $DT_C(\bar{B}, \bar{C})$ have real parts less than 1

The solution is not E-stable if any of these three conditions fail with eigenvalues strictly greater than one.

APPENDIX G

INFINITE HORIZON STABILITY CONDITIONS DERIVATION

We derive the infinite horizon stability conditions in section (4). As mentioned in a footnote in that section, we assume that all exogenous driving processes are mean-zero i.i.d and that $\rho = 0$ in our derivation. We do this for expositional purposes; we can relax these assumptions and obtain stability conditions in the more general model. It should also be noted that all matrix infinite series of the form $\sum_{t\geq 0} A^t$ converge to $(I - A)^{-1}$ if and only if the spectral radius of A is less than one. In the model explored in section 4, this condition is met for nearly all of the parameterizations we consider. When shocks are i.i.d agents employ the following perceived laws of motion for $x \in \{y, \pi, i\}$ and b:

$$
\begin{aligned}
b_t &= a(s_t) + b(s_t)b_{t-1} \\
x_t &= c_x(s_t) + d_x(s_t)b_{t-1}
\end{aligned}
$$

We also define the following matrix for each $z \in \{a, b, c_x, d_x\}$:

$$\hat{z} = diag(z(1), z(2), ..., z(S))$$

where S is the number of Markov states. In much of what follows, we let F_i denote the ith row of the matrix F. Moreover, P is the transition probability matrix and 1_s denotes an S by 1 vector of ones. Let E_t denote expectations formed using agents' subjective beliefs. For $T \geq t$, the following is true:

$$\begin{aligned}
E_t b_{t+1} &= (\sum_{k=0}^{T-t} P^{T-t+1-k} \hat{a} (P\hat{b})^k)_i 1_s + ((P\hat{b})^{T-t+1})_i 1_s E_t b_t \\
E_t x_{t+1} &= P^{T-t+1} \hat{c_x} \\
&+ (\sum_{k=0}^{T-t} P^{T-t+1-k} \hat{a} (P\hat{b})^{k-1} P\hat{d_x})_i 1_s + ((P\hat{b})^{T-t}) P\hat{d_x})_i 1_s E_t b_t
\end{aligned}$$

To proceed, we need to find an expression for $A_T = \sum_{k=0}^{T-t} P^{-k} \hat{a} (P\hat{b})^k$. We accomplish this by first vectorizing A_T:

$$\begin{aligned}
vec(\sum_{k=0}^{T-t} P^{-k} \hat{a} (P\hat{b})^k) &= \sum_{k=0}^{T-t} vec(P^{-k} \hat{a} (P\hat{b})^k) \\
&= \sum_{k=0}^{T-t} (((P\hat{b})^k)' \otimes P^{-k}) vec(\hat{a}) \\
&= \sum_{k=0}^{T-t} ((P\hat{b})' \otimes P^{-1})^k vec(\hat{a}) \\
&= (I - ((P\hat{b})' \otimes P^{-1})^{T-t+1})(I - (P\hat{b})' \otimes P^{-1})^{-1} vec(\hat{a}) \\
&= vec(A_T)
\end{aligned}$$

Returning to the original equation:

$$E_t b_{t+1} = (P^{T-t+1} A_T)_i 1_s + ((P\hat{b})^{T-t+1})_i 1_s E_t b_t$$

This implies:

$$\begin{aligned}
\sum_{T \geq t} \gamma^{T-t} E_t b_{t+1} &= \gamma^{-1} (\sum_{T \geq t} \gamma^{T-t+1} E_t b_{T+1}) \\
&= \gamma^{-1} (\sum_{T \geq t} ((\gamma P)^{T-t+1} A_T)_i 1_S + ((\gamma P\hat{b})^{T-t+1})_i 1_S) E_t b_t
\end{aligned}$$

We now need to calculate:

$$\begin{aligned}
\bar{A} &= \sum_{T\geq t}(\gamma P)^{T-t+1}A_T) \\
\implies vec\bar{A} &= \sum_{T\geq t} I\otimes(\gamma P)^{T-t+1}vec(A_T) \\
vec(\bar{A}) &= ((I\otimes\gamma P)(I-I\otimes\gamma P)^{-1} \\
&- ((P\hat{b})'\otimes\gamma I)(I-(P\hat{b})'\otimes\gamma I)^{-1})(I-(P\hat{b})'\otimes P^{-1})^{-1}vec\hat{a}
\end{aligned}$$

Therefore:

$$\sum_{T\geq t}\gamma^{T-t}E_t b_{T+1} = (\gamma^{-1}\bar{A})_i 1_S + \gamma^{-1}((I-\gamma P\hat{b})^{-1}P\hat{b})_i 1_S E_t b_t$$

for all reduced form discount factors in the infinite horizon model. Similarly, we can show:

$$\sum_{T\geq t}\gamma^{T-t}E_t x_{T+1} = ((I-P\gamma)^{-1}P\hat{c_x})_i 1_s + ((\gamma P)^{-1}\tilde{A}P\hat{d_x})_i 1_S + ((I-\gamma P\hat{b})^{-1}P\hat{d_x})_i 1_S E_t b_t$$

where

$$\begin{aligned}
\tilde{A} &= \sum_{T\geq t}(\gamma P)^{T-t+1}\tilde{A}_T) \\
\tilde{A}_T &= \sum_{k=0}^{T-t-1} P^{-k}\hat{a}(P\hat{b})^k
\end{aligned}$$

In our model, $\gamma \in \{\beta, \alpha\beta\}$. We can write the model as:

$$Y_t = \sum_{\gamma} M_\gamma(s_t)(\sum_{T\geq t}\gamma^{T-t}E_t Y_{T+1}) + \hat{N}(s_t)Y_{t-1} + \epsilon_t$$

where $Y = (y, \pi, i, b)'$ and ϵ is an m by 1 vector of i.i.d shocks. Next, we define $A = (c^y(1)\ c^\pi(1)\ c^i(1)\ a(1)\ c^y(2)\ c^\pi(2)\ c^i(2)\ a(2))'$ and $B = (d^y(1)\ d^y(2)\ d^\pi(1)\ d^\pi(2)\ d^i(1)\ d^i(2)\ b(1)\ b(2))'$. If $E_t b_t = a(s_t) + b(s_t)b_{t-1}$, it follows that:

$$
\begin{aligned}
T(A) \;&=\; \sum_\gamma \begin{pmatrix} M_\gamma(1)\begin{pmatrix} ((I-P\gamma)^{-1}P\hat{c}_y)_1 1_s + ((\gamma P)^{-1}\tilde{A}P\hat{d}_y)_1 1_S \\ ((I-P\gamma)^{-1}P\hat{c}_\pi)_1 1_s + ((\gamma P)^{-1}\tilde{A}P\hat{d}_\pi)_1 1_S \\ ((I-P\gamma)^{-1}P\hat{c}_i)_1 1_s + ((\gamma P)^{-1}\tilde{A}P\hat{d}_i)_1 1_S \\ (\gamma^{-1}\bar{A})_1 1_S + ((I-\gamma P\hat{b})^{-1}P\hat{b})_1 1_S a(1) \end{pmatrix} \\ M_\gamma(2)\begin{pmatrix} ((I-P\gamma)^{-1}P\hat{c}_y)_2 1_s + ((\gamma P)^{-1}\tilde{A}P\hat{d}_y)_2 1_S \\ ((I-P\gamma)^{-1}P\hat{c}_\pi)_2 1_s + ((\gamma P)^{-1}\tilde{A}P\hat{d}_\pi)_2 1_S \\ ((I-P\gamma)^{-1}P\hat{c}_i)_2 1_s + ((\gamma P)^{-1}\tilde{A}P\hat{d}_i)_2 1_S \\ (\gamma^{-1}\bar{A})_2 1_S + ((I-\gamma P\hat{b})^{-1}P\hat{b})_2 1_S a(2) \end{pmatrix} \end{pmatrix} \\
&+\; \sum_\gamma \begin{pmatrix} M_\gamma(1)\begin{pmatrix} ((I-\gamma P\hat{b})^{-1}P\hat{d}_y)_1 1_S a(1) \\ ((I-\gamma P\hat{b})^{-1}P\hat{d}_\pi)_1 1_S a(1) \\ ((I-\gamma P\hat{b})^{-1}P\hat{d}_i)_1 1_S a(1) \\ 0 \end{pmatrix} \\ M_\gamma(2)\begin{pmatrix} ((I-\gamma P\hat{b})^{-1}P\hat{d}_y)_2 1_S a(2) \\ ((I-\gamma P\hat{b})^{-1}P\hat{d}_\pi)_2 1_S a(2) \\ ((I-\gamma P\hat{b})^{-1}P\hat{d}_i)_2 1_S a(2) \\ 0 \end{pmatrix} \end{pmatrix}
\end{aligned}
$$

To simplify this expression in order to obtain E-stability conditions, consider the following manipulation that works, for some correctly defined matrix, for any $\hat{z} \in$

$\{\hat{a}, \hat{b}, \hat{c_x}\}$ where $x \in \{y, \pi, i\}$:

$$
\begin{aligned}
((I - P\gamma)^{-1} P \hat{c_y})_i 1_s &= ((I - P\gamma)^{-1} P (c_y(1), c_y(2))')_i \\
&= ((I - P\gamma)^{-1} P \xi((1,1),(5,2)))_i A
\end{aligned}
$$

where $\xi((h,j),(l,m))$ is a 2 by 8 matrix with ones in the (h,j) and (l,m) entries and zeros elsewhere. Similarly, we define $\xi(h,j)$ as a 2 by 8 matrix with one in its $(4,1)$th entry and zeros elsewhere. Also note that $vec(\bar{A})$ assumes the following form:

$$
\begin{aligned}
vec(\bar{A}) &= Qvec(\hat{a}) \\
\implies \bar{A} &= \begin{pmatrix} Q_{11}a(1) + Q_{14}a(2) & Q_{31}a(1) + Q_{34}a(2) \\ Q_{21}a(1) + Q_{24}a(2) & Q_{41}a(1) + Q_{44}a(2) \end{pmatrix} \\
&= Q_1 \xi((4,1),(8,2)) A \begin{pmatrix} 1 & 0 \end{pmatrix}) + Q_2 \xi((4,1),(8,2)) A \begin{pmatrix} 0 & 1 \end{pmatrix})
\end{aligned}
$$

Similarly:

$$
\tilde{A} = \tilde{Q}_1 \xi((4,1),(8,2)) A \begin{pmatrix} 1 & 0 \end{pmatrix}) + \tilde{Q}_2 \xi((4,1),(8,2)) A \begin{pmatrix} 0 & 1 \end{pmatrix})
$$

For appropriately defined matrices Q_1 and Q_2, which can be recovered from our calculations above. We can also simplify $((\gamma P)^{-1}\bar{A}P\hat{d}_x)_i 1_S$ for $x \in \{y, i, b\}$ as follows:

$$\begin{aligned}
((\gamma P)^{-1}\bar{A}P\hat{d}_x)_i 1_S &= ((\gamma P)^{-1})_i \bar{A}P\hat{d}_x 1_S \\
&= ((\gamma P)^{-1})_i (Q_1 \xi((4,1),(8,2))A \begin{pmatrix} 1 & 0 \end{pmatrix} \\
&+ Q_2 \xi((4,1),(8,2))A \begin{pmatrix} 0 & 1 \end{pmatrix}) P\hat{d}_x 1_S \\
&= (\begin{pmatrix} 1 & 0 \end{pmatrix} P\hat{d}_x 1_S)((\gamma P)^{-1})_i Q_1 \xi((4,1),(8,2))A \\
&+ (\begin{pmatrix} 0 & 1 \end{pmatrix} P\hat{d}_x 1_S)((\gamma P)^{-1})_i Q_2 \xi((4,1),(8,2))A
\end{aligned}$$

Since $\begin{pmatrix} 1 & 0 \end{pmatrix} P\hat{d}_y 1_S$ and $\begin{pmatrix} 0 & 1 \end{pmatrix} P\hat{d}_y 1_S$ are scalars. These simplications allow us to express $T(A) = DT(A,B)A$. Evaluated at the rational expectations equilibrium, where $\xi_i(k)$ is a 1 by 8 matrix of zeros with 1 in the $(k,1)$ entry:

$$DT_A(0,\Omega^*) = \sum_\gamma \begin{pmatrix} M_\gamma(1) \begin{pmatrix} ((I - P\gamma)^{-1}P)_1 \xi((1,1),(5,2)) \\ ((I - P\gamma)^{-1}P)_1 \xi((2,1),(6,2)) \\ ((I - P\gamma)^{-1}P)_1 \xi((3,1),(7,2)) + \\ ((I - \gamma P\hat{b})^{-1}P\hat{b}1_s)_1 \xi_1(4) \end{pmatrix} \\ M_\gamma(2) \begin{pmatrix} ((I - P\gamma)^{-1}P)_2 \xi((1,1),(5,2)) \\ ((I - P\gamma)^{-1}P)_2 \xi((2,1),(6,2)) \\ ((I - P\gamma)^{-1}P)_2 \xi((3,1),(7,2)) \\ ((I - \gamma P\hat{b})^{-1}P\hat{b}1_s)_2 \xi_1(8) \end{pmatrix} \end{pmatrix}$$

$$
+ \sum_{\gamma} \begin{pmatrix} M_\gamma(1) \begin{pmatrix} ((I-\gamma P\hat{b})^{-1} P\hat{d}_y 1_s)_1 \xi_1(4) \\ ((I-\gamma P\hat{b})^{-1} P\hat{d}_\pi 1_s)_1 \xi_1(4) \\ ((I-\gamma P\hat{b})^{-1} P\hat{d}_i 1_s)_1 \xi_1(4) \\ 0 \end{pmatrix} \\ M_\gamma(2) \begin{pmatrix} ((I-\gamma P\hat{b})^{-1} P\hat{d}_y 1_s)_2 \xi_1(8) \\ ((I-\gamma P\hat{b})^{-1} P\hat{d}_\pi 1_s)_2 \xi_1(8) \\ ((I-\gamma P\hat{b})^{-1} P\hat{d}_i 1_s)_2 \xi_1(8) \\ 0 \end{pmatrix} \end{pmatrix}
$$

$$
+ \sum_{\gamma} \begin{pmatrix} M_\gamma(1) \begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} P\hat{d}_y 1_s((\gamma P)^{-1})_1 \tilde{Q}_1 \xi((4,1),(8,2)) \\ \begin{pmatrix} 1 & 0 \end{pmatrix} P\hat{d}_\pi 1_s((\gamma P)^{-1})_1 \tilde{Q}_1 \xi((4,1),(8,2)) \\ \begin{pmatrix} 1 & 0 \end{pmatrix} P\hat{d}_i 1_s((\gamma P)^{-1})_1 \tilde{Q}_1 \xi((4,1),(8,2)) \\ \begin{pmatrix} 1 & 0 \end{pmatrix} 1_s(\gamma)^{-1} Q_1 \xi((4,1),(8,2)) \end{pmatrix} \\ M_\gamma(2) \begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} P\hat{d}_y 1_s((\gamma P)^{-1})_2 \tilde{Q}_1 \xi((4,1),(8,2)) + \\ \begin{pmatrix} 1 & 0 \end{pmatrix} P\hat{d}_\pi 1_s((\gamma P)^{-1})_2 \tilde{Q}_1 \xi((4,1),(8,2)) \\ \begin{pmatrix} 1 & 0 \end{pmatrix} P\hat{d}_i 1_s((\gamma P)^{-1})_2 \tilde{Q}_1 \xi((4,1),(8,2)) \\ \begin{pmatrix} 1 & 0 \end{pmatrix} 1_s(\gamma)^{-1} Q_1 \xi((4,1),(8,2)) \end{pmatrix} \end{pmatrix}
$$

$$
+ \sum_{\gamma} \begin{pmatrix} M_{\gamma}(1) \begin{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} P\hat{d}_y 1_s((\gamma P)^{-1})_1 \tilde{Q}_2 \xi((4,1),(8,2)) \\ \begin{pmatrix} 0 & 1 \end{pmatrix} P\hat{d}_\pi 1_s((\gamma P)^{-1})_1 \tilde{Q}_2 \xi((4,1),(8,2)) \\ \begin{pmatrix} 0 & 1 \end{pmatrix} P\hat{d}_i 1_s((\gamma P)^{-1})_1 \tilde{Q}_2 \xi((4,1),(8,2)) \\ \begin{pmatrix} 0 & 1 \end{pmatrix} 1_s(\gamma)^{-1} Q_2 \xi((4,1),(8,2)) \end{pmatrix} \\ M_{\gamma}(2) \begin{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} P\hat{d}_y 1_s((\gamma P)^{-1})_2 \tilde{Q}_2 \xi((4,1),(8,2)) \\ \begin{pmatrix} 0 & 1 \end{pmatrix} P\hat{d}_\pi 1_s((\gamma P)^{-1})_2 \tilde{Q}_2 \xi((4,1),(8,2)) \\ \begin{pmatrix} 0 & 1 \end{pmatrix} P\hat{d}_i 1_s((\gamma P)^{-1})_2 \tilde{Q}_2 \xi((4,1),(8,2)) \\ \begin{pmatrix} 0 & 1 \end{pmatrix} 1_s(\gamma)^{-1} Q_2 \xi((4,1),(8,2)) \end{pmatrix} \end{pmatrix}
$$

Again, the $\hat{d}_z$ are evaluated at the rational equations equilibrium coefficients. Since the ODE associated to B decouples from the system, it can be studied in isolation. Let $\gamma = 1$ correspond to a discount factor of $\alpha\beta$ and $\gamma = 2$ correspond to a discount factor of β. We make use of the model structure to simplify our analysis as well:

$$
M_1(s_t) = \begin{pmatrix} M_1^{yy}(i) & M_1^{\pi y}(i) & 0 & 0 \\ M_1^{y\pi}(i) & M_1^{\pi\pi}(i) & 0 & 0 \\ M_1^{yi}(i) & M_1^{\pi i}(i) & 0 & 0 \\ M_1^{yb}(i) & M_1^{\pi b}(i) & 0 & 0 \end{pmatrix}
$$

$$
M_2(s_t) = \begin{pmatrix} M_2^{yy}(i) & M_2^{\pi y}(i) & M_2^{iy}(i) & 0 \\ M_2^{y\pi}(i) & M_2^{\pi\pi}(i) & M_2^{i\pi}(i) & 0 \\ M_2^{yi}(i) & M_2^{\pi i}(i) & M_2^{i\pi}(i) & 0 \\ M_2^{yb}(i) & M_2^{\pi b}(i) & M_2^{i\pi}(i) & 0 \end{pmatrix}
$$

Using elements from the above matrices, we form the following:

$$M_z^\gamma(1) = \begin{pmatrix} M_{zy}^\gamma(1) & 0 \\ 0 & 0 \\ M_{z\pi}^\gamma(1) & 0 \\ 0 & 0 \\ M_{zi}^\gamma(1) & 0 \\ 0 & 0 \\ M_{zb}^\gamma(1) & 0 \\ 0 & 0 \end{pmatrix}$$

$$M_z^\gamma(2) = \begin{pmatrix} 0 & 0 \\ 0 & M_{zy}^\gamma(2) \\ 0 & 0 \\ 0 & M_{z\pi}^\gamma(2) \\ 0 & 0 \\ 0 & M_{zi}^\gamma(2) \\ 0 & 0 \\ 0 & M_{zb}^\gamma(2) \end{pmatrix}$$

for $z \in \{\pi, y, i, b\}$. The T-map is then given by:[1]

$$T(B) = \sum_{\gamma} \sum_{i \in \{1,\ldots,S\}} \sum_{z} M_z^\gamma(i)(I - \gamma P\hat{b})^{-1} P\hat{d}_z 1_S N(i) B$$

[1] We exclude $\hat{N}$ terms because they will not affect the stability analysis and because they clutter the proof

where $N(1) = (0\ 0\ 0\ 0\ 0\ 0\ 1\ 0)$ and $N(2) = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 1)$. Define 2 by 8 matrices $N(z,i)$ such that $N(z,1)B = (\hat{d}_z(1), 0)'$ and $N(z,2)B = (0, \hat{d}_z(2))'$ for $z \in \{y, \pi, i, b\}$. This implies:

$$P\hat{d}_z = P(N(z,1)B \begin{pmatrix} 1 & 0 \end{pmatrix} + N(z,2)B \begin{pmatrix} 0 & 1 \end{pmatrix})$$

Using this, we can derive the E-stability matrix associated to B:

$$\begin{aligned} DT_B(B) &= \\ & \sum_{\gamma} \sum_{i \in \{1,\ldots,S\}} \sum_{z} (((1,0)\delta(\gamma,i,z))' \otimes M_z^{\gamma}(i)(I - \gamma\hat{b})^{-1} PN(b,1) \\ &+ ((0,1)\delta(\gamma,i,z))' \otimes M_z^{\gamma}(i)(I - \gamma\hat{b})^{-1} PN(b,2) \\ &+ ((1,0)1_s N(i)B)' \otimes M_z^{\gamma}(i)(I - \gamma\hat{b})^{-1} PN(z,1) \\ &+ ((0,1)1_s N(i)B)' \otimes M_z^{\gamma}(i)(I - \gamma\hat{b})^{-1} PN(z,2) \\ &+ I \otimes M_z^{\gamma}(i)(I - \gamma P\hat{b})^{-1} P\hat{d}_z 1_S N(i)) \end{aligned}$$

where $\delta(\gamma, i, z) = (I - \gamma\hat{b})^{-1} P\hat{d}_z 1_s N(i)B$. The necessary and sufficient conditions for E-stability are that all the eigenvalues of DT_B and DT_A have real parts less than one where both matrices are evaluated at the rational expectations equilibrium.

APPENDIX H

AN AND SCHORFHEIDE (2007) MODEL DERIVATION

We present a simple model that is inspired by An and Schorfheide (2007). As in standard in the New Keynesian literature, the model consists of households, a competitive final goods producing firm, monopolistically competitive intermediate firms, a fiscal authority and a monetary authority. We briefly describe the optimization problems facing agents in this economy, then we collect the equilibrium conditions which are log-linearized and presented in section 2.

Households maximize a lifetime utility functions that depends positively on the level of consumption, C_t and negatively on labor supply, N_t. Additionally, households are subjected to a preference shock, Z_t that directly impacts the contribution of time t utility to overall lifetime utility. Formally:

$$\max_{\{C_t,N_t,W_t\}} E_o \sum_{t\geq 0} \left(\frac{C_t^{1-\sigma} - 1}{1-\sigma} - \chi N_t \right) Z_t$$

subject to

$$P_t C_t + E_t(R_{t,t+1} W_{t+1}^j) \;\; \leq \;\; W_t^j + P_t \omega_t N_t - P_t \tau_t$$

and a tranversality condition of the form:

$$\lim_{t\to\infty} E_t[R_{t,T} W_T] = 0$$

where W_t is wealth at time t, ω is the competitive real wage paid to labor, τ is a lump-sum tax, C is consumption, and $R_{t,t+1}$ is a stochastic discount factor that equals $(C_{t+1}/C_t)^{-\sigma}$ in our model with complete markets. From the first order conditions for W_{t+1}, C_t and W_t we get the familiar necessary intertemporal and intratemporal conditions for the household optimization problem:

$$1 = \beta E_t \left\{ \frac{C_{t+1}}{C_t}^{-\sigma} \frac{Z_{t+1}}{Z_t} \frac{(1+i_t)}{\pi_{t+1}} \right\} \tag{H.1}$$

$$\omega_t = \chi C_t^{\sigma}$$

The perfectly competitive firm has technology described by:

$$Y_t = \left(\int_0^1 Y_t(j)^{1-\eta_t} dj \right)^{\frac{1}{1-\eta_t}}$$

where inputs, $Y_t(j)$, are goods produced by each intermediate firm $j \in [0,1]$, and η_t is a shock to markups. The perfectly competitive firm maximizes profits given by:

$$\Pi_t^{FIN} = P_t Y_t - \int_0^1 P_t(j) Y_t(j) dj$$

This implies the following demand schedule for each intermediate producer's good, $Y_t(j)$:

$$\begin{aligned} Y_t(j) &= \left(\frac{P_t(j)}{P_t} \right)^{-1/\eta_t} Y_t \\ P_t(j) &= \left(\int_0^1 P_t(j)^{\frac{\eta_t - 1}{\eta_t}} \right)^{\frac{\eta_t}{\eta_t - 1}} \end{aligned}$$

Intermediate firms are monopolistically competitive and utilize identical technologies that assume the form:

$$Y_t(j) = N_t(j)$$

To introduce nominal rigidities, we assume that firms face the following adjustment costs:

$$AC_t(j) = \frac{\phi}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - \pi\right)^2 Y_t(j)$$

Firms maximize the present value of firm profits taken real wages, ω_{t+s} as given. Formally, they choose labor inputs and prices to maximize the following:

$$\Pi^{INT} = E_0\left\{\sum_{t\geq 0}\beta^t R_{0,t}\left(\frac{P_t(j)}{P_t}Y_t(j) - \omega_t(j)N_t(j) - AC_t(j)\right)\right\}$$

Substituting the product demand schedule into the profits equation, then optimizing with respect to $P_{t+s}(j)$ and substituting for $\omega_{t+s} = c_t^\sigma$ and $R_{t|0} = (C_t/C_0)^{-\sigma}$ yields the following optimality condition:

$$\left(\frac{1}{\eta_t} - 1\right) = \frac{C_t^\sigma}{\eta_t} - \frac{\phi}{2}\left(2(\pi_t - \pi) - \frac{(\pi_t - \pi)^2}{\eta_t}\right)$$
$$+\beta\phi\left(\left(\frac{C_{t+1}}{C_t}\right)^\sigma(\pi_{t+1} - \pi)\,\pi_{t+1}\frac{Y_{t+1}}{Y_t}\right) \tag{H.2}$$

The fiscal authority only issues a bond portfolio, B_t^m, with a maturity that declines at a rate $\rho \in [0,1]$. Under this maturity structure, the quantity of government debt issued at $t-1$ that matures at $t+j$ is:

$$B_{t-1}(t+j) = B_{t-1}^m \rho^j$$

The evolution of the government's bond portfolio satisfies that following budget constraint:

$$B^m_{t-1}(1 - \sum_{j\geq 0} Q_t(t+j)\rho^j) + P_t G_t = P_t \tau_t + B^m_t \sum_{j\geq 0} Q_t(t+j)\rho^{j-1}$$

where $Q_t(t+j)$ is the price of debt that matures at time $t+j$ and is sold at t. To simplify the government budget constraint, we define the price of the bond portfolio, P^m_t, as:

$$P^m_t = E_t \sum_{j\geq 0} Q_t(t+j)\rho^{j-1}$$

which allows us to rewrite the government budget constraint as

$$B^m_{t-1}(1 + \rho P^m_t) + P_t G_t = P_t \tau_t + P^m_t B^m_t \tag{H.3}$$

Furthermore, we can show that bond prices follow a recursive formulation:

$$P^m_t = Q_t(t+1)(1 + \rho E_t P^m_{t+1}) \tag{H.4}$$

given B^m_{-1}. The government also implements a rule that adjusts real primary surpluses in response to the market value of real debt. In equilibrium, households hold all government debt which requires that the following condition hold $\forall t$:

$$W_t = B^m_{t-1}(1 + \rho P^m_t)$$

The processes for τ_t and G_t are specified. Finally, monetary policy follows the following rule:

$$R_t = R^{\rho_i}_{t-1} \left(R^* \left(\frac{\pi_t}{\pi^*} \right)^{\phi_\pi(s_t)} \left(\frac{Y_t}{Y^*_t} \right)^{\phi_y(s_t)} \right)^{1-\rho_i} \tag{H.5}$$

where $R_t = 1 + i_t$, $R^* = \beta^{-1}$, Y_t^* is potential output defined as the level of output that obtains without nominal rigidities and with constant markups. The log-linearized equilibrium conditions in Chapter 5, are simply log-linearized versions of equations (H.1), (H.2), (H.3)-(H.5). μ_t is a composite of η_t from (H.2), and all other shocks and the fiscal policy rule are described in Chapter 5.

APPENDIX I

PRIOR AND POSTERIOR DISTRIBUTIONS

TABLE 8. Prior and Posterior Distribution Statistics

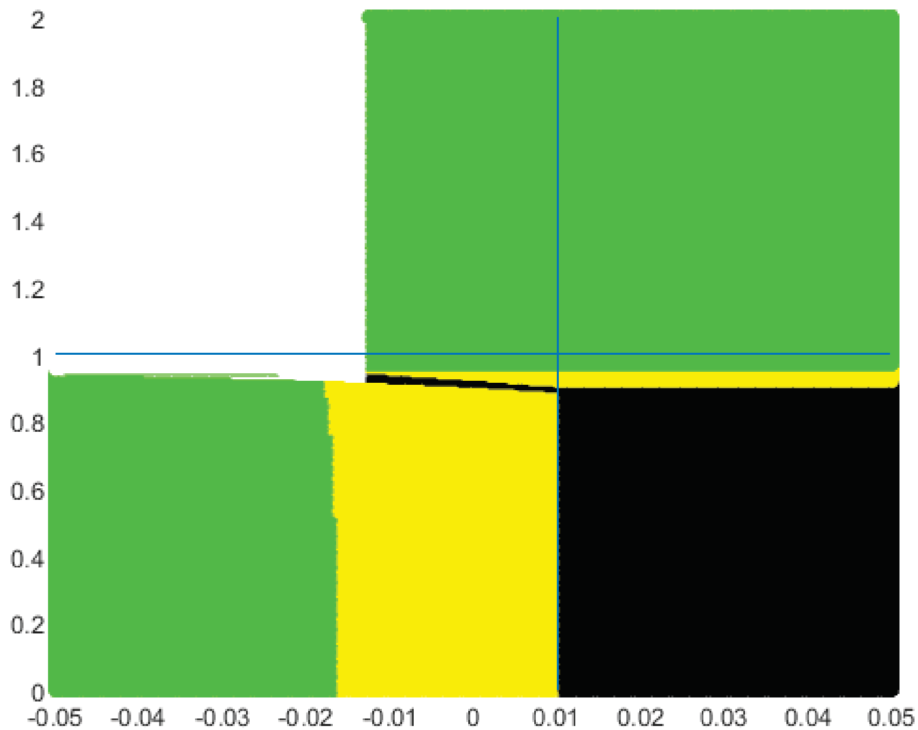
Name	**Prior Density**	**Prior Param (1)**	**Prior Param (2)**	**Posterior Mean**
σ	Gamma	2.00	0.50	2.36
κ	Uniform	0.00	1.00	.91
ϕ_π	Gamma	1.50	0.25	2.16
ϕ_y	Gamma	0.50	0.25	.56
ρ_i	Uniform	0.00	1.00	.71
ρ_g	Uniform	0.00	1.00	.98
ρ_z	Uniform	0.00	1.00	.93
$100\sigma_m$	InvGamma	0.40	4.00	.2
$100\sigma_g$	InvGamma	1.00	4.00	.75
$100\sigma_z$	InvGamma	0.50	4.00	.2

We estimate the An and Schorfheide (2007) model using U.S. data, Q1:1983 to Q3:2007. Param (1) and Param (2) are the lower and upper bounds for the uniform distributions and the mean and standard deviation for the Gamma and Inverse Gamma distributions

APPENDIX J

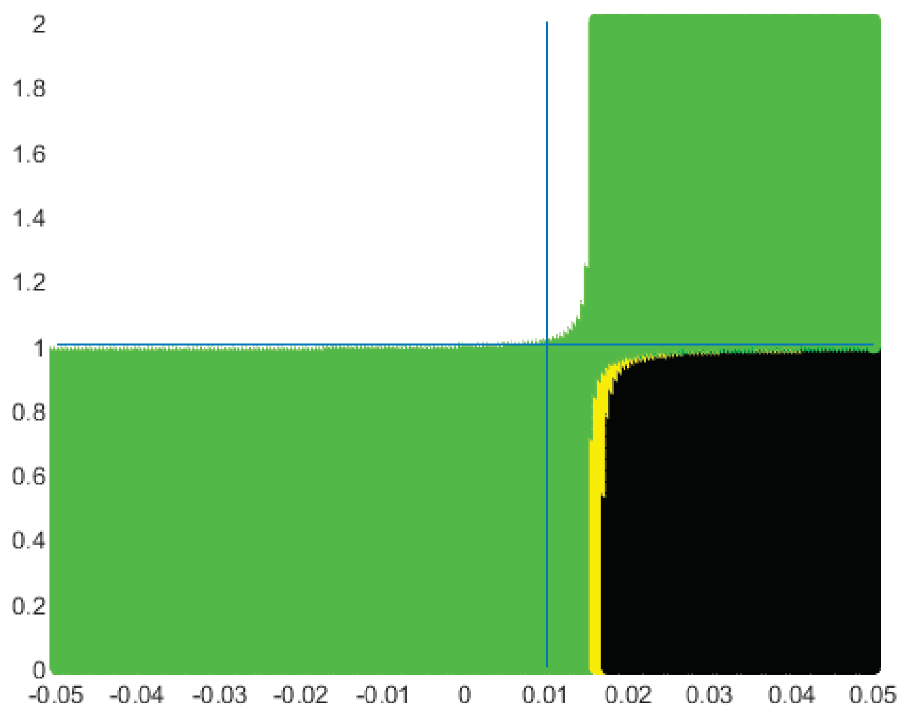
MISCELLANEOUS FIGURES

FIGURE 19. Determinacy and E-Stability (Parameterization 1)



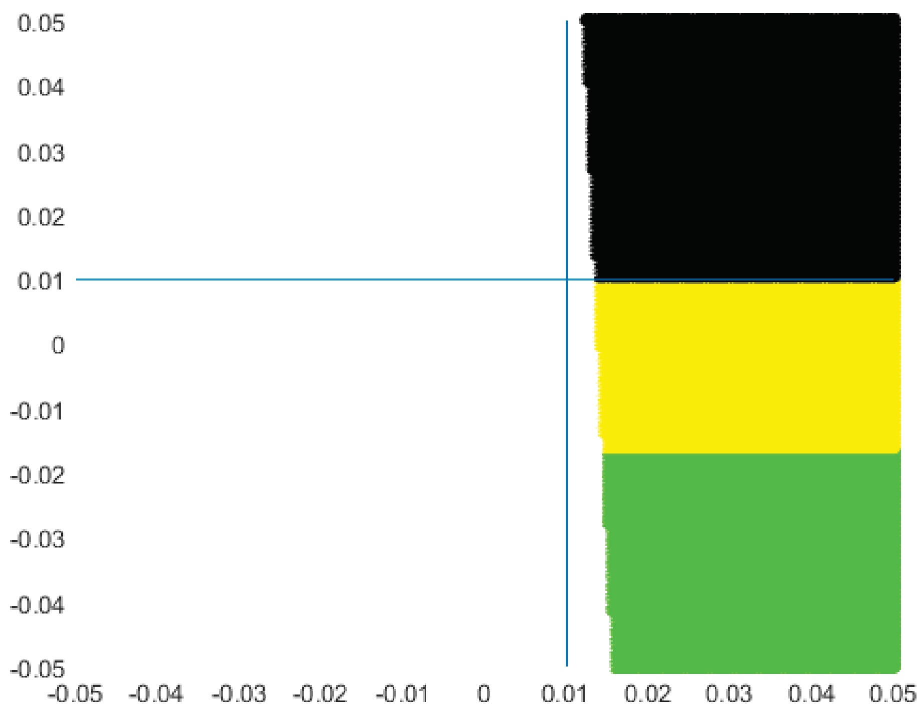
This figures show determinacy and E-stability regions in Regime F when $\phi_\pi(M) = 1.5$ and $\tilde{\gamma}(M) = \gamma(M)(1 - \beta) = .05$. $\phi_\pi(F)$ is on the vertical axis and $\tilde{\gamma}(F)$ is on the horizontal axis. The top right and bottom left quadrants are consistent with determinacy in the fixed regime model. Green regions are determinate and E-stable, yellow regions are indeterminate and E-stable, and black regions of indeterminate and E-unstable

FIGURE 20. Determinacy and E-Stability (Parameterization 2)



This figures show determinacy and E-stability regions in Regime F when $\phi_\pi(F) = 0$ and $\tilde{\gamma}(F) = \gamma(M)(1 - \beta) = -.05$. $\phi_\pi(M)$ is on the vertical axis and $\tilde{\gamma}(M)$ is on the horizontal axis. The top right and bottom left quadrants are consistent with determinacy in the fixed regime model. Green regions are determinate and E-stable, yellow regions are indeterminate and E-stable, and black regions of indeterminate and E-unstable

FIGURE 21. Determinacy and E-Stability (Parameterization 2)



This figures show determinacy and E-stability regions in Regime F when $\phi\pi(M) = 1.5$ and $\phi_\pi(M) = 0$. $\tilde{\gamma}(M) = \gamma(M)(1-\beta)$ is on the horizontal axis and $\tilde{\gamma}(F)$ is on the vertical axis. The bottom right and top left quadrants are consistent with determinacy in the fixed regime model. Green regions are determinate and E-stable, yellow regions are indeterminate and E-stable, and black regions of indeterminate and E-unstable

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