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The Joukowsky Map Reveals the Cubic Equation

Javier Sánchez-Reyes

Abstract. Two canonical polynomials generate all cubics, via linear transformations of the polynomial map and the parameter: the cubic power function, with coincident critical points, and the third Chebyshev polynomial of the first kind, with two distinct critical points. Computing the roots of any cubic boils down to inverting these fundamental maps. In the more general case of distinct critical points, we show that the roots admit a startlingly simple expression in terms of a Joukowsky map and its inverse. Marden's theorem comes as a straightforward consequence, because the roots are the images, under a Joukowsky map, of the vertices of an equilateral triangle.

The first time you come across the closed-form solution to the cubic equation [23], it looks intimidating, with square roots within cube roots and terms coming out of the blue, without apparent geometric meaning. Our goal is to change this perception, by re-examining known results and endowing them with an intuitive interpretation that leads to more compact expressions. To get this new insight, we analyze cubic polynomials as complex transformations, appealing to geometric intuition as exemplified by Needham in his beautifully illustrated book [18].

1. CLASSIFYING CUBIC MAPS. The first question to answer is: How many essentially different cubic maps are there? We consider only true cubics, hence ruling out quadratic or linear maps. The answer arises naturally from a key feature of complex maps, namely their *critical points* z_c, where the conformality of the otherwise conformal mapping breaks down. For a polynomial, critical points correspond to the points z_c where the derivative vanishes. In particular, for a cubic $c(z)$, the equation $c'(z_c) = 0$ is quadratic, furnishing only two cases: coincident or distinct critical points. Thus, we split the set of cubics into two subsets, depending on their number $N = 1, 2$ of distinct critical points. Formally speaking, we define two *equivalence classes*, whereby two cubics are equivalent if they share the same N. This case-by-case analysis will provide a deeper insight and prove simpler than the traditional approach of categorizing cubics according to the nature of their roots [2].

Next, we check that each class is generated by its respective *canonical representative* $\zeta(z)$, via linear transformations of $\zeta(z)$ and the parameter z. In the complex plane $\mathbb{C}$, these transformations have a clear geometric meaning, since they amount to direct similarities [18], that is, combinations of uniform dilations, translations and rotations.

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Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/uamm.

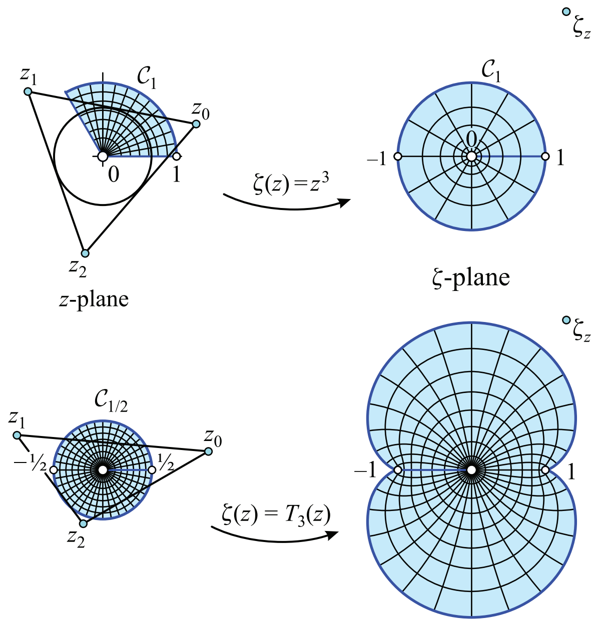

Figure 1. Canonical representative maps $\zeta(z)$.

Coincident critical points. For cubics $c(z)$ with coincident critical points, by shifting the parameter we can always assume that $z_c = 0$. Therefore, $c'(z)$ is proportional to z^2 and, after integration, we get the general expression for such cubics:

$$c(z) = C + D \cdot \zeta(z), \quad \zeta(z) = z^3, \tag{1}$$

with constants $C, D \in \mathbb{C}$ computable as

$$C = c(0), \quad D = c(1) - C. \tag{2}$$

Geometrically, $c(z)$ is a direct similarity of z^3 (a dilative rotation D plus a translation by the critical value C). The map $\zeta(z) = z^3$ is quite simple (Figure 1), displaying the following properties:

(i) The three points $\{-1, 0, 1\}$ on the real axis are fixed, i.e., such that $\zeta(z) = z$.

(ii) The origin $z = 0$ is the critical point of the derivative: $\zeta''(0) = 0$. Equivalently, the cubic $\zeta(z)$ is *depressed* (without quadratic term), thereby generating depressed cubics (1).

(iii) The real and imaginary axes are mapped to themselves.

(iv) The image ζ-plane is covered three times.

(v) The family of origin-centered circles $\mathcal{C}_r$ of radius r transforms to itself.

(vi) Similarly, radial lines through the origin transform to radial lines.

Distinct critical points. Once again, we perform a suitable affine transformation of the parameter, now so that the critical points become $\{-1/2, 1/2\}$. Therefore, $c'(z)$ is proportional to $z^2 - 1/4$ and, after integration, we get a family of cubics generated by linear transformations of a second representative $\zeta(z)$:

$$c(z) = C + D \cdot \zeta(z), \quad \zeta(z) = T_3(z) = 4z^3 - 3z, \tag{3}$$

where $C = c(0)$ and $D = c(1) - C$. We could have chosen any scaled version of $\zeta(z)$, but this normalized version coincides with $T_3(z)$, the *third Chebyshev polynomial of the first kind* [**14**]. Thus, as it shares property (*i*) with z^3, the expressions in (2) hold. In addition, it also shares properties (*ii*), (*iii*), (*iv*). Lebedev [**12**] already noted the possibility of rewriting certain cubics in the form (3) by solving a system of equations.

As shown in Figure 1, T_3 belongs to the family of polynomials, described elsewhere [**10**], mapping circles onto epitrochoids. In particular, $\mathcal{C}_{1/2}$ transforms to an origin-centered nephroid, unsurprisingly with its two cusps at the critical values $T_3(\pm\frac{1}{2}) = \mp 1$. Moreover, T_3 is univalent [**6**] (one-to-one) on the disk bounded by $\mathcal{C}_{1/2}$. However, its more relevant property is not yet apparent and will be revealed by the Joukowsky map. At the risk of ruining the suspense, we advance that T_3 transforms the family of ellipses with foci $\{-1, 1\}$ to itself.

2. INVERTING THE REPRESENTATIVE MAP FOR ROOT FINDING. Given a cubic $c(z)$ generated by its canonical representative $\zeta(z)$, either z^3 or $T_3(z)$, its roots z_k are obtained by setting $c(z_k) = 0$ in (1) or (3):

$$\zeta(z_k) = \zeta_z, \quad \zeta_z = \frac{c(0)}{c(0) - c(1)}. \tag{4}$$

Thus, all roots $\{z_0, z_1, z_2\}$ have a common image ζ_z in the ζ-plane (Figure 1) or, equivalently, the roots of $c(z)$ are those of a translated polynomial $\zeta(z) - \zeta_z$. Therefore, finding these roots is tantamount to reversing ζ and computing the preimages z_k of the point ζ_z. Let us see what happens for our two representatives.

Coincident critical points. Isolating z_k in (4) is a straightforward exercise, as

$$z_k = z_0 e^{2k\pi i/3}, \quad z_0 = \sqrt[3]{\zeta_z}, \quad k = 0, 1, 2,$$

where z_0 denotes the principal cube root of ζ_z. Clearly, these roots z_k satisfy Marden's theorem [**13**], also known as the Siebeck–Marden theorem, since it was proven earlier by J. Siebeck. This celebrated theorem has received ample attention in this MONTHLY [**1**,**4**,**11**,**16**]. It states that the *Steiner inellipse*, inscribed at the midpoints of the triangle of vertices z_k, has as foci the critical points. Indeed, the three roots define an equilateral triangle, so, as Kalman [**11**] notes, its inellipse is actually its incircle, centered at the critical point $z_c = 0$. Admittedly, we changed the original parameter so that z_c lies precisely at the origin, but this is immaterial because the parameter change amounts to a translation.

Distinct critical points. Now, finding the roots turns out to be more challenging because, in principle, we do not know how to reverse the representative $T_3(z)$. Fortunately, the Joukowsky transformation will come to our rescue.

3. THE JOUKOWSKY MAP. Figure 2 displays the Joukowsky map:

$$J(w) = \tfrac{1}{2}\left(w + \tfrac{1}{w}\right), \tag{5}$$

which satisfies (*iii*) and other well-known properties [**22**]:

- It has only two fixed points $\{-1, 1\}$, which are also critical points. Now, the origin $w = 0$ is singular.
- Concentric circles $\mathcal{C}_r$ transform, via nonuniform dilation, to confocal ellipses $\mathcal{E}_r = J(\mathcal{C}_r)$, with foci at $\{-1, 1\}$, known as *Bernstein ellipses* in approximation theory [**24**].

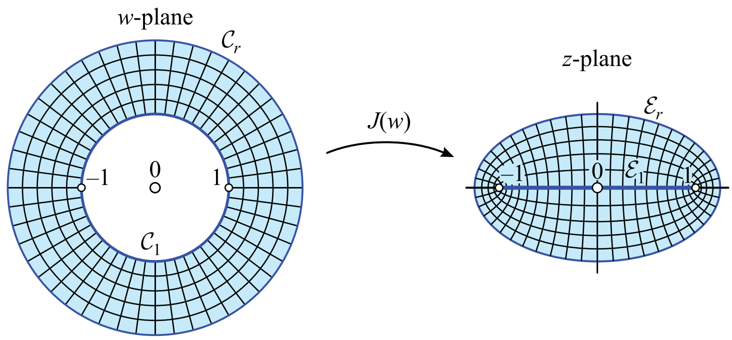

Figure 2. The Joukowsky map $J(w)$.

- In particular, $\mathcal{C}_1$ flattens out to the interval $[-1, 1]$ on the real axis (as a degenerate ellipse $\mathcal{E}_1$), via parallel projection $J(e^{i\theta}) = \cos\theta$.
- Aside from the real or imaginary axes, radial lines through the origin transform to confocal hyperbolas, with foci $\{-1, 1\}$.
- Since $J(w) = J(1/w)$, the map provides a double covering of the image z-plane.

Returning to our original problem, what has the rational quadratic map J to do with the cubic polynomial T_3? A generic nth degree Chebyshev polynomial $T_n(z)$ is closely related to $J(w)$ through the change of variable [**14**]

$$T_n(z) = J(w^n), \quad z = J(w). \tag{6}$$

This relationship, readily verified for the case $n = 3$ depicted in Figure 3, involves the following maps.

- w-plane $\mapsto$ z-plane: The Joukowsky map, transforming circles $\mathcal{C}_r$ to confocal ellipses $\mathcal{E}_r$.
- w-plane $\mapsto$ ζ-plane: Applying properties (v), (vi) to a general power function w^n, we see that $J(w^n)$ behaves in a Joukowsky-like fashion, mapping circles $\mathcal{C}_r$ to confocal ellipses and radial lines to hyperbolas, albeit covering the ζ-plane $2n$ times.
- z-plane $\mapsto$ ζ-plane: The map $T_n(z)$, consequently, transforms the family of ellipses $\mathcal{E}_r$ or the family of hyperbolas with foci $\{-1, 1\}$ to themselves. Intuitively, $T_n(z)$ wraps the ellipses $\mathcal{E}_r$ around the origin n times, yielding ellipses traced n times, much like z^n does with circles $\mathcal{C}_r$.

Observe that z^n, $J(z)$, and $T_n(z)$ thus belong to the remarkable set of maps that transform one family of conics upon another; such were thoroughly analyzed by the Swedish mathematician T. H. Gronwall [**9**] back in 1920. The reader with some insight can identify the polynomials $T_n(z)$ as members of a more general family of maps $J[(J^{-1}(z))^m]$, $m \in \mathbb{R}$, in Gronwall's work.

For the cubic case ($n = 3$), relationship (6) boils down to rewriting the classical rational *Vieta substitution* [**25**] to a monic depressed cubic, in a suitable coordinate system and in terms of Joukowsky maps.

4. FINDING THE ROOTS AND MARDEN'S THEOREM. The stage is finally set to find the roots z_k for the case of distinct critical points. Indeed, relationship (6) furnishes a way to invert $T_3(z)$ via an excursion to the w-plane (Figure 3). If w_k denotes the preimage on the plane of $z_k = J(w_k)$, then the common image ζ_z in (4) can be

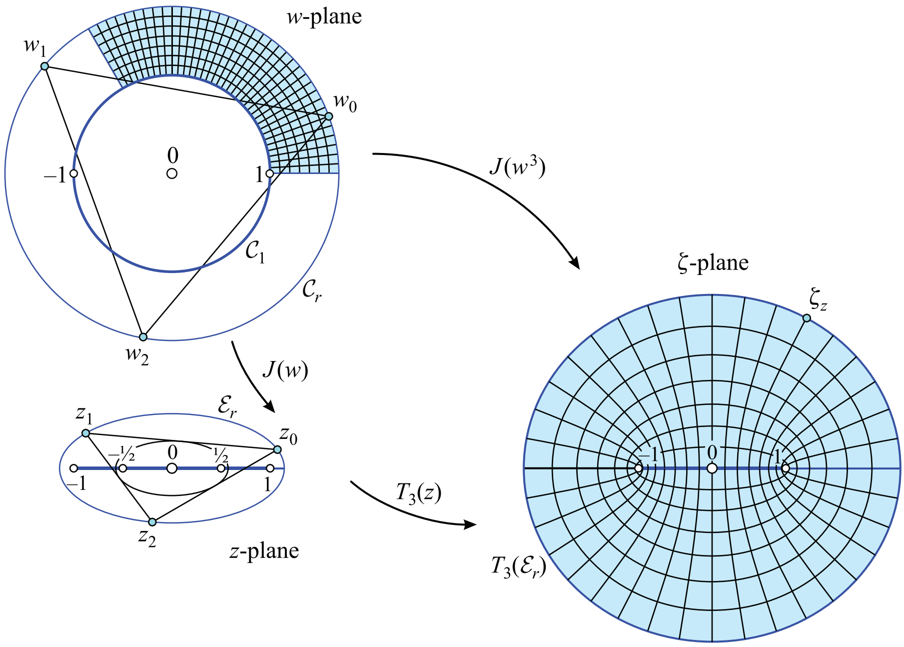

Figure 3. Maps transforming preimages w_k to roots z_k and their common image ζ_z.

expressed as $\zeta_z = J(w_k^3)$, so that

$$w_k = w_0 e^{2k\pi i/3}, \quad w_0 = \sqrt[3]{J^{-1}(\zeta_z)}, \quad k = 0, 1, 2, \tag{7}$$

where w_0 denotes the principal cube root, and the inverse map J^{-1} is given by the two solutions of a quadratic equation [**14**]:

$$J^{-1}(z) = z \pm \sqrt{z^2 - 1}.$$

The choice of sign is immaterial, since swapping it simply inverts the result and hence w_0, which does not affect the outcome $z_0 = J(w_0) = J(1/w_0)$ and only swaps z_1, z_2. Let us summarize our key result.

Theorem 1. *The roots z_k of a cubic (3) $c(z) = C + D \cdot \zeta(z)$, $\zeta(z) = T_3(z)$, with two distinct critical points $\pm 1/2$, are the images $z_k = J(w_k)$, under a Joukowsky map (5), of the vertices w_k (7) of an origin-centered equilateral triangle.*

Marden's theorem comes as a straightforward corollary. Via a nonuniform dilation, $J(w)$ maps w_k to z_k, and the circle $\mathcal{C}_r$ where the w_k's lie maps to a Bernstein ellipse $\mathcal{E}_r$ with foci ± 1, namely the *Steiner circumellipse* of the triangle of vertices z_k. The Steiner circumellipse and inellipse are related through a central dilation of factor $F = 1/2$, so the inellipse has foci $\pm F$ precisely at the critical points $z_c = \pm 1/2$. Note also the following:

- Much like in the discussion for coincident critical points, the parameter change to set z_c at $\pm 1/2$ is immaterial, as it amounts to a similarity S. To adapt Figure 3 to arbitrary critical points in the z-plane, replace J with the composition $S^{-1} \circ J$ in $J(w)$, and T_3 with $T_3 \circ S$. This simply yields a similar geometry on the z-plane, with roots $S^{-1}[J(w_k)]$.
- Whereas $z_k = J(w_k)$, the equilateral triangle of vertices w_k does not map to a triangle of vertices z_k, since $J(w)$ does not preserve straight lines.

In a paper published in this MONTHLY back in 1952, Moppert [17] already solved the cubic equation by conformal mapping. Instead of the Chebyshev polynomial $T_3(z)$, he employs a scaled representative map, and defines a six-sheeted Riemann surface that can be identified as that of $J(w^3)$. More recently, Lebedev [12] and Boyd [5] observe that solving a cubic equation is tantamount to inverting T_3. These previous works reach equivalent formulae for the roots, but do not relate them to Marden's theorem. Finally, in an enlightening work [20], Northshield already pointed out the existence of a nonuniform dilation (in fact, of a linear map) between the vertices of an equilateral triangle and the roots, and utilized this property to prove Marden's theorem and find the roots. We reinterpret this idea in terms of Joukowsky transformations.

Extension to arbitrary degree. We have concluded that finding the roots (4) of any cubic polynomial amounts to computing those of a translated Chebyshev polynomial or power function. Unfortunately, this property does not carry over to higher degrees. Nevertheless, we can characterize the roots z_k of translated polynomials $p_n(z) = T_n(z) - \zeta_z$ for arbitrary degree n. These roots are the images $z_k = J(w_k)$ of the vertices w_k of a origin-centered regular n-gon, in other words, the vertices of such an n-gon after a nonuniform dilation. As Clifford and Lachance [7] note, the polygon $\mathcal{P}$ of vertices z_k still defines a Steiner circumellipse and inellipse, related by a central dilation of factor $F = \cos \pi/n$. Since the circumellipse has foci ± 1, those of the inellipse lie at $\pm F$. From basic properties of Chebyshev polynomials, such foci $\pm F$ are critical points of $T_n(z)$, and hence of $p_n(z)$, too.

By introducing a similarity on the z-plane, this geometry generalizes to roots defining an *affinely-regular* n-gon $\mathcal{P}$, i.e., the affine image of a regular n-gon. Without invoking Chebyshev polynomials, Parish [21] already proved that the foci of the Steiner inellipse of $\mathcal{P}$ are critical points. When $n = 4$, a case studied in detail in [8], an affinely-regular n-gon is always a parallelogram.

5. CASE OF THREE REAL ROOTS.

For distinct critical points, consider the particular case where all $z_k \in \mathbb{R}$, shown in Figure 4 adapting the general configuration of Figure 3. Since all z_k are aligned, their Steiner circumellipse degenerates to a segment, so their preimages w_k lie on a unit circle $\mathcal{C}_1$. The maps work as follows.

- w-plane $\mapsto$ z-plane: $J(w)$ flattens $\mathcal{C}_1$ onto the real segment $[-1, 1]$ as a degenerate Bernstein ellipse $\mathcal{E}_1$, and the vertices w_k in $\mathcal{C}_1$ onto points z_k in this segment.
- w-plane $\mapsto$ ζ-plane: $J(w^3)$ wraps $\mathcal{C}_1$ around the origin three times and then flattens it onto the real segment $[-1, 1]$, mapping all vertices w_k onto a point $\zeta_z \in [-1, 1]$.
- z-plane $\mapsto$ ζ-plane: $T_3(z)$ maps $[-1, 1]$, which contains the roots, onto itself. This transformation is easily understood by plotting the real function $T_3(x)$, restricted to real values $x \in [-1, 1]$. It attains its local extrema at the critical points $\pm 1/2$, and has an inflection point at the origin, the critical point of $T_3'(x)$.

Consequently, well-known properties [3, 19, 20] of this case become evident:

- The roots z_k are the projections onto the real axis of the vertices w_k of an equilateral triangle Δ.
- The center of Δ has the same abscissa ($x = 0$) as the inflection point.
- The inscribed circle $\mathcal{C}_{1/2}$ of Δ projects onto the interval $[-1/2, 1/2]$, bounded by the critical points.

To find the z_k's, we could compute their complex preimages w_k using the general expression (7), but this implies abandoning the real line, despite all roots being real.

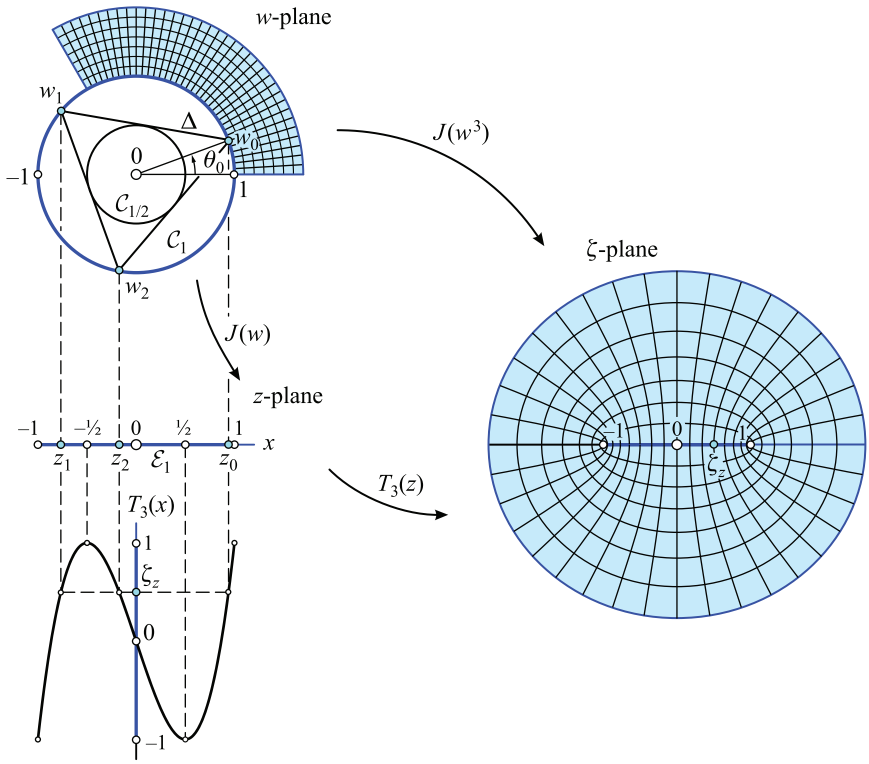

Figure 4. Geometry of the maps in the special case where all roots z_k are real.

However, with little effort we can avoid this nuisance and express z_k in terms of real-valued trigonometric functions, rediscovering the trigonometric solution [**15**, **19**, **23**] to the cubic equation. Indeed, as the preimages $w_k = e^{i\theta_k}$ lie on $\mathcal{C}_1$, the Joukowsky map degenerates to a parallel projection $J(e^{i\theta}) = \cos\theta$. Hence, the roots are expressible in terms of the arguments θ_k as

$$z_k = \cos\theta_k. \tag{8}$$

Moreover, since the w_k's are rotated cube roots of unity by an angle θ_0, and J^{-1} reduces over $[-1, 1]$ to reversing the cosine function, θ_k takes values

$$\theta_k = \theta_0 + \frac{2k\pi}{3}, \qquad \theta_0 = \frac{1}{3}\arccos\zeta_z, \qquad k = 0, 1, 2,$$

where the standard function $\arccos(x) \in [0, \pi]$ yields an argument $\theta_0 \in [0, \pi/3]$. As a final check, if $\zeta_z = 0$, then $\theta_0 = \pi/6$ and the z_k's in (8) coincide with the symmetrical Chebyshev roots [**14**] of T_3.

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