Spring 2012

Analysis of the Family of Generalized Cumulative Sum Type Control Charts

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ANALYSIS OF THE FAMILY OF GENERALIZED CUMULATIVE SUM TYPE CONTROL CHARTS

by

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(Under the Direction of Charles W. Champ)

ABSTRACT

As an aid to the practitioners, various Phase II quality control charts have been developed to monitor for a change in the parameters of the distribution of a quality measurement. In this project, the family of generalized cumulative sum type charts was studied. An equivalent chart version that requires fewer parameters was given. Some useful integral equations were derived for determining the run length distribution of the lower and upper one-sided charts. The Markov chain methods were also given. The parameters unknown version was presented and the performance analysis was studied for the chart for monitoring for a change in the mean of a normal distribution. The design and analysis of a chart when the quality measurement follows a gamma distribution was given, which includes the design and analysis of a chart for monitoring for a change in the standard deviation of a normal distribution.

Key Words: Generalized CUSUM, Average run length, Normal distribution, Gamma distribution

2009 *Mathematics Subject Classification*: 62E15, 62F10, 62N05

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B.S. in Huazhong Normal University

A Thesis Submitted to the Graduate Faculty of Georgia Southern University in Partial Fulfillment
of the Requirement for the Degree

MASTER OF SCIENCE

STATESBORO, GEORGIA

2012

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Electronic Version Approved:

May 2012

ACKNOWLEDGMENTS

First and foremost I express the deepest appreciation to my advisor, Dr. Charles W. Champ, for his advice and guidance during my graduate study, and especially while conducting this research. His encouragement, guidance and support from the initial to the final level enabled me to develop an understanding of the subject. Without his support with his patience and knowledge this thesis would not have been completed or written. Thanks to Dr. Broderick O. Oluyede for his guidance both on my research and in my general academic pursuits. Thanks to Dr. Hani Samawi for graciously lending your time and expertise in content analysis and comments. I am truly grateful for the opportunity to have had such a dynamic and capable thesis committee. I would like to thank Dr. Yan Wu for his attention to my graduate study. In addition, The Department of Mathematical Sciences has provided the support and equipment I have needed to produce and complete my thesis. Finally, I offer my regards and blessings to all of those who supported my in any respect during the completion of my graduate study and research.

TABLE OF CONTENTS

Page

ACKNOWLEDGMENTS v

CHAPTER

1 Introduction 1

2 Generalized CUSUM Type Control Charts 6

2.1 Introduction 6

2.2 Special Cases 7

2.3 Equivalent Forms 9

2.4 Analysis Using Integral Equations 11

2.5 Markov Chain Approximation 22

2.6 Conclusion 35

3 Monitoring for a Change in the Process Mean 36

3.1 Introduction 36

3.2 Parameters Estimated Version 37

3.3 Integral Equations Approach 39

3.4 Markov Chain Approach 44

3.5 Unconditional Run Length Distribution 47

3.6 Conclusion 49

4 Monitoring A Scale Parameter 50

4.1 Introduction 50

4.2 Performance Analysis Using Integral Equations 52

4.3 Markov Chain Approach 60

4.4 Gamma Distributed Data 63

4.5 Monitoring for a Change in the Process Variance 65

4.6 Conclusion 66

5 CONCLUSION 68

5.1 General Conclusions 68

5.2 Areas for Further Research 69

BIBLIOGRAPHY 71

CHAPTER 1

INTRODUCTION

In the early 1920's, Walter A. Shewhart (see Shewhart (1925,1931)) introduced the quality control chart as an aid for practitioners in their efforts to produce quality goods. The control chart is a time plot of one or more summary statistics plotted against the time variable, sample number. Control charts are used in two phases of the production process. In the first phase (Phase I), the practitioner is interested in answering the question "whether is the data collected from an in control process?" Also, in Phase I, the chart is used as an aid in estimating what is meant - by the process being in control. Phase I is also known as the retrospective phase. Control charts used in this phase are known as Phase I or retrospective control charts. Most Phase I charts are similar to the ones proposed by Walter A. Shewhart. It is very important to note that the practitioner does not solely use the Phase I chart to help in answering the posed question.

In the second phase (Phase II or the prospective phase), a control chart is used to aid practitioners in comparing the data to what is meant by the process being in control to answer the question "has the process changed from a in control process to an out-of-control process?". Charts used in this phase are known as Phase II or prospective control charts. A variety of Phase II charts and their modifications are discussed in the literature. The Shewhart charts (with run rules), the run sum chart, the cumulative sum (CUSUM) chart, and the exponentially moving average (EWMA) chart including their various modifications are the most common charts discussed in the literature.

In the univariate case, the quality of the process is described in terms of one or more parameters $\theta_1, \ldots, \theta_q$ of the distribution of a quality measurement X to be taken on an item that is to be output by the process. When X is a continuous measurement, the quality of the process can often be characterized by the mean μ and standard deviation σ of the distribution of X. A simple model of the process being in control is when $\theta_1, \ldots, \theta_q$ are equal, respectively, to the fixed but unknown values $\theta_{1,0}, \ldots, \theta_{q,0}$. Some assumptions are then made about the distribution of X as with most statistical methods. When X is a continuous measurement, the normal distribution is usually entertained as the data model. Further, charts in Phase I are typically developed under the assumption that the X measurements are stochastically independent.

A typical Phase I Shewhart chart is based on the X measurements (univariate case) with m samples $\{X_{i,1}, \ldots, X_{i,n}\}$ each of size n taken from the output of the process, $i = 1, \ldots, m$. The samples are collected periodically (over time) from the output of the process. Assume the quality measurement X is a continuous random variable. At sampling stage (time) i, a statistic

$$Y_i = y\left(X_{i,1}, \ldots, X_{i,n}\right)$$

is determined and plotted versus i for $i = 1, \ldots, m$. Assume that $E\left(Y_i\right) = \theta_j$. Thus, the points (i, Y_i) should locate randomly about the horizontal line that passes through the point $(0, \theta_j)$. Since $\theta_{j,0}$ is not known, then one can only judge from a Phase I chart if the process is stable, that is, if the plotted points appear to be plotting randomly about "some" horizontal lines. If the process is in control, then we have that $\theta_j = \theta_{j,0}$. Assuming that the process is in control, one then estimates the mean and standard deviation of the plotted statistic Y_i and adds the horizontal lines that pass through the points

$$\left(0, \widehat{\theta}_j - k_L \widehat{\sigma}_Y\right), \ \left(0, \widehat{\theta}_j\right), \text{ and } \left(0, \widehat{\theta}_j + k_U \widehat{\sigma}_Y\right),$$

where $\widehat{\sigma}_Y$ is the estimate of the standard deviation of the distribution of Y and the variables k_L and k_U are chart parameters to be selected by the practitioner. The first, second, and third horizontal lines are known, respectively, as the lower control limit (LCL), the center line (CL), and the upper control limit (UCL) of the chart. These lines are provided as aids to the practitioner in attempting to objectively answer the question "were these quality measurements on the output of an in-control process?" Methods for designing Phase I Shewhart control charts were given in Montgomery (2008). Selecting probability limits were discussed in Newton and Champ (1997). Champ and Chou (2003) examined the use of individual limits and compared these charts to those based on standard limits.

In Phase II, one assumes that the process is in a state of statistical in control. It is of interest in this phase to monitor the process for any change from being in control to an out-of-control state. Once again samples are collected periodically from the output of the process and the quality measurement $(X_{t,1}, \ldots, X_{t,n_t})$ is to be taken on each item in the sample. We use the time variable t in this phase with $t = m+1, m+2, m+3, \ldots$. At time t one or more chart statistics are computed and plotted against the time variable t. The plotted points define the chart. It is the practitioner's task to select time between the $(t-1)$th and the tth samples, the sample size n_t at time t, and to use the present as well as the previous sample data to define the chart. Each chart consists of a collection of chart parameters at time t. Chart parameters are selected by the practitioner. These parameters include the sample size and the time between samples. If past data is used to select the chart to be used at the next sampling stage, the chart is known as an adaptive control chart. Adaptive control charts were discussed by Champ (1986). He suggested using more stringent runs rules for detecting a shift in the process if there are evidence the process may be out-of-control and less stringent runs rules otherwise. Since then, adaptive versions of most of the popular control charts found in the literature have been proposed.

A family of Phase II control charting procedures proposed by Champ, Woodall, Moshen (1991) include as subfamilies the Shewhart, CUSUM, and EWMA charts. This family is known as the family of generalized cumulative sum type control charts. Our interest is to study the use of these charts for monitoring for a change in the mean of a quality measurement X that has a normal distribution. Also, we examine the use of these charts when the quality measurement/statistic has a gamma distribution. We extend this family of charting procedures to include parameters estimated versions.

The run length T of a Phase II chart is defined as the first time t in which the chart signals. The most typically used measures of a Phase II chart's performance are parameters of the run length distribution. These parameters include the mean, standard deviation, and percentiles of the run length distribution. The mean is often referred to as the average run length (ARL). Often the performances of two or more charts are compared by their ARLs. Consequently, charts are often designed to have some desired values of the ARL when the process is both in- and for some out-of-control scenario(s). It is common to design charts so that the in-control ARL is some specified value and the out-of-control ARL for a particular out-of-control scenario is a minimum.

There are three methods that are typically used to evaluate the run length distribution – simulation, the Markov chain approximation and integral equations. Simulation is a good method when one is interested in an estimated parameter as the average run length of the run length distribution. Methods for selecting chart parameters that optimize the performance of the chart under some given criteria require more accurate approximations than the estimate provided via simulation. The Markov chain method for approximating the run length distribution of a chart was introduced by Brook and Evans (1972). A chart is observed to be a continuous state, discrete time Markov process which is approximated by a Markov chain. Exact results obtained from the Markov chain become the approximations to the run length distribution and its various parameters. The third method makes use of the fact that the run length distribution and its various parameters can be expressed as exact solutions to integral equations. These solutions are then approximated. It was shown by Champ and Rigdon (1991) that some well known integral equations which are useful in evaluating the run length properties of a chart have approximate solutions that are the exact results obtained by using a Markov chain to approximate the chart.

The family of generalized cumulative sum type control charts is discussed in the next chapter. This includes methods for deriving the run length distribution of the chart. In Chapter 3, a performance analysis is given for the chart used to monitor for a change in the mean of a normal distribution. The design and analysis of a chart when the quality measurement follows a gamma distribution is given in Chapter 4. This chapter includes the design and analysis of a chart for monitoring for a change in the standard deviation of a normal distribution. Some concluding remarks are given in the last chapter along with some areas for further study.

CHAPTER 2

GENERALIZED CUSUM TYPE CONTROL CHARTS

2.1 Introduction

A common practice when monitoring for the mean or standard deviation of a continuous quality measurement is to monitor for increases as well as decreases in these parameters. As an aid to the practitioner, various Phase II quality control charts have been developed for this purpose. Champ, Woodall, and Mohsen (1991) showed that three of the most commonly recommended charts, the Shewhart, CUSUM, and EWMA charts, are members of a family of cumulative sum (CUSUM) type control charting procedures. Also, included in this family are one-sided versions of the EWMA chart. Members of this family are used for monitoring for a decrease in a parameter, or increase, or both. Two-sided charts are the combination of two one-sided charts.

A chart that can be used to monitor for the decrease change in the statistic $Y_{L,t}$ plots the points (t, L_t) for $t = 1, 2, 3, \ldots$, where

$$L_t = \min\left\{b_0, b_1 L_{t-1} + b_2 Y_{L,t} + b_3\right\}$$

with $L_0 = b_4$ and $b_2 > 0$ for some value of t. The chart signals if $L_t \le b_5$. The chart may also be designed to signal if $Y_{L,t} \le b_6$(Shewhart limit). The values b_0, b_1, b_2, b_3, b_4, b_5, and b_6 are known as the chart parameters. This chart is referred to as the lower one-sided generalized cumulative sum type chart. The upper one-sided generalized cumulative sum type chart plots the points (t, U_t) for $t = 1, 2, 3, \ldots$, where

$$U_t = \max\left\{a_0, a_1 U_{t-1} + a_2 Y_{U,t} + a_3\right\}$$

with $U_0 = a_4$ and $a_2 > 0$. The chart signals if $U_t \ge a_5$ or $Y_{U,t} \ge a_6$(Shewhart limit). The chart parameters are a_0, a_1, a_2, a_3, a_4, a_5, and a_6. The chart that results from plotting the points (t, L_t) and (t, U_t) on the same graph is a two-sided generalized cumulative sum type chart.

In this chapter, several special cases of the family of generalized cumulative sum type control charts are given. In section 3, an equivalent version of the chart is presented that requires fewer chart parameters for the practitioner to select. The two methods, Markov chain and integral equations, are discussed for evaluating the run length distribution of a generalized control chart in Section 4 and 5. It is shown that the integral equation approach is equivalent to the Markov chain approach.

2.2 Special Cases

Walter A. Shewhart is credited with the introduction of the quality control chart. A Shewhart chart is designed to signal at time t if

$$Y_t \leq E_0(Y) - k\sqrt{V_0(Y)} \text{ or } Y_t \geq E_0(Y) + k\sqrt{V_0(Y)},$$

with $k > 0$, where $E_0(Y)$ and $V_0(Y)$ are the in-control mean and variance of the statistic Y_t. The two-sided generalized cumulative sum type chart with $Y_{L,t} = Y_{U,t} = Y_t$ is defined by

$$\begin{aligned} L_0 &= 0,\ L_t = \min\{E_0(Y), (0)\, L_{t-1} + Y_t + 0\} = \min\{E_0(Y), Y_t\}, \\ U_0 &= 0, \text{ and } U_t = \max\{E_0(Y), (0)\, U_{t-1} + Y_t + 0\} = \max\{E_0(Y), Y_t\}. \end{aligned}$$

We see that $L_t = Y_t$ or $U_t = Y_t$ at time t. Letting

$$b_5 = E_0(Y) - k\sqrt{V_0(Y)} \text{ and } a_5 = E_0(Y) + k\sqrt{V_0(Y)},$$

the chart would signal if

$$L_t \leq b_5 \text{ or } U_t \geq a_5.$$

This is, of course, the same rule that causes the Shewhart chart based on the statistic Y_t to signal.

The cumulative sum (CUSUM) chart was introduced by Page (1956). This chart signals a potential out-of-control process if

$$L_t \leq -h \text{ or } U_t \geq h,$$

where

$$\begin{aligned} L_0 &= 0,\ L_t = \min\left\{0, L_{t-1} + Y_t + k\right\}, \\ U_0 &= 0, \text{ and } U_t = \max\left\{0, U_{t-1} + Y_t - k\right\}. \end{aligned}$$

This is a generalized control chart with

$$\begin{aligned} Y_{L,t} &= Y_{U,t} = Y_t,\ b_0 = a_0 = 0,\ b_1 = b_2 = a_1 = a_2 = 1, \\ b_3 &= -a_3 = k,\ b_4 = a_4 = 0, \text{ and } b_5 = -a_5 = -h. \end{aligned}$$

Clearly, the CUSUM chart is a member of the family of generalized CUSUM type charts.

Roberts (1971) introduced a family of control charting procedures which he referred to as the geometric moving average charts. These charts are now referred to as exponentially weighted moving average (EWMA) charts. An EWMA chart can be viewed as a special case of the generalized control chart by setting

$$\begin{aligned} Y_{L,t} &= Y_{U,t} = \frac{\overline{X}_t - \mu_0}{\sigma_0/\sqrt{n}},\ b_0 = \infty,\ a_0 = -\infty,\ b_1 = a_1 = 1 - r, \\ b_2 &= a_2 = r,\ b_3 = a_3 = 0,\ b_4 = a_4 = 0, \text{ and } b_5 = -a_5 = -h, \end{aligned}$$

where $0 < r \leq 1$ and $h > 0$.

Shu, Jiang, and Wu (2007) examined a Phase II EWMA chart for monitoring for a change in the mean that plots the points (t, L_t) and (t, U_t), where

$$\begin{aligned} L_0 &= \mu_0 - \frac{\sigma_0}{\sqrt{2\pi}},\ L_t = (1-r)\,L_{t-1} + rX_t^-,\ \text{and} \\ U_0 &= \mu_0 + \frac{\sigma_0}{\sqrt{2\pi}},\ U_t = (1-r)\,U_{t-1} + rX_t^+, \end{aligned}$$

where $0 < r \leq 1$,

$$X_t^- = \min\{\mu_0, X_t\},\ \text{and}\ X_t^+ = \max\{\mu_0, X_t\}.$$

The chart signals at time t if $L_t \leq LCL$ or $U_t \geq UCL$, where

$$\begin{aligned} LCL &= \mu_0 - \left(\frac{1}{\sqrt{2\pi}} + k\sqrt{\frac{r\left[1-(1-r)^{2t}\right]}{2-r}}\sqrt{\frac{1}{2} - \frac{1}{2\pi}}\right)\sigma_0 \ \text{and} \\ UCL &= \mu_0 + \left(\frac{1}{\sqrt{2\pi}} + k\sqrt{\frac{r\left[1-(1-r)^{2t}\right]}{2-r}}\sqrt{\frac{1}{2} - \frac{1}{2\pi}}\right)\sigma_0. \end{aligned}$$

They also discussed using the lower (LCL) and upper (UCL) control limits

$$\begin{aligned} LCL &= \mu_0 - \left(\frac{1}{\sqrt{2\pi}} + k\sqrt{\frac{r}{2-r}}\sqrt{\frac{1}{2} - \frac{1}{2\pi}}\right)\sigma_0 \ \text{and} \\ UCL &= \mu_0 + \left(\frac{1}{\sqrt{2\pi}} + k\sqrt{\frac{r}{2-r}}\sqrt{\frac{1}{2} - \frac{1}{2\pi}}\right)\sigma_0. \end{aligned}$$

One can see that these charts are members of the family of generalized control charts with

$$Y_{L,t} = X_t^-,\ Y_{U,t} = X_t^+,$$

$$b_0 = \infty,\ a_0 = -\infty,\ b_1 = b_2 = a_1 = a_2 = 1,\ b_3 = -a_3 = k,$$

$$b_4 = \mu_0 - \frac{\sigma_0}{\sqrt{2\pi}},\ a_4 = \mu_0 + \frac{\sigma_0}{\sqrt{2\pi}},\ b_5 = LCL,\ \text{and}\ a_5 = UCL.$$

2.3 Equivalent Forms

It will be interesting to examine an equivalent form of a generalized control chart. The following is a definition of equivalence of two forms of a chart.

Definition: Two charts are said to be equivalent if and only if at time t either both charts signal or both charts do not signal for all possible sets of data.

Assume $b_2 > 0$ and define

$$L_t^* = \frac{L_t - b_0}{b_2}.$$

It follows that

$$\begin{aligned}
L_0^* &= \frac{b_4 - b_0}{b_2} = b_4^* \text{ and} \\
L_t^* &= \frac{\min\left\{b_0, b_1 L_{t-1} + b_2 Y_{L,t} + b_3\right\} - b_0}{b_2} \\
&= \min\left\{\frac{b_0 - b_0}{b_2}, \frac{b_1 L_{t-1} + b_2 Y_{L,t} + b_3 - b_0}{b_2}\right\} \\
&= \min\left\{0, b_1 \frac{L_{t-1} - b_0}{b_2} + Y_{L,t} + \frac{b_3 + (b_1 - 1)\, b_0}{b_2}\right\} \\
&= \min\left\{b_0^*, b_1^* L_{t-1}^* + Y_{L,t} + b_3^*\right\},
\end{aligned}$$

where

$$b_0^* = 0,\ b_1^* = b_1,\ b_2^* = 1, \text{ and } b_3^* = \frac{b_3 + (b_1 - 1)\, b_0}{b_2}.$$

The chart based on the sequence $\{L_t^*\}$ signals if

$$L_t^* \le b_5^* = \frac{b_5 - b_0}{b_2}.$$

This chart is equivalent to the chart based on the sequence $\{L_t\}$. It is easy to see that at time t for the same data either both charts signal or both charts do not signal.

Similarly, we can write for $a_2 > 0$

$$U_t^* = \max\left\{0, a_1^* U_{t-1}^* + Y_{U,t} + a_3^*\right\},$$

with $U_0^* = \frac{a_4 - a_0}{a_2}$, where

$$a_0^* = \frac{a_0 - a_0}{a_2} = 0,\ a_1^* = a_1,\ a_2^* = 1,\ a_3^* = \frac{a_3 + (a_1 - 1)\, a_0}{a_2}, \text{ and } a_4^* = \frac{a_4 - a_0}{a_2},$$

and the chart signals if

$$U_t^* \geq a_5^* = \frac{a_5 - a_0}{a_2}.$$

The charts based on the sequences $\{U_t\}$ and $\{U_t^*\}$ are equivalent.

Theorem(Champ and Wang**):** A one-sided generalized control chart with six chart parameters is equivalent to a one-sided generalized control chart with four chart parameters.

Proof: The results follow from the previous discussion. Note that the addition of the Shewhart limit only adds one more chart parameter in both cases.

It may be reasonable to have a relationship between the a_i's and the b_i's. An example of such a relationship is

$$b_0 = -a_0,\ b_1 = a_1,\ b_2 = a_2,\ b_3 = -a_3,\ b_4 = -a_4, \text{ and } b_5 = -a_5$$

which is equivalent to the one used in Aparisi, Lluch, and Luna (2008). We see that

$$b_1^* = b_1 = a_1 = a_1^*,\ b_2^* = 1 = a_2^*,\ b_3^* = -a_3^*,\ b_4^* = -a_4^*, \text{ and } b_5^* = -a_5^*.$$

Thus,

$$b_0^* = -a_0^*,\ b_1^* = a_1^*,\ b_2^* = a_2^*,\ b_3^* = -a_3^*,\ b_4^* = -a_4^*, \text{ and } b_5^* = -a_5^*.$$

Hence, for this example there are only four chart parameters that need to be determined by the practitioner.

2.4 Analysis Using Integral Equations

Champ, Woodall, and Moshen (1991) gave some integral equations useful in analyzing generalized cumulative sum type charts. Champ, Rigdon, and Scharnagl (2001) gave a more general study of integral equations that are useful in analyzing quality control charts. We examine a more general setting integral equations that can be used to analyze a generalized cumulative sum type charts.

First, we look at the run length distribution of the lower one-sided generalized cumulative sum type chart with plotted statistic defined by

$$L_0 = l \text{ and } L_t = \min\{0, b_1 L_{t-1} + Y_{L,t} + b_3\}$$

with $b_5 < l \leq 0$ that signals if $L_t \leq b_5$ or $Y_{L,t} \leq b_6$. Assume that the chart parameters do not depend on t. The probability mass function describing the conditional distribution of the run length T_L given $L_0 = l$ of an lower one-sided generalized chart is represented by

$$pr_L(t\,|l) = P(T_L = t\,|L_0 = l).$$

For the case in which $t = 1$, we have

$$\begin{aligned} pr_L(1\,|l) &= P(L_1 \leq b_5 \text{ or } Y_{L,1} \leq b_6\,|L_0 = l) \\ &= P(Y_{L,1} \leq b_5 \ - b_1 l - b_3 \text{or } Y_{L,1} \leq b_6) \\ &= P(Y_{L,1} \leq \max\{b_5 - b_1 l - b_3, b_6\}) \\ &= F_{Y_L}(\beta_L(l) - b_1 l - b_3), \end{aligned}$$

where $\beta_L(l) = \max\{b_5, b_1 l + b_6 + b_3\}$. Note that if the Shewhart limit is not included, then $\beta_L(l) = b_5$. When $t > 1$ and the support of the distribution of Y_L is the reals, we see that the event $\{T_L = t\}$ can be expressed as

$$\{T_L = t\} = \{T_L = t, L_1 = 0, Y_{L,1} > b_6\} \cup \{T_L = t, b_5 < L_1 < 0, Y_{L,1} > b_6\}.$$

Thus, the probability mass function for $t > 1$ can be written as

$$\begin{aligned} pr_L(t\,|l) &= P(T_L = t, L_1 = 0, Y_{L,1} > b_6\,|L_0 = l) \\ &\quad + P(T_L = t, b_5 < L_1 < 0, Y_{L,1} > b_6\,|L_0 = l). \end{aligned}$$

It then follows that

$$\begin{aligned} pr_L(t\,|l) &= P(T_L - 1 = t-1\,|L_0 = l, L_1 = 0, Y_{L,1} > b_6) \\ &\quad \times P(L_1 = 0, Y_{L,1} > b_6\,|L_0 = l) \\ &\quad + P(T_L - 1 = t-1\,|L_0 = l, b_5 < L_1 < 0, Y_{L,1} > b_6) \\ &\quad \times P(b_5 < L_1 < 0, Y_{L,1} > b_6\,|L_0 = l). \end{aligned}$$

Conditioned on the chart not signaling at time $t = 1$, the random variable $T_L - 1$ is the remaining run length and its distribution given the value of L_1 is the same as the distribution of T_L given the value of L_0. Thus,

$$\begin{aligned} P(T_L - 1 = t-1\,|L_0 = l, L_1 = 0) &= pr_L(t-1\,|0) \text{ and} \\ P(T_L - 1 = t-1\,|L_0 = l, L_1 = l_1) &= pr_L(t-1\,|l_1). \end{aligned}$$

Observe that

$$\begin{aligned} \{L_1 = 0, Y_{L,1} > b_6\} &= \{b_1 l + Y_{L,1} + b_3 \geq 0, b_1 l + Y_{L,1} + b_3 > b_1 l + b_6 + b_3\} \\ &= \{L_1 = 0\}, \end{aligned}$$

since regardless of whether the value of $b_1 l + b_6 + b_3$ is negative, zero, or positive for the event to occur, the value of $b_1 l + Y_{L,1} + b_3$ is nonnegative. We now can write

$$\begin{aligned} pr_L(t\,|l) &= P(T_L - 1 = t-1\,|L_0 = l, L_1 = 0)\, P(L_1 = 0\,|L_0 = l) \\ &\quad + P(T_L - 1 = t-1\,|L_0 = l, b_5 < L_1 < 0, L_1 > b_1 l + b_6 + b_3) \\ &\quad \times P(b_5 < L_1 < 0, L_1 > b_1 l + b_6 + b_3\,|L_0 = l) \\ &= P(T_L - 1 = t-1\,|L_0 = l, L_1 = 0)\, P(L_1 = 0\,|L_0 = l) \\ &\quad + \int_{\max\{b_5, b_1 l + b_6 + b_3\}}^{0} P(T_L - 1 = t-1\,|L_0 = l, L_1 = l_1)\, f_{L_1|L_0}(l_1\,|l)\, dl_1 \\ &= pr_L(t-1\,|0)\, f_{L_1|L_0}(0\,|l) + \int_{\beta_L(l)}^{0} pr_L(t-1\,|l_1)\, f_{L_1|L_0}(l_1\,|l)\, dl_1. \end{aligned}$$

Next, consider the conditional probability mass function describing the distribution of L_1 given $L_0 = l$.

$$
\begin{aligned}
f_{L_1|L_0}\left(l_1\,|l\,\right) &= P\left(L_1 = l_1\,|L_0 = l\,\right) = P\left(\min\left\{0, b_1 l + Y_{L,1} + b_3\right\} = l_1\right) \\
&= \begin{cases} 0, & \text{if } l_1 > 0 \\ P\left(b_1 l + Y_{L,1} + b_3 \geq 0\right), & \text{if } l_1 = 0 \\ P\left(b_1 l + Y_{L,1} + b_3 = l_1\right), & \text{if } l_1 < 0. \end{cases} \\
&= \begin{cases} 0, & \text{if } l_1 > 0 \\ 1 - F_{Y_L}\left(-b_1 l - b_3\right), & \text{if } l_1 = 0 \\ f_{Y_L}\left(l_1 - b_1 l - b_3\right), & \text{if } l_1 < 0. \end{cases} \\
&= \left[1 - F_{Y_L}\left(-b_1 l - b_3\right)\right] I_{\{0\}}\left(l_1\right) + f_{Y_L}\left(l_1 - b_1 l - b_3\right) I_{(-\infty,0)}\left(l_1\right).
\end{aligned}
$$

Note that the argument $l_1 - b_1 l - b_3$ must be in the support of the distribution of Y_L.

Thus, for the case in which the support of the distribution of Y_L is the reals, we can express the probability mass function of the run length distribution conditioned on $L_0 = l$ with $b_5 < l < 0$ as

$$
\begin{aligned}
pr_L\left(1\,|l\,\right) &= F_{Y_L}\left(\beta_L\left(l\right) - b_1 l - b_3\right) \text{ and} \\
pr_L\left(t\,|l\,\right) &= pr_L\left(t-1\,|0\right)\left[1 - F_{Y_L}\left(-b_1 l - b_3\right)\right] \\
&\quad + \int_{\beta_L(l)}^{0} pr_L\left(t-1\,|l_1\,\right) f_{Y_L}\left(l_1 - b_1 l - b_3\right) dl_1.
\end{aligned}
$$

For the case in which the support of the distribution of Y_L is the positive reals, we have

$$
\begin{aligned}
pr_L\left(1\,|l\,\right) &= F_{Y_L}\left(\beta_L\left(l\right) - b_1 l - b_3\right) \text{ and} \\
pr_L\left(t\,|l\,\right) &= pr_L\left(t-1\,|0\right)\left[1 - F_{Y_L}\left(-b_1 l - b_3\right)\right] \\
&\quad + \int_{\beta_L(l)}^{\alpha_L(l)} pr_L\left(t-1\,|l_1\,\right) f_{Y_L}\left(l_1 - b_1 l - b_3\right) dl_1,
\end{aligned}
$$

where $\alpha_L\left(l\right) = \min\left\{0, b_1 l + b_3\right\}$.

Two parameters of the run length distribution that are the most commonly reported are the mean (referred to as the average run length (ARL)) and the standard deviation $(SDRL)$. These parameters are functions of the starting value $L_0 = l$ for lower one-sided chart and $U_0 = u$ for upper one-sided chart. It is convenient to let the $M_L(l)$ represent the ARL of the lower one-sided chart, then

$$M_L(l) = \sum_{t=1}^{\infty} t pr_L(t\,|l\,).$$

We can write $M_L(l)$ as

$$\begin{aligned} M_L(l) &= pr_L(1\,|l\,) + \sum_{t=2}^{\infty} t pr_L(t\,|l\,) \\ &= pr_L(1\,|l\,) + \sum_{t=1}^{\infty} (1+t)\, pr_L(1+t\,|l\,) \\ &= pr_L(1\,|l\,) + \sum_{t=1}^{\infty} pr_L(1+t\,|l\,) + \sum_{t=1}^{\infty} t pr_L(1+t\,|l\,) \\ &= 1 + \sum_{t=1}^{\infty} t pr_L(1+t\,|l\,). \end{aligned}$$

Using previous results, we see that

$$\begin{aligned} pr_L(1+t\,|l\,) &= pr_L(t\,|0)\left[1 - F_{Y_L}(-b_1 l - b_3)\right] \\ &\quad + \int_{\beta_L(l)}^{\alpha_L} pr_L(t\,|l_1\,)\, f_{Y_L}(l_1 - b_1 l - b_3)\, dl_1, \end{aligned}$$

where $\alpha_L = 0$ or $\alpha_L = \alpha_L(l)$ depending on if the support of the distribution of Y_L is the reals or the positive reals. Hence, we have that $M_L(l)$ is the solution to the following integral equation

$$\begin{aligned} M_L(l) &= 1+\sum_{t=1}^{\infty} t\,pr_L(t\,|0)\left[1-F_{Y_L}(-b_1 l-b_3)\right] \\ &\quad +\sum_{t=1}^{\infty} t\int_{\beta_L(l)}^{\alpha_L} pr_L(t\,|l_1)\,f_{Y_L}(l_1-b_1 l-b_3)\,dl_1 \\ &= 1+M_L(0)\left[1-F_{Y_L}(-b_1 l-b_3)\right] \\ &\quad +\int_{\beta_L(l)}^{\alpha_L}\sum_{t=1}^{\infty} t\,pr_L(t\,|l_1)\,f_{Y_L}(l_1-b_1 l-b_3)\,dl_1 \\ &= 1+M_L(0)\left[1-F_{Y_L}(-b_1 l-b_3)\right] \\ &\quad +\int_{\beta_L(l)}^{\alpha_L} M_L(l_1)\,f_{Y_L}(l_1-b_1 l-b_3)\,dl_1. \end{aligned}$$

To determine $SDRL$, we first find

$$M_L^2(l)=E(T_L^2\,|L_0=l\,)=\sum_{t=1}^{\infty} t^2 pr_L(t\,|l\,),$$

which can be written as

$$\begin{aligned} M_L^2(l) &= pr_L(1\,|l\,)+\sum_{t=2}^{\infty} t^2 pr_L(t\,|l\,) \\ &= pr_L(1\,|l\,)+\sum_{t=1}^{\infty}(t+1)^2 pr_L(t+1\,|l\,) \\ &= pr_L(1\,|l\,)+\sum_{t=1}^{\infty}\left(1+2t+t^2\right) pr_L(t+1\,|l\,) \\ &= 1+2\sum_{t=1}^{\infty} t\,pr_L(t+1\,|l\,)+\sum_{t=1}^{\infty} t^2 pr_L(t+1\,|l\,) \\ &= 2M_L(l)-1+\sum_{t=1}^{\infty} t^2 pr_L(t+1\,|l\,). \end{aligned}$$

As we continue, we see that

$$\begin{aligned}
M_L^2(l) &= 2M_L(l) - 1 + \sum_{t=1}^{\infty} t^2 pr_L(t\,|0)\left[1 - F_{Y_L}(-b_1 l - b_3)\right] \\
&\quad + \sum_{t=1}^{\infty} t^2 \int_{\beta_L(l)}^{\alpha_L} pr_L(t\,|l_1)\, f_{Y_L}(l_1 - b_1 l - b_3)\, dl_1 \\
&= 2M_L(l) - 1 + M_L^2(0)\left[1 - F_{Y_L}(-b_1 l - b_3)\right] \\
&\quad + \int_{\beta_L(l)}^{\alpha_L} \sum_{t=1}^{\infty} t^2 pr_L(t\,|l_1)\, f_{Y_L}(l_1 - b_1 l - b_3)\, dl_1 \\
&= 2M_L(l) - 1 + M_L^2(0)\left[1 - F_{Y_L}(-b_1 l - b_3)\right] \\
&\quad + \int_{\beta_L(l)}^{\alpha_L} M_L^2(l_1)\, f_{Y_L}(l_1 - b_1 l - b_3)\, dl_1.
\end{aligned}$$

We now can find the $SDRL$ using the familiar formula

$$\sqrt{M_L^2(l) - (M_L(l))^2}.$$

There are similar integral equations that describe the distribution of the run length T_U of a upper one-sided generalized cumulative sum type chart with plotted statistics defined by

$$U_0 = u \text{ and } U_t = \max\{0, a_1 U_{t-1} + Y_{U,t} + a_3\}$$

with $0 \leq u < a_5$ that signals if $U_t \geq a_5$ or $Y_{U,t} \geq a_6$. We represent the probability mass function describing the conditional distribution of T_U given $U_0 = u$ by

$$pr_U(t\,|u) = P(T_U = t\,|U_0 = u).$$

We see that for $t = 1$,

$$\begin{aligned}
pr_U(1\,|u) &= P(U_1 \geq a_5 \text{ or } Y_{U,1} \geq a_6\,|U_0 = u) \\
&= P(Y_{U,1} \geq a_5 - a_1 u - a_3 \text{ or } Y_{U,1} \geq a_6) \\
&= 1 - F_{Y_U}(\min\{a_5 - a_1 u - a_{3,} a_6\}) \\
&= 1 - F_{Y_U}(\beta_U(u) - a_1 u - a_{3,}),
\end{aligned}$$

where $\beta_U(u) = \min\{a_5, a_1u + a_6 + a_3\}$. If the Shewhart limit is not used, then $\beta_U(u) = a_5$. If $t > 1$, then $0 \le U_1 < a_5$ and $Y_{U,1} < a_6$. Thus, we can express the event $\{T_U = t\}$ as

$$\{T_U = t\} = \{T_U = t, U_1 = 0, Y_{U,1} < a_6\} \cup \{T_U = t, 0 < U_1 < a_5, Y_{U,1} < a_6\}.$$

It follows for $t > 1$ we have that

$$\begin{aligned}
pr_U(t\,|u) &= P(T_U = t, U_1 = 0, Y_{U,1} < a_6\,|U_0 = u) \\
&\quad + P(T_U = t, 0 < U_1 < a_5, Y_{U,1} < a_6\,|U_0 = u) \\
&= P(T_U - 1 = t - 1\,|U_0 = u, U_1 = 0, Y_{U,1} < a_6) \\
&\quad \times P(U_1 = 0, Y_{U,1} < a_6\,|U_0 = u) \\
&\quad + P(T_U - 1 = t - 1\,|U_0 = u, 0 < U_1 < a_5, Y_{U,1} < a_6) \\
&\quad \times P(0 < U_1 < a_5, Y_{U,1} < a_6\,|U_0 = u).
\end{aligned}$$

The random variable $T_U - 1$ is the remaining run length and its distribution given the value of U_1 is the same as the distribution of T_U given the value of U_0. Thus,

$$\begin{aligned}
P(T_U - 1 = t - 1\,|U_0 = u, U_1 = 0) &= pr_U(t - 1\,|0) \text{ and} \\
P(T_U - 1 = t - 1\,|U_0 = u, U_1 = u_1) &= pr_U(t - 1\,|u_1).
\end{aligned}$$

Also observe that

$$\{U_1 = 0, Y_{U,1} < a_6\} = \{a_1u + Y_{U,1} + a_3 \le 0, U_1 < a_1u + a_6 + a_3\} = \{U_1 = 0\}.$$

If $U_1 = 0$, then we have that $a_1u + Y_{U,1} + a_3 \le 0$. Thus, regardless of the value of

$a_1u + a_6 + a_3$, the statistic U_1 would be set to zero. We now can write

$$\begin{aligned} pr_U(t\,|u) &= P(T_U - 1 = t-1\,|U_0 = u, U_1 = 0)\,P(U_1 = 0\,|U_0 = u) \\ &\quad + P(T_U - 1 = t-1\,|U_0 = u, 0 < U_1 < a_5, U_1 < a_1u + a_6 + a_3) \\ &\quad \times P(0 < U_1 < a_5, U_1 < a_1u + a_6 + a_3\,|U_0 = u) \\ &= pr_U(t-1\,|0)\,f_{U_1|U_0}(0\,|u) + \int_0^{\beta_U(u)} pr_U(t-1\,|u_1)\,f_{U_1|U_0}(u_1\,|u)\,du_1, \end{aligned}$$

where $\beta_U(u) = \min\{a_5, a_1u + a_6 + a_3\}$. The conditional probability mass function of U_1 given $U_0 = u$ can be written as follows:

$$\begin{aligned} f_{U_1|U_0}(u_1\,|u) &= P(U_1 = u_1\,|U_0 = u) = P(\max\{0, a_1u + Y_{U,1} + a_3\} = u_1) \\ &= \begin{cases} 0, & \text{if } u_1 < 0 \\ P(a_1u + Y_{U,1} + a_3 \le 0), & \text{if } u_1 = 0 \\ P(a_1u + Y_{U,1} + a_3 = u_1), & \text{if } u_1 > 0\ . \end{cases} \\ &= \begin{cases} 0, & \text{if } u_1 < 0 \\ F_{Y_U}(-a_1u - a_3), & \text{if } u_1 = 0 \\ f_{Y_U}(u_1 - a_1u - a_3), & \text{if } u_1 > 0\ . \end{cases} \\ &= F_{Y_U}(-a_1u - a_3)\,I_{\{0\}}(u_1) + f_{Y_U}(u_1 - a_1u - a_3)\,I_{(0,\infty)}(u_1)\,. \end{aligned}$$

Thus, if the support of the distribution of Y_U is the reals, we have

$$\begin{aligned} pr_U(1\,|u) &= 1 - F_{Y_U}(\beta_U(u) - a_1u - a_3) \text{ and} \\ pr_U(t\,|u) &= pr_U(t-1\,|0)\,F_{Y_U}(-a_1u - a_3) \\ &\quad + \int_0^{\beta_U(u)} pr_U(t-1\,|u_1)\,f_{Y_U}(u_1 - a_1u - a_3)\,du_1. \end{aligned}$$

If the support of the distribution of Y_U is the positive reals, then we have

$$\begin{aligned} pr_U(1\,|u) &= 1 - F_{Y_U}(\beta_U(u) - a_1 u - a_3) \text{ and} \\ pr_U(t\,|u) &= pr_U(t-1\,|0)\, F_{Y_U}(-a_1 u - a_3) \\ &\quad + \int_{\alpha_U(u)}^{\beta_U(u)} pr_U(t-1\,|u_1)\, f_{Y_U}(u_1 - a_1 u - a_3)\, du_1, \end{aligned}$$

where $\alpha_U(u) = \max\{0, a_1 u + a_3\}$.

The average run length $M_U(u)$ of the upper one-sided control chart with $U_0 = u$ is represented by

$$\begin{aligned} M_U(u) &= \sum_{t=1}^{\infty} t pr_U(t\,|u) \\ &= pr_U(1\,|u) + \sum_{t=1}^{\infty} (1+t)\, pr_U(1+t\,|u) \\ &= pr_U(1\,|u) + \sum_{t=1}^{\infty} pr_U(1+t\,|u) + \sum_{t=1}^{\infty} t pr_U(1+t\,|u) \\ &= 1 + \sum_{t=1}^{\infty} t pr_U(1+t\,|u). \end{aligned}$$

Using our previous result, $M_U(u)$ is then given as the solution to the following integral equation

$$\begin{aligned} M_U(u) &= 1 + \sum_{t=1}^{\infty} t pr_U(t\,|0)\, F_{Y_U}(-a_1 u - a_3) \\ &\quad + \sum_{t=1}^{\infty} t \int_{\alpha_U}^{\beta_U(u)} pr_U(t\,|u_1)\, f_{Y_U}(u_1 - a_1 u - a_3)\, du_1 \\ &= 1 + M_U(0)\, F_{Y_U}(-a_1 u - a_3) \\ &\quad + \int_{\alpha_U}^{\beta_U(u)} \sum_{t=1}^{\infty} t pr_U(t\,|u_1)\, f_{Y_U}(u_1 - a_1 u - a_3)\, du_1 \\ &= 1 + M_U(0)\, F_{Y_U}(-a_1 u - a_3) \\ &\quad + \int_{\alpha_U}^{\beta_U(u)} M_U(u_1)\, f_{Y_U}(u_1 - a_1 u - a_3)\, du_1. \end{aligned}$$

where $\alpha_U = 0$ or $\alpha_U = \alpha_U(u)$ depending on if the support of the distribution of Y_U is the reals or the positive reals.

Further, the expected value $M_U^2(u)$ of the square of the run length T_U^2 given $U_0 = u$ is the solution to the following integral equation

$$\begin{aligned}
M_U^2(u) &= \sum_{t=1}^{\infty} t^2 pr_U(t|u) \\
&= pr_U(1|u) + \sum_{t=1}^{\infty} (1+t)^2 pr_U(1+t|u) \\
&= pr_U(1|u) + \sum_{t=1}^{\infty} \left(1+2t+t^2\right) pr_U(1+t|u) \\
&= 1 + 2\sum_{t=1}^{\infty} t pr_U(1+t|u) + \sum_{t=1}^{\infty} t^2 pr_U(1+t|u) \\
&= 2M_U(u) - 1 + \sum_{t=1}^{\infty} t^2 pr_U(1+t|u).
\end{aligned}$$

Using previous results, it follows that

$$\begin{aligned}
M_U^2(u) &= 2M_U(u) - 1 + \sum_{t=1}^{\infty} t^2 pr_U(t|0) F_{Y_U}(-a_1 u - a_3) \\
&\quad + \sum_{t=1}^{\infty} t^2 \int_{\alpha_U}^{\beta_U(u)} pr_U(t|u_1) f_{Y_U}(u_1 - a_1 u - a_3)\, du_1 \\
&= 2M_U(u) - 1 + M_U^2(0) F_{Y_U}(-a_1 u - a_3) \\
&\quad + \int_{\alpha_U}^{\beta_U(u)} \sum_{t=1}^{\infty} t^2 pr_U(t|u_1) f_{Y_U}(u_1 - a_1 u - a_3)\, du_1 \\
&= 2M_U(u) - 1 + M_U^2(0) F_{Y_U}(-a_1 u - a_3) \\
&\quad + \int_{\alpha_U}^{\beta_U(u)} M_U^2(u_1) f_{Y_U}(u_1 - a_1 u - a_3)\, du_1.
\end{aligned}$$

We then can find the $SDRL$ of the upper one-sided chart using

$$\sqrt{M_U^2(u) - (M_U(u))^2}.$$

Exact solutions for the aforementioned integral equations do not exist. On the other hand, very good approximations can be obtained using numerical methods. If the limits of the integral do not depend on u, the integral equation is a Fredholm equation of the second kind. Using a quadrature method to approximate the integral leads to a system of equations whose solution is the desired parameter of the run length distribution. For those equations in which the one or both of the limits of the integral is a function of u, the equations are Volterra equations of the second kind. In this case, the collocation method gives a good approximation of the desired parameter of the run length distribution. The collocation method approximates the desired parameter with a polynomial which is determined exactly by the selected values of u.

2.5 Markov Chain Approximation

When the quality measurement X is a continuous random variable, then the generalized cumulative sum type chart is a continuous state discrete time Markov process. Brook and Evans (1972) presented a method for evaluating the run length distribution of a one-sided CUSUM chart using a Markov chain approximation when the quality measurement is continuous. The run length distribution of the Markov chain is used as an approximation to the run length distribution of the chart. This method has been demonstrated to work quite well in approximating the run length distribution of various other quality control charts. In this section, we will develop Markov chain approximations to both the one- and two-sided generalized control charts. The Shewhart limit discussion is not included in this section. Champ and Rigdon (1992) showed that for the one-sided CUSUM chart the Markov chain and integral equation methods are equivalent in approximating the solution of the integral equations. An integral equation method for two-sided generalized cumulative sum type charts has not been developed. On the other hand, a Markov chain approach has been developed for the two-sided CUSUM chart by Woodall (1984). In this section, we will develop the Markov chain method for both one- and two-sided generalized control charts when the support of the Y is the reals.

First we discuss the lower one-sided chart. We select two sets of numbers $\nu_{L,0}, \nu_{L,1}, \ldots, \nu_{L,\eta_L}$ and $\xi_{L,0}, \xi_{L,1}, \ldots, \xi_{L,\eta_L}$ that have the following constraints:

$$\xi_{L,0} = 0 > \nu_{L,0} > \xi_{L,1} > \nu_{L,1} > \ldots > \nu_{L,\eta_L - 1} > \xi_{L,\eta_L} > \nu_{L,\eta_L} = b_5$$

with η_L a positive integer. We label the values $\xi_{L,0}, \xi_{L,1}, \ldots, \xi_{L,\eta_L}$ as the $\eta_L + 1$ non-absorbing states of the Markov chain L_t^* which will be used to approximate the chart statistic L_t as follows,

$$L_t^* = \begin{cases} \xi_{L,0}, & \text{if } L_t > v_{L,0} \\ \xi_{L,i}, & \text{if } L_t \in (v_{L,i}, v_{L,i-1}] \text{ for } i = 1, 2, \cdots, \eta_L \end{cases} .$$

If $L_t \leq \nu_{L,\eta_L}$, we label the state to be ξ_{L,η_L+1} which is an absorbing state because the chart signals and the sampling stops at this time point.

The method presented in Brook and Evans (1972) for approximating the CUSUM chart as a Markov chain selects these number as

$$\xi_{L,k} = k\left(\frac{b_5}{\eta_L + \frac{1}{2}}\right) \text{ and } \nu_{L,k} = \left(k + \frac{1}{2}\right)\left(\frac{b_5}{\eta_L + \frac{1}{2}}\right)$$

for $k = 0, 1, \ldots, \eta_L$. Another possibility for selecting these values is to choose $\xi_{L,k}$'s as

$$\xi_{L,0} = 0 \text{ and } \xi_{L,k} = b_5 + \frac{-b_5}{2}\left(x_k + 1\right),$$

where $x_1, \ldots, x_{\eta_L}$ are the Legendre quadrature points for $k = 1, \ldots, \eta_L$. We then take the values of the $\nu_{L,k}$'s to be

$$\nu_{L,k} = \frac{\xi_{L,k} + \xi_{L,k+1}}{2} \text{ and } \nu_{L,\eta_L} = b_5$$

for $k = 0, \ldots, \eta_L - 1$.

A third method is useful when the support of the distribution of Y_L is the positive real numbers. The $\xi_{L,k}$'s and the $v_{L,k}$'s are selected to satisfy

$$\begin{aligned}
\xi_{L,0} &= 0 > \nu_{L,0} > \xi_{L,1} > \nu_{L,1} > \ldots > \nu_{L,\eta_{L,1}-1} > \xi_{L,\eta_{L,1}} > \nu_{L,\eta_{L,1}} = -b_3/b_1 \\
&> \xi_{L,\eta_{L,1}+1} > \nu_{L,\eta_{L,1}+1} > \ldots > \nu_{L,\eta_{L,1}+\eta_{L,2}-1} > \xi_{L,\eta_{L,1}+\eta_{L,2}} > \nu_{L,\eta_{L,1}+\eta_{L,2}} = b_5,
\end{aligned}$$

where $\eta_{L,1}$ and $\eta_{L,2}$ are positive integers with $\eta_{L,1}+\eta_{L,2} = \eta_L$. Applying a method similar to Brook and Evans (1972) method, we select

$$\xi_{L,k} = k\left(\frac{-b_3/b_1}{\eta_{L,1}+\frac{1}{2}}\right) \text{ and } \nu_{L,k} = \left(k+\frac{1}{2}\right)\left(\frac{-b_3/b_1}{\eta_{L,1}+\frac{1}{2}}\right),$$

for $k = 0, 1, \ldots, \eta_{L,1}$; and

$$\xi_{L,k} = -\frac{b_3}{b_1} + k\left(\frac{b_5+b_3/b_1}{\eta_{L,2}+\frac{1}{2}}\right) \text{ and } \nu_{L,k} = -\frac{b_3}{b_1} + \left(k+\frac{1}{2}\right)\left(\frac{b_5+b_3/b_1}{\eta_{L,2}+\frac{1}{2}}\right),$$

for $k = 0, 1, \ldots, \eta_{L,2}$.

The nonabsorbing states $\xi_{L,i}$ are numbered i ($i = 0, 1, \ldots, \eta_L$), and the absorbing state ξ_{L,η_L+1} is numbered $\eta_L + 1$. Then the approximate conditional probability of transitioning from the nonabsorbing state k to the nonabsorbing state j is

$$\begin{aligned}
(\mathbf{P}_L)_{k,0} &= P\left(L^*_{t+1} = \xi_{L,0}\,|L^*_t = \xi_{L,k}\right) \\
&= P\left(b_1 L_t + Y_{L,t} + b_3 > \nu_{L,0}\,|L^*_t = \xi_{L,k}\right) \\
&\approx P\left(b_1\xi_{L,k} + Y_{L,t} + b_3 > \nu_{L,0}\right) \\
&= 1 - F_{Y_{L,t}}\left(\nu_{L,0} - b_1\xi_{L,k} - b_3\right), \text{ and} \\
(\mathbf{P}_L)_{k,j} &= P\left(L^*_{t+1} = \xi_{L,j}\,|L^*_t = \xi_{L,k}\right) \\
&= P\left(v_{L,j} < b_1 L_t + Y_{L,t} + b_3 \le v_{L,j-1}\,|L^*_t = \xi_{L,k}\right) \\
&\approx P\left(v_{L,j} < b_1\xi_{L,k} + Y_{L,t} + b_3 \le v_{L,j-1}\right) \\
&= F_{Y_{L,t}}\left(\nu_{L,j-1} - b_1\xi_{L,k} - b_3\right) - F_{Y_{L,t}}\left(\nu_{L,j} - b_1\xi_{L,k} - b_3\right)
\end{aligned}$$

for $j = 1, \ldots, \eta_L$. The approximate probability of transitioning from a nonabsorbing state to the absorbing state is

$$\begin{aligned} (\mathbf{P}_L)_{k,\eta_L+1} &= P\left(L^*_{t+1} = \xi_{L,\eta_L+1} \,|L^*_t = \xi_{L,k}\right) \\ &= P\left(b_1 L_t + Y_{L,t} + b_3 \leq \nu_{L,\eta_L} \,|L^*_t = \xi_{L,k}\right) \\ &\approx P\left(b_1 \xi_{L,k} + Y_{L,t} + b_3 \leq \nu_{L,\eta_L}\right) \\ &= F_{Y_{L,t}}\left(\nu_{L,\eta_L} - b_1 \xi_{L,k} - b_3\right). \end{aligned}$$

The probability of transitioning from the absorbing state to a nonabsorbing state, and from the absorbing state to the absorbing state are given, respectively, by

$$\begin{aligned} (\mathbf{P}_L)_{\eta_L+1,j} &= 0, \text{ and} \\ (\mathbf{P}_L)_{\eta_L+1,\eta_L+1} &= 1. \end{aligned}$$

For the $(\eta_L + 2) \times (\eta_L + 2)$ matrix $\mathbf{P}_L$ whose (k, j)th component is the conditional probability $(\mathbf{P}_L)_{k,j}$ to be a transition matrix, it must be the case that for all $k = 0, 1, \ldots, \eta_L + 1$

$$\sum_{j=0}^{\eta_L+1} (\mathbf{P}_L)_{k,j} = 1.$$

Clearly, this holds for $k = \eta_L + 1$. For the case in which $k = 0, 1, \ldots, \eta_L$, we see that

$$\begin{aligned}
\sum_{j=0}^{\eta_L+1} (\mathbf{P}_L)_{k,j} &= (\mathbf{P}_L)_{k,0} + \sum_{j=1}^{\eta_L} (\mathbf{P}_L)_{k,j} + (\mathbf{P}_L)_{k,\eta_L+1} \\
&= 1 - F_{Y_{L,t}} (\nu_{L,0} - b_1 \xi_{L,k} - b_3) \\
&\quad + \sum_{j=1}^{\eta_L} \left[F_{Y_{L,t}} (\nu_{L,j-1} - b_1 \xi_{L,k} - b_3) - F_{Y_{L,t}} (\nu_{L,j} - b_1 \xi_{L,k} - b_3) \right] \\
&\quad + F_{Y_{L,t}} (\nu_{L,\eta_L} - b_1 \xi_{L,k} - b_3) \\
&= \int_{\nu_{L,0} - b_1 \xi_{L,k} - b_3}^{\infty} f_{Y_{L,t}}(y)\, dy + \sum_{j=1}^{\eta_L} \int_{\nu_{L,j} - b_1 \xi_{L,k} - b_3}^{\nu_{L,j-1} - b_1 \xi_{L,k} - b_3} f_{Y_{L,t}}(y)\, dy \\
&\quad + \int_{-\infty}^{\nu_{L,\eta_L} - b_1 \xi_{L,k} - b_3} f_{Y_{L,t}}(y)\, dy \\
&= \int_{-\infty}^{\infty} f_{Y_{L,t}}(y)\, dy = 1.
\end{aligned}$$

Hence, the matrix $\mathbf{P}_L$ is a transition matrix. Let $\mathbf{Q}_L$ be the $(\eta_L + 1) \times (\eta_L + 1)$ matrix obtained from the transition probability matrix $\mathbf{P}_L$ by deleting the final row and column, it is then not difficult to see that all the information in $\mathbf{P}_L$ is contained in the sub-matrix $\mathbf{Q}_L$.

The number of transitions $T_{L,k}$ of the chain that begins in one of the nonabsorbing states k until it first enters the absorbing state is a random variable known as the run length. By Brook and Evans (1972), the conditional probability mass functions of $T_{L,k}$ $(k = 0, 1, \cdots, \eta_L)$ are determined as

$$\begin{bmatrix} P(T_{L,0} = t \,|L_0 = \xi_{L,0}) \\ P(T_{L,1} = t \,|L_0 = \xi_{L,1}) \\ \vdots \\ P(T_{L,\eta_L} = t \,|L_0 = \xi_{L,\eta_L}) \end{bmatrix} = \mathbf{Q}_L^{t-1} (\mathbf{I} - \mathbf{Q}_L) \mathbf{1}$$

for $t = 1, 2, 3, \ldots$, where $\mathbf{I}$ is the $(\eta + 1) \times (\eta + 1)$ identity matrix and $\mathbf{1}$ is a vector that has each of its $(\eta + 1)$ elements equal to unity. The kth component of $\mathbf{Q}_L^{t-1} (\mathbf{I} - \mathbf{Q}_L) \mathbf{1}$ is then the conditional probability mass function of the run length $T_{L,k}$. The conditional expectations of these $\eta_L + 1$ random variables are determined by

$$\begin{bmatrix} E\left(T_{L,0}\left|L_0 = \xi_{L,0}\right.\right) \\ E\left(T_{L,1}\left|L_0 = \xi_{L,1}\right.\right) \\ \vdots \\ E\left(T_{L,\eta_L}\left|L_0 = \xi_{L,\eta_L}\right.\right) \end{bmatrix} = \sum_{t=1}^{\infty} t\mathbf{Q}_L^{t-1}\left(\mathbf{I}-\mathbf{Q}_L\right)\mathbf{1} = \left(\mathbf{I}-\mathbf{Q}_L\right)^{-1}\mathbf{1}.$$

Further, we see that

$$\begin{aligned} \begin{bmatrix} E\left(T_{L,0}^2\left|L_0 = \xi_{L,0}\right.\right) \\ E\left(T_{L,1}^2\left|L_0 = \xi_{L,1}\right.\right) \\ \vdots \\ E\left(T_{L,\eta_L}^2\left|L_0 = \xi_{L,\eta_L}\right.\right) \end{bmatrix} &= \sum_{t=1}^{\infty} t^2\mathbf{Q}_L^{t-1}\left(\mathbf{I}-\mathbf{Q}_L\right)\mathbf{1}, \\ &= \sum_{t=1}^{\infty} t^2\mathbf{Q}_L^{t-1}\mathbf{1} + \sum_{t=1}^{\infty} t^2\mathbf{Q}_L^{t}\mathbf{1} \\ &= \left(\mathbf{I}-\mathbf{Q}_L\right)^{-2}\mathbf{1} + \mathbf{Q}_L\left(\mathbf{I}-\mathbf{Q}_L\right)^{-2}\mathbf{1} \\ &= \left(\mathbf{I}+\mathbf{Q}_L\right)\left(\mathbf{I}-\mathbf{Q}_L\right)^{-2}\mathbf{1}. \end{aligned}$$

Hence, one can find the standard deviations of the run length distributions using the well known formula

$$\sqrt{E\left(T_{L,k}^2\left|L_0 = \xi_{L,k}\right.\right) - \left[E\left(T_{L,k}\left|L_0 = \xi_{L,k}\right.\right)\right]^2}$$

for $k = 0, 1, 2, \ldots, \eta_L$.

For the upper one-sided chart, the values $\nu_{U,0}, \nu_{U,1}, \ldots, \nu_{U,\eta_U}$ and $\xi_{U,0}, \xi_{U,1}, \ldots, \xi_{U,\eta_U}$ are selected with the following constraints:

$$\xi_{U,0} = 0 < \nu_{U,0} < \xi_{U,1} < \nu_{U,1} < \ldots < \nu_{U,\eta_U - 1} < \xi_{U,\eta_U} < \nu_{U,\eta_U} = a_5$$

with η_U a positive integer. The values $\xi_{U,0}, \xi_{U,1}, \ldots, \xi_{U,\eta_L}$ are the $\eta_U + 1$ nonabsorbing states of the Markov chain U_t^* that is used to approximate of the chart statistic U_t,

$$U_t^* = \begin{cases} \xi_{U,0}, & \text{if } U_t < v_{U,0} \\ \xi_{U,i}, & \text{if } U_t \in [v_{U,i-1}, v_{U,i}), (i = 1, 2, \cdots, \eta_L) \end{cases}.$$

If $U_t \geq \nu_{U,\eta_u}$, the state is set to be ξ_{U,η_U+1} which is an absorbing state.

Brook and Evans (1972) used the values

$$\xi_{U,k} = k\left(\frac{a_5}{\eta_U + \frac{1}{2}}\right) \text{ and } \nu_{U,k} = \left(k + \frac{1}{2}\right)\left(\frac{a_5}{\eta_U + \frac{1}{2}}\right)$$

when approximating the upper one-sided CUSUM chart as a Markov chain for $k = 0, 1, \ldots, \eta_U$. Another method for the selection of these values is to choose $\xi_{L,k}$'s as

$$\xi_{U,0} = 0 \text{ and } \xi_{U,k} = a_5 + \frac{-a_5}{2}\left(x_k + 1\right),$$

where $x_1, \ldots, x_{\eta_U}$ are the Legendre quadrature points for $k = 1, \ldots, \eta_U$. The values of the $\nu_{U,k}$'s are then taken to be

$$\nu_{U,k} = \frac{\xi_{U,k} + \xi_{U,k+1}}{2} \text{ and } \nu_{U,\eta_U} = a_5$$

for $k = 0, \ldots, \eta_U - 1$.

As with the lower one-sided chart, a third method is useful when the support of the distribution of Y_U is the positive reals. The $\xi_{U,k}$'s and the $v_{U,k}$'s are selected as

$$\begin{aligned} \xi_{U,0} &= 0 < \nu_{U,0} < \xi_{U,1} < \nu_{U,1} < \ldots < \nu_{U,\eta_{U,1}-1} < \xi_{U,\eta_{U,1}} < \nu_{U,\eta_{U,1}} = -a_3/a_1 \\ &< \xi_{U,\eta_{U,1}+1} < \nu_{U,\eta_{U,1}+1} < \ldots < \nu_{U,\eta_{U,1}+\eta_{U,2}-1} < \xi_{U,\eta_{U,1}+\eta_{U,2}} < \nu_{U,\eta_{U,1}+\eta_{U,2}} = a_5, \end{aligned}$$

where $\eta_{U,1}$ and $\eta_{U,2}$ are positive integers with $\eta_{U,1} + \eta_{U,2} = \eta_U$. Then we select

$$\xi_{U,k} = k\left(\frac{-a_3/a_1}{\eta_{U,1} + \frac{1}{2}}\right) \text{ and } \nu_{U,k} = \left(k + \frac{1}{2}\right)\left(\frac{-a_3/a_1}{\eta_{U,1} + \frac{1}{2}}\right),$$

for $k = 0, 1, \ldots, \eta_{U,1}$; and

$$\xi_{U,k} = -\frac{a_3}{a_1} + k\left(\frac{a_5 + a_3/a_1}{\eta_{U,2} + \frac{1}{2}}\right) \text{ and } \nu_{L,k} = -\frac{a_3}{a_1} + \left(k + \frac{1}{2}\right)\left(\frac{a_5 + a_3/a_1}{\eta_{U,2} + \frac{1}{2}}\right),$$

for $k = 0, 1, \ldots, \eta_{U,2}$.

Similar to the approximation to the lower one-sided chart, the nonabsorbing states are numbered $0, 1, \ldots, \eta_U$, and the absorbing state is numbered $\eta_U + 1$. The approximate conditional transitioning probability from the nonabsorbing state k to the nonabsorbing state j is given by

$$\begin{aligned}
(\mathbf{P}_U)_{k,0} &= P\left(U^*_{t+1} = \xi_{U,0} \,|U^*_t = \xi_{U,k}\right) \\
&= P\left(a_1 U_t + Y_{U,t} + a_3 < v_{U,0} \,|U^*_t = \xi_{U,k}\right) \\
&\approx P\left(a_1 \xi_{U,k} + Y_{U,t} + a_3 < \nu_{U,0}\right) \\
&= F_{Y_{U,t}}\left(\nu_{U,0} - a_1 \xi_{U,k} - a_3\right), \text{ and} \\
(\mathbf{P}_U)_{k,j} &= P\left(U^*_{t+1} = \xi_{U,j} \,|U^*_t = \xi_{U,k}\right) \\
&= P\left(v_{U,j-1} \le a_1 U_t + Y_{U,t} + a_3 < v_{U,j} \,|U^*_t = \xi_{U,k}\right) \\
&\approx P\left(\nu_{U,j-1} \le a_1 \xi_{U,k} + Y_{U,t} + a_3 < \nu_{U,j}\right) \\
&= F_{Y_{U,t}}\left(\nu_{U,j} - a_1 \xi_{U,k} - a_3\right) - F_{Y_{U,t}}\left(\nu_{U,j-1} - a_1 \xi_{U,k} - a_3\right)
\end{aligned}$$

for $j = 1, \ldots, \eta_U$. The probability of transitioning from a nonabsorbing state to the absorbing state is

$$\begin{aligned}
(\mathbf{P}_U)_{k,\eta_U+1} &= P\left(U^*_{t+1} = \xi_{U,\eta_U+1} \,|U^*_t = \xi_{U,k}\right) \\
&= P\left(a_1 U_t + Y_{U,t} + a_3 \ge \nu_{U,\eta_u} \,|U^*_t = \xi_{U,k}\right) \\
&\approx P\left(a_1 \xi_{U,k} + Y_{U,t} + a_3 \ge \nu_{U,\eta_u}\right) \\
&= 1 - F_{Y_{U,t}}\left(\nu_{U,\eta_U} - a_1 \xi_{U,k} - a_3\right),
\end{aligned}$$

And the probability of transitioning from the absorbing state to absorbing state, and from the absorbing state to the absorbing state are given, respectively, by

$$\begin{aligned}
(\mathbf{P}_U)_{\eta_U+1,j} &= 0 \text{ ; and} \\
(\mathbf{P}_U)_{\eta_U+1,\eta_U+1} &= 1.
\end{aligned}$$

The $(\eta_U + 2) \times (\eta_U + 2)$ matrix $\mathbf{P}_U$ whose (k, j)th component is the conditional probability $(\mathbf{P}_U)_{k,j}$ is a transition matrix. Clearly, for $k = \eta_U + 1$, we have

$$\sum_{j=0}^{\eta_U+1} (\mathbf{P}_U)_{k,j} = 1;$$

and for $k = 0, 1, \ldots, \eta_U$, we have

$$\begin{aligned}
\sum_{j=0}^{\eta_L+1} (\mathbf{P}_U)_{k,j} &= (\mathbf{P}_U)_{k,0} + \sum_{j=1}^{\eta_U} (\mathbf{P}_U)_{k,j} + (\mathbf{P}_U)_{k,\eta_L+1} \\
&= F_{Y_{U,t}} (\nu_{U,0} - a_1\xi_{U,k} - a_3) \\
&\quad + \sum_{j=1}^{\eta_U} \left[F_{Y_{U,t}} (\nu_{U,j} - a_1\xi_{U,k} - a_3) - F_{Y_{U,t}} (\nu_{U,j-1} - a_1\xi_{U,k} - a_3) \right] \\
&\quad + 1 - F_{Y_{U,t}} (\nu_{U,\eta_U} - a_1\xi_{U,k} - a_3) \\
&= \int_{\nu_{U,0}-a_1\xi_{U,k}-a_3}^{-\infty} f_{Y_{U,t}} (y)\, dy + \sum_{j=1}^{\eta_U} \int_{\nu_{U,j}-a_1\xi_{U,k}-a_3}^{\nu_{U,j-1}-a_1\xi_{U,k}-a_3} f_{Y_{U,t}} (y)\, dy \\
&\quad + \int_{\infty}^{\nu_{U,\eta_U}-a_1\xi_{U,k}-a_3} f_{Y_{U,t}} (y)\, dy \\
&= \int_{\infty}^{-\infty} f_{Y_{U,t}} (y)\, dy = 1.
\end{aligned}$$

All the information contained in $\mathbf{P}_U$ can be obtained from the $(\eta_U + 1) \times (\eta_U + 1)$ matrix $\mathbf{Q}_U$ by excluding the last row and last column of $\mathbf{P}_U$. The conditional probability mass function of the run length $T_{U,k}$ when the chain begins in one of the nonabsorbing states

k is then determined as

$$\begin{bmatrix} P\left(T_{U,0} = t \left| U_0 = \xi_{U,0}\right.\right) \\ P\left(T_{U,1} = t \left| U_0 = \xi_{U,1}\right.\right) \\ \vdots \\ P\left(T_{U,\eta_U} = t \left| U_0 = \xi_{U,\eta_U}\right.\right) \end{bmatrix} = \mathbf{Q}_U^{t-1}\left(\mathbf{I} - \mathbf{Q}_U\right)\mathbf{1},$$

for $t = 1, 2, 3, \ldots$, where the kth component is the conditional probability mass function of the run length $T_{U,k}$. The conditional expectations of these $\eta_U + 1$ random variables are given by

$$\begin{bmatrix} E\left(T_{U,0} \left| U_0 = \xi_{U,0}\right.\right) \\ E\left(T_{U,1} \left| U_0 = \xi_{U,1}\right.\right) \\ \vdots \\ E\left(T_{U,\eta_U} \left| U_0 = \xi_{U,\eta_U}\right.\right) \end{bmatrix} = \sum_{t=1}^{\infty} t\mathbf{Q}_U^{t-1}\left(\mathbf{I} - \mathbf{Q}_U\right)\mathbf{1} = \left(\mathbf{I} - \mathbf{Q}_U\right)^{-1}\mathbf{1}.$$

And we have

$$\begin{bmatrix} E\left(T_{U,0}^2 \left| L_0 = \xi_{U,0}\right.\right) \\ E\left(T_{U,1}^2 \left| L_0 = \xi_{U,1}\right.\right) \\ \vdots \\ E\left(T_{U,\eta_U}^2 \left| L_0 = \xi_{U,\eta_U}\right.\right) \end{bmatrix} = \sum_{t=1}^{\infty} t^2\mathbf{Q}_U^{t-1}\left(\mathbf{I} - \mathbf{Q}_U\right)\mathbf{1} = \left(\mathbf{I} + \mathbf{Q}_U\right)\left(\mathbf{I} - \mathbf{Q}_U\right)^{-2}\mathbf{1}.$$

Then the standard deviations of the run length distributions can be obtained by the formula

$$\sqrt{E\left(T_{U,k}^2 \left| U_0 = \xi_{U,k}\right.\right) - \left[E\left(T_{U,k} \left| U_0 = \xi_{U,k}\right.\right)\right]^2}$$

for $k = 0, 1, 2, \ldots, \eta_U$.

The Markov chain approximation of a two-sided generalized cumulative sum type chart builds on the methods used to construct the Markov chain approximations of the one-sided charts. A nonabsorbing state of the two-sided chart is expressed as the order pair $(\xi_{L,i_L}, \xi_{U,i_U})$ of nonabsorbing states of the lower and upper one-sided charts, respectively, for $i_L = 0, 1, \ldots, \eta_L$ and $i_U = 0, 1, \ldots, \eta_U$. If either ξ_{L,i_L} or ξ_{U,i_U} is the absorbing state, i.e. $i_L = \eta_L + 1$ or $i_U = \eta_U + 1$, the order pair $(\xi_{L,i_L}, \xi_{U,i_U})$ is an absorbing state of the two-sided chart. There are $(\eta_L + 2) \times (\eta_U + 2)$ states of a two-sided chart and the transition matrix is the $[(\eta_L + 2) \times (\eta_U + 2)]^2$ matrix $\mathbf{P}$ with (i, j)th component $P_{i,j}$ being determined by

$$P_{i,j} = P\left[(\xi_{L,i_L}, \xi_{U,i_U}) \to (\xi_{L,j_L}, \xi_{U,j_U})\right]$$

with $i = i_L(\eta_U + 1) + i_U$ and $j = j_L(\eta_U + 1) + j_U$, where $i_L, j_L = 0, 1, \ldots, \eta_L, \eta_L + 1$ and $i_U, j_U = 0, 1, \ldots, \eta_U, \eta_U + 1$.

Since all the information we acquire from transition matrix $\mathbf{P}$ can be obtained by working with $\mathbf{Q}$ which is the sub-matrix of $\mathbf{P}$ by excluding the rows and columns that respond to the absorbing states, we here only consider the transition probabilities among nonabsorbing states. To determine these probabilities, some special cases must be considered when $j_L = 0$ and/or $j_U = 0$. We first consider the case in which both $j_L = 0$ and $j_U = 0$. In this case, we are interested in the probability

$$P_{i,0} = P\left[(\xi_{L,i_L}, \xi_{U,i_U}) \to (\xi_{L,0}, \xi_{U,0})\right].$$

It follows that

$$\begin{aligned}
P_{i,0} &= P\left(L^*_{t+1} = \xi_{L,0}, U^*_{t+1} = \xi_{U,0} \,|L^*_t = \xi_{L,i_L}, U^*_t = \xi_{U,i_U}\right) \\
&\approx P\left(b_1\xi_{L,i_L} + Y_{L,t} + b_3 > v_{L,0}, a_1\xi_{U,i_U} + Y_{U,t} + a_3 < v_{U,0}\right) \\
&= P\left(Y_{L,t} > \nu_{L,0} - b_1\xi_{L,i_L} - b_3, Y_{U,t} < \nu_{U,0} - a_1\xi_{U,i_U} - a_3\right).
\end{aligned}$$

Next, we examine the case in which $j_L = 0$ and $j_U \neq 0$. That is,

$$\begin{aligned}
P_{i,j_U} &= P\left[\left(\xi_{L,i_L}, \xi_{U,i_U}\right) \to \left(\xi_{L,0}, \xi_{U,j_U}\right)\right] \\
&= P\left(L^*_{t+1} = \xi_{L,0}, U^*_{t+1} = \xi_{U,j_U} \left| L^*_t = \xi_{L,i_L}, U^*_t = \xi_{U,i_U}\right.\right) \\
&\approx P\left(b_1\xi_{L,i_L} + Y_{L,t} + b_3 > \nu_{L,0}, \nu_{U,j_U-1} \leq a_1\xi_{U,i_U} + Y_{U,t} + a_3 < \nu_{U,j_U}\right) \\
&= P\begin{pmatrix} Y_{L,t} > \nu_{L,0} - b_1\xi_{L,i_L} - b_3 \\ \nu_{U,j_U-1} - a_1\xi_{U,i_U} - a_3 \leq Y_{U,t} < \nu_{U,j_U} - a_1\xi_{U,i_U} - a_3 \end{pmatrix}.
\end{aligned}$$

Our third case is concerned with $j_L \neq 0$ and $j_U = 0$, in which $j = j_L\left(\eta_U + 1\right)$ and the transition probability is

$$\begin{aligned}
P_{i,j_L(\eta_U+1)} &= P\left[\left(\xi_{L,i_L}, \xi_{U,i_U}\right) \to \left(\xi_{L,j_L}, \xi_{U,0}\right)\right] \\
&= P\left(L^*_{t+1} = \xi_{L,j_L}, U^*_{t+1} = \xi_{U,0} \left| L^*_t = \xi_{L,i_L}, U^*_t = \xi_{U,i_U}\right.\right) \\
&\approx P\left(\nu_{L,j_L} < b_1\xi_{L,i_L} + Y_{L,t} + b_3 \leq \nu_{L,j_L-1}, a_1\xi_{U,i_U} + Y_{U,t} + a_3 < \nu_{U,0}\right) \\
&= P\begin{pmatrix} \nu_{L,j_L} - b_1\xi_{L,i_L} - b_3 < Y_{L,t} \leq \nu_{L,j_L-1} - b_1\xi_{L,i_L} - b_3 \\ Y_{U,t} \leq \nu_{U,0} - a_1\xi_{U,i_U} - a_3 \end{pmatrix}.
\end{aligned}$$

If both $j_L \neq 0$ and $j_U \neq 0$, the transition probability is then given by

$$\begin{aligned}
P_{i,j} &= P\left[\left(\xi_{L,i_L}, \xi_{U,i_U}\right) \to \left(\xi_{L,j_L}, \xi_{U,j_U}\right)\right] \\
&= P\left(v_{L,j_L} < L_{t+1} \leq v_{L,j_L-1}, \nu_{U,j_U-1} \leq U_{t+1} < \nu_{U,j_U} \left| L^*_t = \xi_{L,i_L}, U^*_t = \xi_{U,i_U}\right.\right) \\
&\approx P\left(\nu_{L,j_L} < b_1\xi_{L,i_L} + Y_{L,t} + b_3 \leq \nu_{L,j_L-1}, \nu_{U,j_U-1} \leq a_1\xi_{U,i_U} + Y_{U,t} + a_3 < \nu_{U,j_U}\right) \\
&= P\begin{pmatrix} \nu_{L,j_L} - b_1\xi_{L,i_L} - b_3 < Y_{L,t} \leq \nu_{L,j_L-1} - b_1\xi_{L,i_L} - b_3, \\ \nu_{U,j_U-1} - a_1\xi_{U,i_U} - a_3 \leq Y_{U,t} < \nu_{U,j_U} - a_1\xi_{U,i_U} - a_3 \end{pmatrix}.
\end{aligned}$$

Let

$$\begin{aligned}
c_0 &= \nu_{L,0} - b_1\xi_{L,i_L} - b_3,\ d_0 = \nu_{U,0} - a_1\xi_{U,i_U} - a_3, \\
c_L &= \nu_{L,j_L} - b_1\xi_{L,i_L} - b_3,\ d_L = \nu_{L,j_L-1} - b_1\xi_{L,i_L} - b_3, \\
c_U &= \nu_{U,j_U-1} - a_1\xi_{U,i_U} - a_3,\ d_U = \nu_{U,j_U} - a_1\xi_{U,i_U} - a_3,
\end{aligned}$$

then when $Y_{L,t} = Y_{U,t} = Y_t$, the transition probabilities among the nonabsorbing states will be expressed as

$$P_{i,j} = \begin{cases} P\left(c_0 \leq Y_t \leq d_0\right), & \text{if } j_L = 0 \text{ and } j_U = 0; \\ P\left(\max\left\{c_0, c_U\right\} < Y_t < d_U\right), & \text{if } j_L = 0 \text{ and } j_U \neq 0; \\ P\left(d_L \leq Y_t < \min\left\{c_L, d_0\right\}\right), & \text{if } j_L \neq 0 \text{ and } j_U = 0; \\ P\left(\max\left\{c_0, c_U\right\} < Y_t < \min\left\{c_L, d_0\right\}\right), & \text{if } j_L \neq 0 \text{ and } j_U \neq 0; \end{cases}$$

where $i = i_L\left(\eta_U + 1\right) + i_U$ and $j = j_L\left(\eta_U + 1\right) + j_U$ with $i_L, j_L \in \{0, 1, \ldots, \eta_L\}$ and $i_U, j_U \in \{0, 1, \ldots, \eta_U\}$.

The run length when the chain starts in one of the nonabsorbing states $(\xi_{L,i}, \xi_{U,j})$ ($i = 0, 1, \ldots, \eta_L$ and $j \in 0, 1, \ldots, \eta_U$) is $T_{i(\eta_U+1)+j}$ of which the conditional probability mass functions are determined as

$$\begin{bmatrix} P\left(T_0 = t\left|(L_0, U_0) = (\xi_{L,0}, \xi_{U,0}\right)\right) \\ P\left(T_1 = t\left|(L_0, U_0) = (\xi_{L,0}, \xi_{U,1}\right)\right) \\ \vdots \\ P\left(T_\eta = t\left|(L_0, U_0) = (\xi_{L,\eta_L}, \xi_{U,\eta_U}\right)\right) \end{bmatrix} = \mathbf{Q}^{t-1}\left(\mathbf{I} - \mathbf{Q}\right)\mathbf{1},$$

for $t = 1, 2, 3, \ldots$, where the kth component is the conditional probability mass function of the run length T_k and $\eta = \left(\eta_L + 1\right)\left(\eta_U + 1\right) - 1$. The conditional expectations of T_k are determined by

$$\begin{bmatrix} E\left(T_0\left|(L_0, U_0) = (\xi_{L,0}, \xi_{U,0}\right)\right) \\ E\left(T_1\left|(L_0, U_0) = (\xi_{L,0}, \xi_{U,1}\right)\right) \\ \vdots \\ E\left(T_\eta\left|(L_0, U_0) = (\xi_{L,\eta_L+1}, \xi_{U,\eta_U+1}\right)\right) \end{bmatrix} = \sum_{t=1}^{\infty} t\mathbf{Q}^{t-1}\left(\mathbf{I} - \mathbf{Q}\right)\mathbf{1} = \left(\mathbf{I} - \mathbf{Q}\right)^{-1}\mathbf{1}.$$

And the conditional expectations of T_k^2 are computed as

$$\begin{bmatrix} E\left(T_0^2\,|(L_0,U_0)=(\xi_{L,0},\xi_{U,0})\right) \\ E\left(T_1^2\,|(L_0,U_0)=(\xi_{L,0},\xi_{U,1})\right) \\ \vdots \\ E\left(T_\eta^2\,|(L_0,U_0)=(\xi_{L,\eta_L+1},\xi_{U,\eta_U+1})\right) \end{bmatrix} = \sum_{t=1}^{\infty} t^2\mathbf{Q}^{t-1}\left(\mathbf{I}-\mathbf{Q}\right)\mathbf{1} = \left(\mathbf{I}+\mathbf{Q}\right)\left(\mathbf{I}-\mathbf{Q}\right)^{-2}\mathbf{1}.$$

Similarly, the standard deviations of the run length distributions are then given by

$$\sqrt{E\left(T_k^2\,|(L_0,U_0)=(\xi_{L,i},\xi_{U,j})\right)-\left[E\left(T_k\,|(L_0,U_0)=(\xi_{L,i},\xi_{U,j})\right)\right]^2},$$

where $k=i\left(\eta_U+1\right)+j$ for $i=0,1,\ldots,\eta_L$ and $j=0,1,\ldots,\eta_U$.

2.6 Conclusion

It was show that the generalized family of cumulative sum type charts introduced by Champ, Woodall, and Mohsen (1991) requires two less parameters to be specified by the practitioner. The run length performance of these charts can be studied using simulation, integral equations and a Markov chain approximation. We have given integral equations useful in determining the run length distribution of the lower and upper one-sided charts. The Markov chain methods for the one- and two-sided charts were given.

CHAPTER 3

MONITORING FOR A CHANGE IN THE PROCESS MEAN

3.1 Introduction

One of the most common uses of control charting procedures is to monitor for the mean of a continuous quality measurement X on the output of a production process. In this setting, the parameter μ is often referred to as the process mean. We make the assumption that X has a normal distribution with mean μ and standard deviation σ. This is a common assumption in control chart development when the quality measurement is a continuous random variable. The process is assumed to be statistically in a state of in control if $\mu = \mu_0$ and $\sigma = \sigma_0$ for fixed constants μ_0 and σ_0. The data available to the practitioner to make a decision about the quality of the process comes in the form of (1) the X measurements $\{X_{i,1}, \ldots, X_{i,n}\}$ on m samples each of size n from the output of the process of a Phase I study for $i = 1, \ldots, m$ and (2) the X measurements $\{X_{t,1}, \ldots, X_{t,n}\}$ taken from fixed periods on the output of the process in Phase II for $t = 1, 2, 3, \ldots$. We assume these measurements are independent. It is our interest to study the use of a generalized cumulative sum type chart for monitoring for a change in the process mean μ from its in-control value μ_0.

In the next section, we will discuss our estimates for the in-control values μ_0 and σ_0. The generalized cumulative sum type $\overline{X}$ chart with estimated parameters will be introduced. An outline is given in Section 3 for using integral equations to evaluate the run length distribution of the one-sided generalized cumulative sum type $\overline{X}$ charts. In Section 4, a run length performance analysis of a two-sided generalized cumulative sum type $\overline{X}$ chart is given using a Markov chain approximation. Some conclusions are given in the final section.

3.2 Parameters Estimated Version

The in-control parameters are typically not known and must be estimated. During Phase I in which the practitioner brings the process into a state of statistical in-control, we assume there will be available from this phase m independent random samples each of size n when the process is believed to be in a state of statistical in control. We represent these data as $X_{i,1}, \ldots, X_{i,n}$ for $i = 1, \ldots, m$. From these data, we will estimate μ_0 and σ_0^2 respectively with the statistics

$$\overline{\overline{X}} = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{i,j} \text{ and } \overline{V} = \frac{1}{m(n-1)} \sum_{i=1}^{m} \sum_{j=1}^{n} \left(X_{i,j} - \overline{X}_i \right)^2,$$

where $\overline{X}_i$ is the mean of the ith sample. It is not difficult to show under the independent normal model that

$$\overline{\overline{X}} \sim N\left(\mu_0, \frac{\sigma_0^2}{mn}\right) \text{ and } \frac{m(n-1)\overline{V}}{\sigma_0^2} \sim \chi^2_{m(n-1)}$$

are independent. Also, we have that $E\left(\overline{\overline{X}}\right) = \mu_0$ and $E\left(\overline{V}\right) = \sigma_0^2$.

A parameters estimated version of the lower and upper one-sided generalized cumulative sum type $\overline{X}$ control chart defines L_t and U_t by

$$L_0 = b_4,\ L_t = \max\{0, b_1 L_{t-1} + Y_t + b_3\}$$

$$U_0 = a_4, \text{ and } U_t = \max\{0, a_1 U_{t-1} + Y_t + a_3\},$$

where

$$Y_t = \frac{\overline{X}_t - \overline{\overline{X}}}{\overline{V}^{1/2}/\sqrt{n}}.$$

The two-sided chart signals if $L_t \leq b_5$, or $Y_t \leq b_6$, or $U_t \geq a_5$, or $Y_t \geq a_6$.

We note that $\overline{V}^{1/2}$ is a biased estimator of σ_0 and could be replaced with the unbiased estimator $\overline{V}^{1/2}/c$, where c is an unbiasing constant that depends only on m and n. This is not necessary because the chart can be designed so that the unbiased constant c is absorbed into the chart parameters. Note that

$$\frac{\overline{X}_t - \overline{\overline{X}}}{\overline{V}^{1/2}/\sqrt{n}} = \frac{1}{\overline{V}^{1/2}/\sigma_0}\left(\frac{\sigma}{\sigma_0}\frac{\overline{X}_t - \mu}{\sigma/\sqrt{n}} + \sqrt{n}\frac{\mu-\mu_0}{\sigma_0} - \frac{\overline{\overline{X}} - \mu_0}{\sigma_0/\sqrt{mn}}/\sqrt{m}\right).$$

Letting

$$Z_0 = \frac{\overline{\overline{X}} - \mu_0}{\sigma_0/\sqrt{mn}} \text{ and } W_0 = \frac{\overline{V}^{1/2}}{\sigma_0},$$

we can write

$$\frac{\overline{X}_t - \overline{\overline{X}}}{\overline{V}^{1/2}/\sqrt{n}} = W_0^{-1}\left(\lambda Z_t + \sqrt{n}\delta - Z_0/\sqrt{m}\right).$$

Under the independent normal model, we have that

$$Z_0 \sim N(0,1) \text{ and } m(n-1)\,W_0^2 \sim \chi^2_{m(n-1)}$$

are independent. The values of L_t and U_t can now be expressed as

$$\begin{aligned}
&L_0 = b_4,\ L_t = \max\left\{0, b_1 L_{t-1} + W_0^{-1}\left(\lambda Z_t + \sqrt{n}\delta - Z_0/\sqrt{m}\right) + b_3\right\}\\
&U_0 = a_4, \text{ and } U_t = \max\left\{0, a_1 U_{t-1} + W_0^{-1}\left(\lambda Z_t + \sqrt{n}\delta - Z_0/\sqrt{m}\right) + a_3\right\},
\end{aligned}$$

which give us a clearer picture of how these statistics behave stochastically. Note that by setting $Z_0 = 0$ and $W_0 = 1$, we obtain the statistics used as the in-control process parameters μ_0 and σ_0 are known.

In what follows, the joint distribution of Z_0 and W_0 will be needed. From our previous results, we find that

$$\begin{aligned}
F_{W_0}(w_0) &= P(W_0 \le w_0) = P\left(m(n-1)\,W_0^2 \le m(n-1)\,w_0^2\right)\\
&= P\left(\chi^2_{m(n-1)} \le m(n-1)\,w_0^2\right) = F_{\chi^2_{m(n-1)}}\left(m(n-1)\,w_0^2\right).
\end{aligned}$$

Hence the probability density function describing the distribution of W_0 is given by

$$f_{W_0}(w_0) = 2m(n-1)\, w_0 f_{\chi^2_{m(n-1)}}\left(m(n-1)\, w_0^2\right).$$

Since $Z_0 \sim N(0,1)$ and W_0 are independent, the joint probability density function describing their joint distribution is

$$\begin{aligned} f_{Z_0,W_0}(z_0, w_0) &= f_{Z_0}(z_0)\, f_{W_0}(w_0) = 2\nu w_0 \phi(z_0)\, f_{\chi^2_{m(n-1)}}\left(\nu w_0^2\right) \\ &= \frac{2}{\sqrt{2\pi}\Gamma\left(\frac{\nu}{2}\right)\left(\frac{2}{\nu}\right)^{\nu}} e^{-z_0^2/2} w_0^{\nu-1} e^{-\left(w_0^2/\nu\right)/2}, \end{aligned}$$

where $\nu = m(n-1)$ and $\phi(z_0) = f_{Z_0}(z_0)$ is the probability density function of a standard normal distribution.

3.3 Integral Equations Approach

Assume for both the lower and upper one-sided charts, the Y statistics are the same. In particular, we have

$$Y_{L,t} = Y_{U,t} = Y_t = W_0^{-1}\left(\lambda Z_t + \sqrt{n}\delta - Z_0/\sqrt{m}\right).$$

Using the results from the previous chapter, the probability mass functions of the run lengths of lower and upper one-sided charts given $Z_0 = z_0$ and $W_0 = w_0$ are

$$\begin{aligned} pr_L(1\,|l, z_0, w_0) &= G_{L,z_0,w_0}(b_5, l)\ , \\ pr_L(t\,|l, z_0, w_0) &= pr_L(t-1\,|0, z_0, w_0)\left[1 - G_{L,z_0,w_0}(0, l)\right] \\ &\quad + \int_{b_5}^{0} pr_L(t-1\,|l_1, z_0, w_0)\, g_{L,z_0,w_0}(l_1, l)\,; \end{aligned}$$

and

$$\begin{aligned} pr_U(1\,|u, z_0, w_0) &= 1 - G_{U,z_0,w_0}(a_5, u)\,, \\ pr_U(t\,|u, z_0, w_0) &= pr_U(t-1\,|0, z_0, w_0)\, G_{U,z_0,w_0}(0, u) \\ &\quad + \int_0^{a_5} pr_U(t-1\,|u_1, z_0, w_0)\, g_{U,z_0,w_0}(u_1, u)\, du_1; \end{aligned}$$

where

$$G_{L,z_0,w_0}(y,l) = \Phi\left(\frac{w_0(y - b_1 l - b_3) - \sqrt{n}\delta + z_0/\sqrt{m}}{\lambda}\right),$$
$$g_{L,z_0,w_0}(l_1,l) = \phi\left(\frac{w_0(l_1 - b_1 l - b_3) - \sqrt{n}\delta + z_0/\sqrt{m}}{\lambda}\right);$$

and

$$G_{U,z_0,w_0}(y,u) = \Phi\left(\frac{w_0(y - a_1 u - a_3) - \sqrt{n}\delta + z_0/\sqrt{m}}{\lambda}\right),$$
$$g_{U,z_0,w_0}(u_1,u) = \phi\left(\frac{w_0(u_1 - a_1 u - a_3) - \sqrt{n}\delta + z_0/\sqrt{m}}{\lambda}\right).$$

Note we do not consider the Shewhart limit here. The unconditional probability mass functions are obtained as

$$\begin{aligned} pr_L(t\,|l) &= \int_0^\infty \int_{-\infty}^\infty pr_L(t\,|l, z_0, w_0)\, f_{Z_0,W_0}(z_0,w_0)\, dz_0 dw_0 \text{ and} \\ pr_U(t\,|u) &= \int_0^\infty \int_{-\infty}^\infty pr_U(t\,|u, z_0, w_0)\, f_{Z_0,W_0}(z_0,w_0)\, dz_0 dw_0 \end{aligned}$$

for $t = 1, 2, 3, \ldots$.

Woodall (1983) gave an argument that can be used to show that for large value of t the "tail" of the probability mass function $pr_U(t\,|u)$ ($pr_L(t\,|l)$) can be approximated by a geometric distribution. That is,

$$pr_U(t^* + t\,|u) \approx \theta_U^t pr_U(t^*\,|u) \;\; (pr_L(t^* + t\,|l) \approx \theta_L^t pr_L(t^*\,|l)).$$

He recommended approximating θ_U by

$$\begin{aligned} \widehat{\theta}_U &= \frac{pr_U(t^*\,|u)}{pr_U(t^* - 1\,|u)} \text{ and } \widetilde{\theta}_U = \frac{\sum_{t=1}^{t^*} pr_U(t\,|u)}{\sum_{t=1}^{t^*} pr_U(t-1\,|u)}, \\ &\left(\widehat{\theta}_L = \frac{pr_L(t^*\,|l)}{pr_L(t^* - 1\,|l)} \text{ and } \widetilde{\theta}_L = \frac{\sum_{t=1}^{t^*} pr_L(t\,|l)}{\sum_{t=1}^{t^*} pr_L(t-1\,|l)}\right). \end{aligned}$$

When the values of $\widehat{\theta}_U$ and $\widetilde{\theta}_U$ ($\widehat{\theta}_L$ and $\widetilde{\theta}_L$) are "close," Woodall recommended using $\widetilde{\theta}_U$ ($\widetilde{\theta}_L$) to approximate θ_U (θ_L).

The ARL for the upper one-sided charts conditioned on $U_0 = u$, $Z_0 = z_0$, and $W_0 = w_0$ is determined by

$$M_U(u, z_0, w_0) = \sum_{t=1}^{\infty} tP(T_U = t \,|U_0 = u, Z_0 = z_0, W_0 = w_0) = \sum_{t=1}^{\infty} tpr_U(t \,|u, z_0, w_0).$$

As with the previous discussion, we have

$$\begin{aligned}
M_U(u, z_0, w_0) &= 1 + \sum_{t=1}^{\infty} tP(T_U = 1 + t \,|U_0 = u, Z_0 = z_0, W_0 = w_0) \\
&= 1 + \sum_{t=1}^{\infty} tP(T_U - 1 = t, U_1 = 0 \,|U_0 = u, Z_0 = z_0, W_0 = w_0) \\
&+ \sum_{t=1}^{\infty} tP(T_U - 1 = t, 0 < U_1 < a_5 \,|U_0 = u, Z_0 = z_0, W_0 = w_0) \\
&= 1 + \sum_{t=1}^{\infty} tP(T_U - 1 = t \,|U_0 = u, U_1 = 0, Z_0 = z_0, W_0 = w_0) \\
&\times P(U_1 = 0 \,|U_0 = u) \\
&+ \sum_{t=1}^{\infty} tP(T_U - 1 = t \,|U_0 = u, 0 < U_1 < a_5, Z_0 = z_0, W_0 = w_0) \\
&\times P(0 < U_1 < a_5 \,|U_0 = u) \\
&= 1 + M_U(0, z_0, w_0) F_{U_1|U_0}(0 \,|u) + \sum_{t=1}^{\infty} t \int_0^{a_5} pr_U(t \,|u_1, z_0, w_0) f_{U_1|U_0}(u_1 \,|u) \, du_1 \\
&= 1 + M_U(0, z_0, w_0) F_{U_1|U_0}(0 \,|u) + \int_0^{a_5} M_U(u_1, z_0, w_0) f_{U_1|U_0}(u_1 \,|u) \, du_1.
\end{aligned}$$

In summary, the conditional ARL, $M_U(u, z_0, w_0)$, of the upper one-sided generalized control chart given $U_0 = u$, $Z_0 = z_0$, and $W_0 = w_0$ is the solution to the integral equation

$$\begin{aligned}
M_U(u, z_0, w_0) &= 1 + M_U(0, z_0, w_0) F_{Y_t}(-a_1 u - a_3) + \int_0^{a_5} M_U(u_1, z_0, w_0) f_{Y_t}(u_1 - a_1 u - a_3) \\
&= 1 + M_U(0, z_0, w_0) G_{U,z_0,w_0}(0, u) + \int_0^{a_5} M_U(u_1, z_0, w_0) g_{U,z_0,w_0}(u_1, u) \, du_1.
\end{aligned}$$

Similarly, we can derive that the conditional ARL, $M_L(l, z_0, w_0)$, of the lower one-sided generalized control chart given $L_0 = l$, $Z_0 = z_0$, and $W_0 = w_0$ is the solution to the integral equation

$$\begin{aligned}
M_L(l, z_0, w_0) &= 1 + M_L(0, z_0, w_0)\left[1 - F_{L_1|L_0}(0\,|l\,)\right] + \int_{b_5}^{0} M_L(l_1, z_0, w_0)\, f_{L_1|L_0}(l_1\,|l\,)\, dl_1 \\
&= 1 + M_L(0, z_0, w_0)\left[1 - G_{L,z_0,w_0}(0, l)\right] + \int_{b_5}^{0} M_L(l_1, z_0, w_0)\, g_{L,z_0,w_0}(l_1, l)\, dl_1.
\end{aligned}$$

Since the exact solution for the aforementioned integral equations does not exist, we will use the numerical quadrature to obtain approximations. For the upper one-sided chart, let $(\xi_{U,j}, \omega_{U,j})$ $(j = 1, \ldots, \eta_U)$ be the ordered pairs of the exact nodes and weights of a numerical quadrature method. This leads to the following system of equations,

$$\begin{aligned}
&pr_U(1\,|\xi_{U,i}, z_0, w_0\,) = 1 - G_{U,z_0,w_0}(a_5, \xi_{U,i}) \text{ and} \\
&pr_U(t\,|\xi_{U,i}, z_0, w_0\,) \approx pr_U(t-1\,|\xi_{U,0}, z_0, w_0\,)\, G_{U,z_0,w_0}(\xi_{U,0}, \xi_{U,i}) \\
&\qquad\qquad + \sum_{j=1}^{\eta_U} pr_U(t-1\,|\xi_{U,j}, , z_0, w_0\,)\, g_{U,z_0,w_0}(\xi_{U,j}, \xi_{U,i})\, \omega_{U,j}
\end{aligned}$$

for $t > 1$ and $\xi_{U,0} = 0$, where $i = 0, 1, \ldots, \eta_U$.

Letting

$$\mathbf{P}_{U,t} = \begin{bmatrix} pr_U(t\,|\xi_{U,0}, z_0, w_0\,) \\ pr_U(t\,|\xi_{U,1}, z_0, w_0\,) \\ \vdots \\ pr_U(t\,|\xi_{U,\eta_U}, z_0, w_0\,) \end{bmatrix}$$

and

$$\mathbf{Q}_{U,z_0,w_0} = \begin{bmatrix} G_{U,z_0,w_0}(\xi_{U,0}, \xi_{U,0}) & g_{U,z_0,w_0}(\xi_{U,1}, \xi_{U,0})\,\omega_{U,1} & \cdots & g_{U,z_0,w_0}(\xi_{U,\eta_U}, \xi_{U,0})\,\omega_{U,\eta_U} \\ G_{U,z_0,w_0}(\xi_{U,0}, \xi_{U,1}) & g_{U,z_0,w_0}(\xi_{U,1}, \xi_{U,1})\,\omega_{U,1} & \cdots & g_{U,z_0,w_0}(\xi_{U,\eta_U}, \xi_{U,1})\,\omega_{U,\eta_U} \\ \vdots & \vdots & \ddots & \vdots \\ G_{U,z_0,w_0}(\xi_{U,0}, \xi_{U,\eta_U}) & g_{U,z_0,w_0}(\xi_{U,1}, \xi_{U,\eta_U})\,\omega_{U,1} & \cdots & g_{U,z_0,w_0}(\xi_{U,\eta_U}, \xi_{U,\eta_U})\,\omega_{U,\eta_U} \end{bmatrix}.$$

Then we note that

$$\left(\mathbf{I}-\mathbf{Q}_{U,z_0,w_0}\right)\mathbf{1}=\begin{bmatrix} 1-G_{U,z_0,w_0}\left(\xi_{U,0},\xi_{U,0}\right)-\sum_{i=1}^{\eta_U} g_{U,z_0,w_0}\left(\xi_{U,i},\xi_{U,0}\right)\omega_{U,i} \\ 1-G_{U,z_0,w_0}\left(\xi_{U,0},\xi_{U,1}\right)-\sum_{i=1}^{\eta_U} g_{U,z_0,w_0}\left(\xi_{U,i},\xi_{U,1}\right)\omega_{U,i} \\ \vdots \\ 1-G_{U,z_0,w_0}\left(\xi_{U,0},\xi_{U,\eta_U}\right)-\sum_{i=1}^{\eta_U} g_{U,z_0,w_0}\left(\xi_{U,i},\xi_{U,\eta_U}\right)\omega_{U,i} \end{bmatrix}.$$

Thus the system of equations can be written in vector notations as

$$\mathbf{P}_{U,1}=\left(\mathbf{I}-\mathbf{Q}_{U,z_0,w_0}\right)\mathbf{1} \text{ and } \mathbf{P}_{U,t}=\mathbf{Q}_{U,z_0,w_0}\mathbf{P}_{U,t-1}$$

for $t>1$, where $\mathbf{1}$ is an $(\eta_U+1)\times 1$ vector of ones.

Making the transformation $y=u_1/a_5$, we have that $u_1=a_5y$ and $du_1=a_5dy$. If follows that

$$\begin{aligned} pr_U\left(t\,|u,z_0,w_0\right) &= pr_U\left(t-1\,|0,z_0,w_0\right)G_{U,z_0,w_0}\left(0,u\right) \\ &\quad+\int_{-1}^{1} pr_U\left(t-1\,|a_5y,z_0,w_0\right)g_{U,z_0,w_0}\left(a_5y,u\right)a_5dy \\ &= pr_U\left(t-1\,|0,z_0,w_0\right)G_{U,z_0,w_0}\left(0,u\right) \\ &\quad+\int_{-1}^{1} pr_U\left(t-1\,|a_5y,z_0,w_0\right)g_{U,z_0,w_0}\left(a_5y,u\right)a_5dy. \end{aligned}$$

For the lower one-sided chart, the associated system of equations is

$$\begin{aligned} pr_L\left(1\,|\xi_{L,i},z_0,w_0\right) &= G_{L,z_0,w_0}\left(b_5,\xi_{L,i}\right) \text{ and} \\ pr_L\left(t\,|\xi_{L,i},z_0,w_0\right) &\approx pr_L\left(t-1\,|\xi_{L,0},z_0,w_0\right)\left[1-G_{L,z_0,w_0}\left(\xi_{L,0},\xi_{L,i}\right)\right] \\ &\quad+\sum_{j=1}^{\eta_L} pr_L\left(t-1\,|\xi_{L,j},z_0,w_0\right)g_{L,z_0,w_0}\left(\xi_{L,j},\xi_{L,i}\right)\omega_{L.j}, \end{aligned}$$

which can be expressed as

$$\mathbf{P}_{L,1}\approx\mathbf{Q}_{L,z_0,w_0}\mathbf{1} \text{ and } \mathbf{P}_{L,t}\approx\mathbf{Q}_{L,z_0,w_0}\mathbf{P}_{L,t-1}$$

for $t > 1$, where $\mathbf{1}$ is an $(\eta_L + 1) \times 1$ vector of ones, and

$$\mathbf{P}_{L,t} = \begin{bmatrix} pr_L\left(t\,|\xi_{L,0}, z_0, w_0\right) \\ pr_L\left(t\,|\xi_{L,1}, z_0, w_0\right) \\ \vdots \\ pr_L\left(t\,|\xi_{L,\eta_L}, z_0, w_0\right) \end{bmatrix},$$

and

$$\mathbf{Q}_{L,z_0,w_0} = \begin{bmatrix} G_{L,z_0,w_0}\left(\xi_{L,0}, \xi_{L,0}\right) & g_{L,z_0,w_0}\left(\xi_{L,1}, \xi_{L,0}\right)\omega_{L,1} & \cdots & g_{L,z_0,w_0}\left(\xi_{L,\eta_L}, \xi_{L,0}\right)\omega_{L,\eta_L} \\ G_{L,z_0,w_0}\left(\xi_{L,0}, \xi_{L,1}\right) & g_{L,z_0,w_0}\left(\xi_{L,1}, \xi_{L,1}\right)\omega_{L,1} & \cdots & g_{L,z_0,w_0}\left(\xi_{L,\eta_L}, \xi_{L,1}\right)\omega_{L,\eta_L} \\ \vdots & \vdots & \ddots & \vdots \\ G_{L,z_0,w_0}\left(\xi_{L,0}, \xi_{L,\eta_L}\right) & g_{L,z_0,w_0}\left(\xi_{L,1}, \xi_{L,\eta_L}\right)\omega_{L,1} & \cdots & g_{L,z_0,w_0}\left(\xi_{L,\eta_L}, \xi_{L,\eta_L}\right)\omega_{L,\eta_L} \end{bmatrix}.$$

3.4 Markov Chain Approach

For the two-sided charts, If $Y_{L,t} = Y_{U,t} = Y_t = W_0^{-1}\left(\lambda Z_t + \sqrt{n}\delta - Z_0/\sqrt{m}\right)$, then as in the previous discussions, the conditional transition probabilities among the nonabsorbing states given $Z_0 = z_0$ and $W_0 = w_0$ will be expressed as follows.

$$P_{i,j|Z_0,W_0} = \begin{cases} P\left(c_0 \leq W_0^{-1}\left(\lambda Z_t + \sqrt{n}\delta - Z_0/\sqrt{m}\right) \leq d_0\,|Z_0, W_0\right), \\ \text{if } j_L = 0 \text{ and } j_U = 0; \\ P\left(\max\left\{c_0, c_U\right\} < W_0^{-1}\left(\lambda Z_t + \sqrt{n}\delta - Z_0/\sqrt{m}\right) < d_U\,|Z_0, W_0\right), \\ \text{if } j_L = 0 \text{ and } j_U \neq 0; \\ P\left(d_L \leq W_0^{-1}\left(\lambda Z_t + \sqrt{n}\delta - Z_0/\sqrt{m}\right) < \min\left\{c_L, d_0\right\}|Z_0, W_0\right), \\ \text{if } j_L \neq 0 \text{ and } j_U = 0; \\ P\left(\max\left\{c_0, c_U\right\} < W_0^{-1}\left(\lambda Z_t + \sqrt{n}\delta - Z_0/\sqrt{m}\right) < \min\left\{c_L, d_0\right\}|Z_0, W_0\right), \\ \text{if } j_L \neq 0 \text{ and } j_U \neq 0; \end{cases}$$

which can be written as

$$P_{i,j|Z_0,W_0} = \begin{cases} P\left(\frac{w_0c_0-\sqrt{n}\delta+z_0/\sqrt{m}}{\lambda} \le Z_t \le \frac{w_0d_0-\sqrt{n}\delta+z_0/\sqrt{m}}{\lambda}\right), \\ \text{if } j_L = 0 \text{ and } j_U = 0; \\ P\left(\frac{w_0\max\{c_0,c_U\}-\sqrt{n}\delta+z_0/\sqrt{m}}{\lambda} < Z_t < \frac{w_0d_U-\sqrt{n}\delta+z_0/\sqrt{m}}{\lambda}\right), \\ \text{if } j_L = 0 \text{ and } j_U \neq 0; \\ P\left(\frac{w_0d_L-\sqrt{n}\delta+z_0/\sqrt{m}}{\lambda} \le Z_t < \frac{w_0\min\{c_L,d_0\}-\sqrt{n}\delta+z_0/\sqrt{m}}{\lambda}\right), \\ \text{if } j_L \neq 0 \text{ and } j_U = 0; \\ P\left(\frac{w_0\max\{c_0,c_U\}-\sqrt{n}\delta+z_0/\sqrt{m}}{\lambda} < Z_t < \frac{w_0\min\{c_L,d_0\}-\sqrt{n}\delta+z_0/\sqrt{m}}{\lambda}\right), \\ \text{if } j_L \neq 0 \text{ and } j_U \neq 0; \end{cases} .$$

Since Z_t has standard normal distribution, then

$$P_{i,j|Z_0,W_0} = \begin{cases} \Phi\left(\frac{w_0d_0-\sqrt{n}\delta+z_0/\sqrt{m}}{\lambda}\right) - \Phi\left(\frac{w_0c_0-\sqrt{n}\delta+z_0/\sqrt{m}}{\lambda}\right), \\ \text{if } j_L = 0 \text{ and } j_U = 0; \\ \Phi\left(\frac{w_0d_U-\sqrt{n}\delta+z_0/\sqrt{m}}{\lambda}\right) - \Phi\left(\frac{w_0\max\{c_0,c_U\}-\sqrt{n}\delta+z_0/\sqrt{m}}{\lambda}\right), \\ \text{if } j_L = 0 \text{ and } j_U \neq 0; \\ \Phi\left(\frac{w_0\min\{c_L,d_0\}-\sqrt{n}\delta+z_0/\sqrt{m}}{\lambda}\right) - \Phi\left(\frac{w_0d_L-\sqrt{n}\delta+z_0/\sqrt{m}}{\lambda}\right), \\ \text{if } j_L \neq 0 \text{ and } j_U = 0; \\ \Phi\left(\frac{w_0\min\{c_L,d_0\}-\sqrt{n}\delta+z_0/\sqrt{m}}{\lambda}\right) - \Phi\left(\frac{w_0\max\{c_0,c_U\}-\sqrt{n}\delta+z_0/\sqrt{m}}{\lambda}\right), \\ \text{if } j_L \neq 0 \text{ and } j_U \neq 0; \end{cases}$$

where

$$\begin{aligned} c_0 &= \nu_{L,0} - b_1\xi_{L,i_L} - b_3,\ d_0 = \nu_{U,0} - a_1\xi_{U,i_U} - a_3, \\ c_L &= \nu_{L,j_L} - b_1\xi_{L,i_L} - b_3,\ d_L = \nu_{L,j_L-1} - b_1\xi_{L,i_L} - b_3, \\ c_U &= \nu_{U,j_U-1} - a_1\xi_{U,i_U} - a_3,\ d_U = \nu_{U,j_U} - a_1\xi_{U,i_U} - a_3, \end{aligned}$$

and $i = i_L(\eta_U + 1) + i_U$, $j = j_L(\eta_U + 1) + j_U$ with $i_L, j_L \in \{0, 1, \ldots, \eta_L\}$, $i_U, j_U \in \{0, 1, \ldots, \eta_U\}$.

For the $[(\eta_L + 2) \times (\eta_U + 2)]^2$ transition matrix $\mathbf{P}_{z_0,w_0}$ whose (i,j)th component is the conditional probability $P_{i,j|z_0,w_0}$, all the information contained in $\mathbf{P}_{z_0,w_0}$ can be obtained in the sub-matrix $\mathbf{Q}_{z_0,w_0}$ by excluding the rows and columns relevant to absorbing states.

Random variable $T_{k|z_0,w_0}$ is the conditional average run length of the two sided-chart that starts at one of the nonabsorbing state k. For convenience, we define the $(\eta_L + 1)(\eta_U + 1) \times 1$ vectors $\mathbf{T}$ and $\mathbf{t}$ by

$$\mathbf{T} = \begin{bmatrix} T_{0|z_0,w_0} \\ T_{1|z_0,w_0} \\ \vdots \\ T_{\eta|z_0,w_0} \end{bmatrix} \text{ and } \mathbf{t} = \begin{bmatrix} t \\ t \\ \vdots \\ t \end{bmatrix} = t\mathbf{1},$$

where $\mathbf{1}$ is an $(\eta_L + 1)(\eta_U + 1) \times 1$ vector of ones. The conditional probability mass function of $\mathbf{T}$ has the form

$$P(\mathbf{T} = \mathbf{t} \,| Z_0 = z_0, W_0 = w_0) = \mathbf{Q}_{z_0,w_0}^{t-1} (\mathbf{I} - \mathbf{Q}_{z_0,w_0}) \mathbf{1},$$

for $t = 1, 2, 3, \ldots$. The unconditional transition matrix is

$$\mathbf{Q} = \int_0^{\infty} \int_{-\infty}^{\infty} \mathbf{Q}_{z_0,w_0} f_{Z_0,W_0}(z_0, w_0)\, dz_0 dw_0.$$

Thus, the unconditional probability mass function of $\mathbf{T}$ is given by

$$P(\mathbf{T} = \mathbf{t}) = \mathbf{Q}^{t-1} (\mathbf{I} - \mathbf{Q}) \mathbf{1},$$

The conditional expectations of these $\eta_L + 1$ random variables is determined by

$$
\begin{aligned}
E\left(\mathbf{T}_L \,|Z_0 = z_0, W_0 = w_0\right) &= \begin{bmatrix} E\left(T_{L,0} \,|Z_0 = z_0, W_0 = w_0\right) \\ E\left(T_{L,1} \,|Z_0 = z_0, W_0 = w_0\right) \\ \vdots \\ E\left(T_{L,\eta_L} \,|Z_0 = z_0, W_0 = w_0\right) \end{bmatrix} \\
&= \sum_{t=1}^{\infty} t \mathbf{Q}_{z_0,w_0}^{t-1} \left(\mathbf{I} - \mathbf{Q}_{z_0,w_0}\right) \mathbf{1} \\
&= \left(\mathbf{I} - \mathbf{Q}_{z_0,w_0}\right) \mathbf{1}.
\end{aligned}
$$

The unconditional expectations are given by

$$E\left(\mathbf{T}\right) = \left(\mathbf{I} - \mathbf{Q}\right) \mathbf{1}.$$

3.5 Unconditional Run Length Distribution

The condition probability mass function given $Z_0 = z_0$ and $W_0 = w_0$ describing the distribution of the run length T_L is expressed as

$$pr_L\left(t \,|l, \delta, \lambda, m, n, z_0, w_0\right) = P\left(T_L = t \,|L_0 = l, \delta, \lambda, m, n, Z_0 = z_0, W_0 = w_0\right).$$

Its unconditional probability mass function is determined by

$$pr_L\left(t \,|l, \delta, \lambda, m, n\right) = \int_0^{\infty} \int_{-\infty}^{\infty} pr_L\left(t \,|l, \delta, \lambda, m, n, z_0, w_0\right) f_{Z_0,W_0}\left(z_0, w_0 \,|m, n\right) dz_0 dw_0.$$

Similarly, the conditional and the unconditional probability mass functions describing the distribution of the run length T_U of the upper one-sided generalized control chart are expressed, respectively, by

$$pr_U\left(t \,|u, \delta, \lambda, m, n, z_0, w_0\right) = P\left(T_U = t \,|U_0 = u, \delta, \lambda, m, n, Z_0 = z_0, W_0 = w_0\right) \text{ and}$$

$$pr_U\left(t\left|u,\delta,\lambda,m,n\right.\right)=\int_0^{\infty}\int_{-\infty}^{\infty}pr_U\left(t\left|u,\delta,\lambda,m,n,z_0,w_0\right.\right)f_{Z_0,W_0}\left(z_0,w_0\left|m,n\right.\right)dz_0dw_0.$$

The run length T of a two-sided generalized control chart is defined as

$$T=\min\left\{T_L,T_U\right\}.$$

The conditional and unconditional probability mass functions describing the distribution of T are, respectively,

$$pr\left(t\left|l,u,\delta,\lambda,m,n,z_0,w_0\right.\right)=P\left(T=t\left|L_0=l,U_0=u,\delta,\lambda,m,n,Z_0=z_0,W_0=w_0\right.\right)\text{ and}$$

$$pr\left(t\left|l,u,\delta,\lambda,m,n\right.\right)=\int_0^{\infty}\int_{-\infty}^{\infty}pr\left(t\left|l,u,\delta,\lambda,m,n,z_0,w_0\right.\right)f_{Z_0,W_0}\left(z_0,w_0\left|m,n\right.\right)dz_0dw_0.$$

Woodall (1983) gives a method for obtaining a geometric distribution approximation to the run length distribution for “large” values of t. The large values of t and the geometric parameters for the conditional probability mass functions will depend on the values of z_0 and w_0. They also depend on the values of l, u, δ, λ, m, and n; but these values are considered to be fixed in our discussion. For the lower one-sided, upper one-sided, and two-sided charts, we will represent the pair consisting of the large value of t and the estimated geometric parameter by $\left(t^*_{L,z_0,w_0},\widehat{\theta}_{L,z_0,w_0}\right)$, $\left(t^*_{U,z_0,w_0},\widehat{\theta}_{U,z_0,w_0}\right)$, and $\left(t^*_{z_0,w_0},\widehat{\theta}_{z_0,w_0}\right)$, respectively. According to Woodall (1983), the upper tail probabilities can be approximated by

$$\begin{aligned}
pr_L\left(t^*_L+t\left|l,\delta,\lambda,m,n,z_0,w_0\right.\right) &\approx \widehat{\theta}^t_{L,z_0,w_0}pr_L\left(t^*_{L,z_0,w_0}\left|l,\delta,\lambda,m,n,z_0,w_0\right.\right),\\
pr_U\left(t^*_U+t\left|u,\delta,\lambda,m,n,z_0,w_0\right.\right) &\approx \widehat{\theta}^t_{U,z_0,w_0}pr_U\left(t^*_{U,z_0,w_0}\left|u,\delta,\lambda,m,n,z_0,w_0\right.\right),\text{ and}\\
pr\left(t^*+t\left|l,u,\delta,\lambda,m,n,z_0,w_0\right.\right) &\approx \widehat{\theta}^t_{z_0,w_0}pr\left(t^*_{z_0,w_0}\left|l,u,\delta,\lambda,m,n,z_0,w_0\right.\right).
\end{aligned}$$

This method yields probability mass functions that are approximation to the probability mass functions of the run length distributions of the lower one-side, upper one-sided, and two-sided generalized control charts. For each chart, the average of these probability mass functions over the weighted values of Z_0 and W_0 give the unconditional probability mass function of the distribution of the run length. Woodall (1983) method can again be used to approximate the tail probabilities of the unconditional probability mass functions. This results in the following approximations.

$$\begin{aligned} pr_L\left(t\,|l,\delta,\lambda,m,n\right) &\approx \widehat{\theta}_L^t pr_L\left(t_L^*\,|l,\delta,\lambda,m,n,z_0,w_0\right); \\ pr_U\left(t\,|u,\delta,\lambda,m,n\right) &\approx \widehat{\theta}_U^t pr_U\left(t_U^*\,|u,\delta,\lambda,m,n,z_0,w_0\right); \text{ and} \\ pr\left(t\,|l,u,\delta,\lambda,m,n\right) &\approx \widehat{\theta}^t pr\left(t^*\,|l,u,\delta,\lambda,m,n,z_0,w_0\right). \end{aligned}$$

In the Appendix, we give a program that implements this method. Simulation is used to examine the accuracy of the programs.

3.6 Conclusion

In this chapter we discussed the generalized cumulative sum type $\overline{X}$ control charts for monitoring for the mean of a continuous quality measurement X which has a normal distribution with mean μ and standard deviation σ. First estimates for the in-control values μ_0 and σ_0 were given. Then the integral equation method was outlined for evaluating the run length performance of the one-sided generalized cumulative type $\overline{X}$ charts followed by the Markov chain approach for approximating the average run length of a two-sided generalized cumulative sum type $\overline{X}$ chart. The unconditional run length distribution was discussed as a method for measuring the performance of the chart when parameters are estimated.

CHAPTER 4

MONITORING A SCALE PARAMETER

4.1 Introduction

A gamma distribution can be defined by its probability density function given by

$$f_{\kappa,\theta}\left(y\right)=\frac{1}{\Gamma\left(\kappa\right)\theta^{\kappa}}y^{\kappa-1}e^{-y/\theta}I_{(0,\infty)}\left(y\right),$$

where $\kappa > 0$ and $\theta > 0$ are referred to as the shape and scale parameters, respectively. As can be seen, the support of the distribution of X is the positive reals. The exponential ($\kappa = 1$), the Erlang (κ a positive integer), and the Chi Square ($\kappa = \nu/2$ and $\theta = 2$) distributions are special cases of the gamma distribution.

The exponential distribution appears as a model of the quality characteristic X when modelling the occurrence rate of rare events with a homogeneous Poisson process. This model results in the time X between events having an exponential distribution. Further, the X values are independent. When modelling the lifetime X of an item with an exponential distribution, a sample of n items are placed on test. Because of time constraints in making a decision about the process, only the minimum lifetime $X_{1:n}$ is to be observed. One can show that the distribution of $X_{1:n}$ is an exponential distribution with parameter θ/n. The Erlang distribution is the distribution of the sum of κ (a positive integer) independent random variables each having an exponential distribution with parameter θ. In the previous example, if all the lifetimes $X_1, \ldots, X_n$ are observed then their sum $X_1 + \ldots + X_n$ has an Erlang distribution with shape parameter n and scale parameter $n\theta$.

It is often of interest to monitor for a change in the variance σ^2 of a continuous quality characteristic. Since the sample variance S^2 is an unbiased estimator of σ^2, a statistic on which one could based the chart is S_t^2/σ_0^2 if the in-control value σ_0^2 of the variance is known or the statistic $S_t^2/\widehat{\sigma}_0^2$ where $\widehat{\sigma}_0^2$ is an estimator for the unknown parameter σ_0^2. Under the independent normal model, the distribution of the statistic

$$\begin{aligned}
\frac{S_t^2}{\sigma_0^2} &\sim GAMMA\left(\frac{n-1}{2}, \frac{2\sigma^2}{\sigma_0^2}\right) \text{ and} \\
\frac{S_t^2}{\widehat{\sigma}_0^2}\left|\frac{\widehat{\sigma}_0^2}{\sigma_0^2} = w_0\right. &\sim GAMMA\left(\frac{n-1}{2}, \frac{2\sigma^2/\sigma_0^2}{w_0}\right).
\end{aligned}$$

Our interest is to monitor for a change in a scale parameter. Two examples that appear in the literature are (1) when monitoring for a change in shape parameter of an Erlang distribution with positive integer scale parameter that is known and (2) when monitoring for a change in the variance of a normal distribution. These examples are special cases of a general setting in which the basic statistic Y_t on which the chart is based can be expressed in the form

$$Y_t = \frac{W_t}{W_0},$$

where W_t and W_t given $W_0 = w_0$ have gamma distributions. The variability described by the distribution of W_0 is determined by the variability found in the data to be obtained from a Phase I study. The variability described by the distribution of W_t is determined by the variability found in the data to be obtained at time t from Phase II. The data to be obtained from the Phase I study is assumed to be from an in-control process. We note here that the parameters known case is equivalent to setting w_0 equal to 1.

Because the parameters for the charts we will study for monitoring for a change in the scale parameter are slightly different than for a change in the mean, we redefine the generalized cumulative sum type charts. The lower one-sided chart is defined as the plot the statistic L_t versus the sampling stage number t with

$$L_0 = b_4 \text{ and } L_t = \min\left\{0, b_1 L_{t-1} + Y_t + b_3\right\},$$

where $b_5 < b_4 \leq 0$, $b_1 > 0$, and $b_3 < 0$. The chart signals at time t if $L_t \leq b_5 < 0$ or $Y_t \leq b_6$, with $b_6 > 0$. The upper one-sided generalized control chart plots the statistic U_t versus t with

$$U_0 = a_4 \text{ and } U_t = \max\left\{0, a_1 U_{t-1} + Y_t + a_3\right\},$$

where $0 \leq a_4 < a_5$, $a_1 > 0$, and $a_3 > 0$. The chart signals at time t if $U_t \geq a_5 > 0$ or $Y_t \geq a_6 > 0$. For convenience, we express the conditional probability mass functions describing the distributions of the run lengths T_L and T_U of the lower and upper one-sided chart given $Y_0 = y_0$, respectively, by

$$\begin{aligned} pr_L\left(t\left|l, y_0\right.\right) &= P\left(T_L = t\left|L_0 = l, Y_0 = y_0\right.\right) \text{ and} \\ pr_U\left(t\left|u, y_0\right.\right) &= P\left(T_U = t\left|U_0 = u, Y_0 = y_0\right.\right), \end{aligned}$$

where $l = b_4$ and $u = a_4$.

In the next section, integral equations are given that are useful in determining the run length performance of the one-sided charts. The Markov chain method is used to analyze the performance of the two-sided charts and this method is presented in Section 3. In Section 4, we apply our results to two examples of monitoring for a change in the shape parameter of an Erlang distribution and the variance of a normally distributed quality measurement.

4.2 Performance Analysis Using Integral Equations

In this section, we will derive integral equations whose exact solutions are the conditional probability mass functions $pr_L\left(t\left|l, w_0\right.\right)$ and $pr_U\left(t\left|u, w_0\right.\right)$. We begin with the function

$pr_L(t\,|l, w_0)$. For the case in which $t = 1$ we have

$$\begin{aligned} pr_L(1\,|l, w_0) &= P\left(b_1 l + \frac{W_1}{W_0} + b_3 \le b_5 \text{ or } \frac{W_1}{W_0} \le b_6 \;\; |W_0 = w_0\right) \\ &= P\left(W_1 \le W_0 \max\{b_5 - b_1 l - b_3, b_6\} \;\; |W_0 = w_0\right) \\ &= F_{W_1|W_0}\left(w_0 \max\{b_5 - b_1 l - b_3, b_6\} \;\; |w_0\right). \end{aligned}$$

Next, we consider the case when $t > 1$. $pr_L(t\,|l, w_0)$ can be written as

$$\begin{aligned} pr_L(t\,|l, w_0) &= P\left(T_L = t, L_1 = 0, \frac{W_1}{W_0} > b_6\,|L_0 = l, W_0 = w_0\right) \\ &\quad + P\left(T_L = t, b_5 < L_1 < 0, \frac{W_1}{W_0} > b_6\,|L_0 = l, W_0 = w_0\right) \\ &= P\left(T_L - 1 = t - 1\,|L_0 = l, L_1 = 0, W_0 = w_0, L_1 > b_1 l + b_6 + b_3\right) \\ &\quad \times P\left(L_1 = 0, L_1 > b_1 l + b_6 + b_3\,|L_0 = l, W_0 = w_0\right) \\ &\quad + P\left(T_L - 1 = t - 1\,|L_0 = l, b_5 < L_1 < 0, W_0 = w_0, L_1 > b_1 l + b_6 + b_3\right) \\ &\quad \times P\left(b_5 < L_1 < 0, L_1 > b_1 l + b_6 + b_3\,|L_0 = l, W_0 = w_0\right). \end{aligned}$$

Since the remaining run length $T_L - 1$ given $L_1 = l_1$ and $W_0 = w_0$ has the same distribution as T_L given $L_0 = l_1$ and $W_0 = w_0$, we have that

$$P\left(T_L - 1 = t - 1\,|L_0 = l, L_1 = l_1, W_0 = w_0\right) = pr_L\left(t - 1\,|l_1, y_0\right).$$

And

$$\begin{aligned} &P\left(L_1 = 0, L_1 > b_1 l + b_6 + b_3\,|L_0 = l, W_0 = w_0\right) \\ = \;& P\left(b_1 l + \frac{W_1}{W_0} + b_3 \ge 0, \frac{W_1}{W_0} > b_6\,|W_0 = w_0\right) \\ = \;& P\left(W_1 \ge W_0 \max\{-b_1 l - b_3, b_6\}\,|W_0 = w_0\right) \\ = \;& \overline{F}_{W_1|W_0}\left(w_0 \max\{-b_1 l - b_3, b_6\}\,|w_0\right), \end{aligned}$$

where $\overline{F}_{W_1|W_0}(w_1\,|w_0) = 1 - F_{W_1|W_0}(w_1\,|w_0)$. Thus,

$$\begin{aligned} pr_L(t\,|l, w_0) &= pr_L(t - 1\,|0, w_0)\,\overline{F}_{W_1|W_0}\left(w_0 \max\{-b_1 l - b_3, b_6\}\,|w_0\right) \\ &\quad + \int_{\max\{b_5, b_1 l + b_6 + b_3\}}^{0} pr_L(t - 1\,|l_1, w_0)\, f_{L_1|L_0, W_0}(l_1\,|l, w_0)\, dl_1. \end{aligned}$$

Note that setting $b_6 = 0$ is equivalent to the chart having no Shewhart limit.

Observe that

$$\begin{aligned}
F_{L_1|L_0,w_0}\left(l_1\,|l,w_0\right) &= P\left(L_1 \le l_1\,|L_0 = l, W_0 = w_0\right) \\
&= P\left(b_1 l + \frac{W_1}{W_0} + b_3 \le l_1\,|W_0 = w_0\right) \\
&= P\left(W_1 \le W_0\left(l_1 - b_1 l - b_3\right)|W_0 = w_0\right) \\
&= F_{W_1|W_0}\left(w_0\left(l_1 - b_1 l - b_3\right)|w_0\right),
\end{aligned}$$

so the probability density function is

$$f_{L_1|L_0,W_0}\left(l_1\,|l,w_0\right) = w_0 f_{W_1|W_0}\left(w_0\left(l_1 - b_1 l - b_3\right)|w_0\right).$$

Hence,

$$\begin{aligned}
pr_L\left(t\,|l,w_0\right) &= pr_L\left(t-1\,|0,w_0\right)\overline{F}_{W_1|W_0}\left(w_0 \max\left\{-b_1 l - b_3, b_6\right\}|w_0\right) \\
&\quad + \int_{\max\{b_5, b_1 l + b_6 + b_3\}}^{0} pr_L\left(t-1\,|l_1,w_0\right) w_0 f_{W_1|W_0}\left(w_0\left(l_1 - b_1 l - b_3\right)|w_0\right) dl_1.
\end{aligned}$$

Recall that $b_5 < l \le 0$. It then follows that

$$\begin{aligned}
pr_L\left(t\,|l,w_0\right) &= pr_L\left(t-1\,|0,w_0\right)\overline{F}_{W_1|W_0}\left(w_0 \max\left\{-b_1 l - b_3, b_6\right\}|w_0\right) \\
&\quad + \int_{\max\{b_5, b_1 l + b_6 + b_3\}}^{0} pr_L\left(t-1\,|l_1,w_0\right) w_0 f_{W_1|W_0}\left(w_0\left(l_1 - b_1 l - b_3\right)|w_0\right) dl_1. \\
&= \begin{cases}
pr_L\left(t-1\,|0,w_0\right)\overline{F}_{W_1|W_0}\left(w_0 \max\left\{-b_1 l - b_3, b_6\right\}|w_0\right) \\
+ \int_{b_5}^{0} pr_L\left(t-1\,|l_1,y_0\right) pr_L\left(t-1\,|l_1,w_0\right) \\
\times w_0 f_{W_1|W_0}\left(w_0\left(l_1 - b_1 l - b_3\right)|w_0\right) dl_1, \\
b_5 < l \le \left(b_5 - b_6 - b_3\right)/b_1; \\
pr_L\left(t-1\,|0,w_0\right)\overline{F}_{W_1|W_0}\left(w_0 \max\left\{-b_1 l - b_3, b_6\right\}|w_0\right) \\
+ \int_{b_1 l + b_6 + b_3}^{0} pr_L\left(t-1\,|l_1,w_0\right) \\
\times w_0 f_{W_1|W_0}\left(w_0\left(l_1 - b_1 l - b_3\right)|w_0\right) dl_1, \\
\left(b_5 - b_6 - b_3\right)/b_1 < l \le 0.
\end{cases}
\end{aligned}$$

As previously mentioned, the ARL of a chart is a commonly used measure for the performance of the chart. An integral equation useful in determining this parameter can be derived as follows. Since the ARL for the lower chart depends on the variables l and w_0, we will represent the ARL by $M_L(l\,|w_0)$. It follows that

$$\begin{aligned} M_L(l\,|w_0) &= \sum_{t=1}^{\infty} tpr_L(t\,|l, w_0) = pr_L(1\,|l, w_0) + \sum_{t=2}^{\infty} tpr_L(t\,|l, w_0) \\ &= pr_L(1\,|l, w_0) + \sum_{t=1}^{\infty} (1+t)\, pr_L(1+t\,|l, w_0) \\ &= \sum_{t=1}^{\infty} pr_L(t\,|l, w_0) + \sum_{t=1}^{\infty} tpr_L(1+t\,|l, w_0) \\ &= 1 + \sum_{t=1}^{\infty} tpr_L(t\,|0, w_0)\, \overline{F}_{W_1|W_0}(w_0 \max\{-b_1 l - b_3, b_6\}\,|w_0) \\ &\quad + \int_{\max\{b_5, b_1 l + b_6 + b_3\}}^{0} \sum_{t=1}^{\infty} tpr_L(t\,|l_1, w_0) \\ &\quad \times w_0 f_{W_1|W_0}(w_0(l_1 - b_1 l - b_3)\,|w_0)\, dl_1. \end{aligned}$$

Hence, the function $M_L(l\,|w_0)$ is the exact solution to the integral equation

$$\begin{aligned} M_L(l\,|w_0) &= 1 + M_L(0\,|w_0)\, \overline{F}_{W_1|W_0}(w_0 \max\{-b_1 l - b_3, b_6\}\,|w_0) \\ &\quad + \int_{\max\{b_5, b_1 l + b_6 + b_3\}}^{0} M_L(l_1\,|w_0)\, w_0 f_{W_1|W_0}(w_0(l_1 - b_1 l - b_3)\,|w_0)\, dl_1. \end{aligned}$$

The standard deviation of the run length ($SDRL$) distribution is sometimes reported when a performance analysis of the chart is given. To highlight that this parameter is a function of the variables l and w_0, we will represent the SDRL by $SDRL(l\,|w_0)$. This parameter is determined by

$$SDRL(l\,|w_0) = \sqrt{M_{L,2}(l\,|w_0) - M_L^2(l\,|w_0)},$$

where $M_{L,2}(l\,|w_0) = E(T_L^2\,|L_0 = l, W_0 = w_0)$. Observe that

$$\begin{aligned}
M_{L,2}(l\,|w_0) &= \sum_{t=1}^{\infty} t^2 pr_L(t\,|l, w_0) = pr_L(1\,|l, w_0) + \sum_{t=1}^{\infty} (1+t)^2 pr_L(1+t\,|l, w_0) \\
&= 1 + 2\sum_{t=1}^{\infty} t pr_L(1+t\,|l, w_0) + \sum_{t=1}^{\infty} t^2 pr_L(1+t\,|l, w_0) \\
&= 1 + 2M_L(0\,|w_0)\,\overline{F}_{W_1|W_0}(w_0 \max\{-b_1 l - b_3, b_6\}\,|w_0) \\
&\quad +2\int_{\max\{b_5, b_1 l + b_6 + b_3\}}^{0} M_L(l_1\,|w_0)\, w_0 f_{W_1|W_0}(w_0(l_1 - b_1 l - b_3)\,|w_0)\, dl_1 \\
&\quad +M_{L,2}(0\,|w_0)\,\overline{F}_{W_1|W_0}(w_0 \max\{-b_1 l - b_3, b_6\}\,|w_0) \\
&\quad +\int_{\max\{b_5, b_1 l + b_6 + b_3\}}^{0} M_{L,2}(l_1\,|w_0)\, w_0 f_{W_1|W_0}(w_0(l_1 - b_1 l - b_3)\,|w_0)\, dl_1.
\end{aligned}$$

Hence, the function $M_{L,2}(l\,|w_0)$ is the exact solution to the integral equation

$$\begin{aligned}
M_{L,2}(l\,|w_0) &= 2M_L(l\,|w_0) - 1 \\
&\quad +M_{L,2}(0\,|w_0)\,\overline{F}_{W_1|W_0}(w_0 \max\{-b_1 l - b_3, b_6\}\,|w_0) \\
&\quad +\int_{\max\{b_5, b_1 l + b_6 + b_3\}}^{0} M_{L,2}(l_1\,|w_0)\, w_0 f_{W_1|W_0}(w_0(l_1 - b_1 l - b_3)\,|w_0)\, dl_1.
\end{aligned}$$

Similar results can be obtained for the upper one-sided chart. For the case in which $t = 1$, we see that

$$\begin{aligned}
pr_U(1\,|u, y_0) &= P\left(a_1 u + \frac{W_1}{W_0} + a_3 \geq a_5 \text{ or } \frac{W_1}{W_0} \geq a_6\,|W_0 = w_0\right) \\
&= P(W_1 \geq W_0 \min\{a_5 - a_1 u - a_3, a_6\}\,|W_0 = w_0) \\
&= 1 - F_{W_1|W_0}(w_0 \min\{a_5 - a_1 u - a_3, a_6\}\,|w_0).
\end{aligned}$$

For the case in which $t > 1$, we derive the following sequence of integral equation of which the function $pr_U\left(t\left|u,w_0\right.\right)$ is the exact solution. We have

$$\begin{aligned}
pr_U\left(t\left|u,w_0\right.\right) &= P\left(T_U=t,U_1=0,\frac{W_1}{W_0}<a_6\left|U_0=u,W_0=w_0\right.\right)\\
&\quad +P\left(T_U=t,0<U_1<a_5,\frac{W_1}{W_0}<a_6\left|U_0=u,W_0=w_0\right.\right)\\
&= P\left(T_U=t\left|U_0=u,U_1=0,\frac{W_1}{W_0}<a_6,W_0=w_0\right.\right)\\
&\quad \times P\left(U_1=0,\frac{W_1}{W_0}<a_6\left|U_0=u,W_0=w_0\right.\right)\\
&\quad +P\left(T_U=t,0<U_1<a_5,\frac{W_1}{W_0}<a_6\left|U_0=u,W_0=w_0\right.\right)\\
&\quad \times P\left(0<U_1<a_5,\frac{W_1}{W_0}<a_6\left|U_0=u,W_0=w_0\right.\right).
\end{aligned}$$

Observe that

$$\begin{aligned}
pr_U\left(t-1\left|0,w_0\right.\right) &= P\left(T_U=t\left|U_0=u,U_1=0,\frac{W_1}{W_0}<a_6,W_0=w_0\right.\right);\\
F_{W_1|W_0}\left(w_0c_U\left|w_0\right.\right) &= P\left(U_1=0,\frac{W_1}{W_0}<a_6\left|U_0=u,W_0=w_0\right.\right);\\
pr_U\left(t-1\left|u_1,w_0\right.\right) &= P\left(T_U=t,U_1=u_1,\frac{W_1}{W_0}<a_6\left|U_0=u,W_0=w_0\right.\right),
\end{aligned}$$

where $c_U=\min\left\{a_5-a_1u-a_3,a_6\right\}$. Thus, we can write

$$\begin{aligned}
pr_U\left(t\left|u,w_0\right.\right) &= pr_U\left(t-1\left|0,w_0\right.\right)F_{W_1|W_0}\left(w_0c_U\left|w_0\right.\right)\\
&\quad +\int_0^{\min\{a_5-a_1u-a_3,a_6\}} pr_U\left(t-1\left|u_1,w_0\right.\right)f_{U_1|U_0,W_0}\left(u_1\left|u,w_0\right.\right)du_1.
\end{aligned}$$

Next, we observe that

$$\begin{aligned}
F_{U_1|U_0,W_0}\left(u_1\left|u,w_0\right.\right) &= P\left(U_1\le u_1\left|U_0=u,W_0=w_0\right.\right)\\
&= P\left(a_1u+\frac{W_1}{W_0}+a_3\le u_1\left|W_0=w_0\right.\right)\\
&\quad \times F_{W_1|W_0}\left(w_0\left(u_1-a_1u-a_3\right)\left|w_0\right.\right).
\end{aligned}$$

Hence,

$$f_{U_1|U_0,W_0}\left(u_1\left|u,w_0\right.\right)=w_0f_{W_1|W_0}\left(w_0\left(u_1-a_1u-a_3\right)\left|w_0\right.\right).$$

We now see that the function $pr_U(t\,|u, w_0)$ for $t > 1$ is the exact solution to the sequence of integral equations

$$\begin{aligned} pr_U(t\,|u,w_0) &= pr_U(t-1\,|0,w_0)\,F_{W_1|W_0}(w_0c_U\,|w_0) \\ &+ \int_{\max\{0,a_1u+a_3\}}^{\min\{a_5,a_1u+a_6+a_3\}} pr_U(t-1\,|u_1,w_0) \\ &\times w_0 f_{W_1|W_0}(w_0(u_1-a_1u-a_3)\,|w_0)\,du_1. \end{aligned}$$

As with the lower one-sided chart, the ARL $M_U(u\,|w_0)$ and $SDRL$ of the upper one-sided chart are functions of the variables u and w_0. We represent the ARL and the expected value of the square of the run length by $M_U(u\,|w_0)$ and $M_U^2(u\,|w_0)$, respectively. The function $M_U(u\,|w_0)$ is the exact solution to the following derived integral equation.

$$\begin{aligned} M_U(u\,|w_0) &= \sum_{t=1}^{\infty} t\,pr_U(t\,|u,w_0) = 1 + \sum_{t=1}^{\infty} t\,pr_U(1+t\,|u,w_0) \\ &= 1 + \sum_{t=1}^{\infty} t\,pr_U(t\,|0,w_0)\,F_{W_1|W_0}(w_0c_U\,|w_0) \\ &+ \int_{\max\{0,a_1u+a_3\}}^{\min\{a_5,a_1u+a_6+a_3\}} \sum_{t=1}^{\infty} t\,pr_U(t\,|u_1,w_0) \\ &\times w_0 f_{W_1|W_0}(w_0(u_1-a_1u-a_3)\,|w_0)\,du_1 \\ &= 1 + M_U(0\,|w_0)\,F_{W_1|W_0}(w_0c_U\,|w_0) \\ &+ \int_{\max\{0,a_1u+a_3\}}^{\min\{a_5,a_1u+a_6+a_3\}} M_U(u_1\,|w_0)\,w_0 f_{W_1|W_0}(w_0(u_1-a_1u-a_3)\,|w_0)\,du_1. \end{aligned}$$

The function $M_U^2\left(u\left|w_0\right.\right)$ is the exact solution to the following integral equation.

$$\begin{aligned}
M_U^2\left(u\left|w_0\right.\right) &= \sum_{t=1}^{\infty} t^2 pr_U\left(t\left|u,w_0\right.\right) = 1 + \sum_{t=1}^{\infty}\left(2t+t^2\right) pr_U\left(1+t\left|u,w_0\right.\right) \\
&= 1 + 2M_U\left(0\left|w_0\right.\right) F_{W_1|W_0}\left(w_0 c_U\left|w_0\right.\right) + M_{U,2}\left(0\left|w_0\right.\right) F_{W_1|W_0}\left(w_0 c_U\left|w_0\right.\right) \\
&\quad +2\int_{\max\{0,a_1u+a_3\}}^{\min\{a_5,a_1u+a_6+a_3\}} M_U\left(u_1\left|w_0\right.\right) w_0 f_{W_1|W_0}\left(w_0\left(u_1-a_1u-a_3\right)\left|w_0\right.\right) du_1 \\
&\quad +\int_{\max\{0,a_1u+a_3\}}^{\min\{a_5,a_1u+a_6+a_3\}} M_{U,2}\left(u_1\left|w_0\right.\right) w_0 f_{W_1|W_0}\left(w_0\left(u_1-a_1u-a_3\right)\left|w_0\right.\right) du_1 \\
&= 2M_U\left(u\left|w_0\right.\right) - 1 + M_U^2\left(0\left|w_0\right.\right) F_{W_1|W_0}\left(w_0 c_U\left|w_0\right.\right) \\
&\quad +\int_{\max\{0,a_1u+a_3\}}^{\min\{a_5,a_1u+a_6+a_3\}} M_U^2\left(u_1\left|w_0\right.\right) w_0 f_{W_1|W_0}\left(w_0\left(u_1-a_1u-a_3\right)\left|w_0\right.\right) du_1.
\end{aligned}$$

One method that can be used to approximate the run length distributions for the lower and upper one-sided charts is the collocation method. This methods uses a polynomial to approximate the function of interest. For example, the probability mass function of the lower and upper one-sided chart can be approximated by

$$\begin{aligned}
pr_L\left(t\left|l,w_0\right.\right) &\approx d_{L,t,w_0,0} + \sum_{i=1}^{\eta_L} d_{L,t,w_0,i} l^i \text{ and} \\
pr_U\left(t\left|u,w_0\right.\right) &\approx d_{U,t,w_0,0} \sum_{i=1}^{\eta_U} d_{U,t,w_0,i} u^i.
\end{aligned}$$

Note that the coefficients of these polynomials are functions of the variables t and w_0. It would follow for the lower one-sided chart for each $t > 1$, the coefficients of the approximating polynomial satisfy the equation

$$\begin{aligned}
d_{L,t,w_0,0} + \sum_{i=1}^{\eta_L} d_{L,t,w_0,i} l^i &= d_{L,t,w_0,0} \overline{F}_{W_1|W_0}\left(w_0 \max\left\{-b_1 l - b_3, b_6\right\}\left|w_0\right.\right) \\
&\quad + \int_{\max\{b_5,b_1l+b_6+b_3\}}^{0} \left(d_{L,t,w_0,0} + \sum_{j=1}^{\eta_L} d_{L,t,w_0,j} l_1^j\right) \\
&\quad \times w_0 f_{W_1|W_0}\left(w_0\left(l_1-b_1l-b_3\right)\left|w_0\right.\right) dl_1
\end{aligned}$$

for any value of $b_5 < l \leq 0$ given $W_0 = w_0$. For the lower one-sided chart for each $t > 1$, the coefficients of the approximating polynomial satisfy the equation

$$\begin{aligned} d_{U,t,w_0,0}\sum_{i=1}^{\eta_U} d_{U,t,w_0,i}u^i &= d_{U,t,w_0,0}F_{W_1|W_0}\left(w_0 \min\left\{-a_1u-a_3, a_6\right\}|w_0\right) \\ &+ \int_{\max\{0,a_1u+a_3\}}^{\min\{a_5,a_1u+a_6+a_3\}} \left(d_{U,t,w_0,0}+\sum_{j=1}^{\eta_U} d_{U,t,w_0,j}u_1^j\right) \\ &\times w_0 f_{W_1|W_0}\left(w_0\left(u_1-a_1u-a_3\right)|w_0\right)du_1. \end{aligned}$$

for any value of $0 \le u < a_5$ given $W_0 = w_0$.

4.3 Markov Chain Approach

In this section, we will apply the Markov Chain approximate method discussed in Chapter 2 to analyze the performance of the two-sided charts that are used to monitor for the scale parameter of gamma distribution. That is,

$$Y_{L,t} = Y_{U,t} = Y_t = \frac{W_t}{W_0}.$$

As previous discussion, the conditional transition probabilities among the nonabsorbing states given $W_0 = w_0$ are expressed as

$$P_{i,j|W_0} = \begin{cases} P\left(c_0 \le \frac{W_t}{W_0} \le d_0 \,|W_0 = w_0\right), & \text{if } j_L = 0 \text{ and } j_U = 0; \\ P\left(\max\left\{c_0, c_U\right\} < \frac{W_t}{W_0} < d_{U0} \,|W_0 = w_0\right), & \text{if } j_L = 0 \text{ and } j_U \ne 0; \\ P\left(d_L \le \frac{W_t}{W_0} < \min\left\{c_L, d_0\right\}_0 |W_0 = w_0\right), & \text{if } j_L \ne 0 \text{ and } j_U = 0; \\ P\left(\max\left\{c_0, c_U\right\} < \frac{W_t}{W_0} < \min\left\{c_L, d_0\right\}_0 |W_0 = w_0\right), & \text{if } j_L \ne 0 \text{ and } j_U \ne 0; \end{cases}$$

It follows that

$$
\begin{aligned}
P_{i,j|W_0} &= \begin{cases} P\left(w_0 c_0 \le W_t \le w_0 d_0\right), & \text{if } j_L = 0 \text{ and } j_U = 0; \\ P\left(w_0 \max\left\{c_0, c_U\right\} < W_t < w_0 d_U\right), & \text{if } j_L = 0 \text{ and } j_U \ne 0; \\ P\left(w_0 d_L \le W_t < w_0 \min\left\{c_L, d_0\right\}\right), & \text{if } j_L \ne 0 \text{ and } j_U = 0; \\ P\left(w_0 \max\left\{c_0, c_U\right\} < W_t < w_0 \min\left\{c_L, d_0\right\}\right), & \text{if } j_L \ne 0 \text{ and } j_U \ne 0; \end{cases} \\
&= \begin{cases} F_{W_t}\left(w_0 d_0\right) - F_{W_t}\left(w_0 c_0\right), & \text{if } j_L = 0 \text{ and } j_U = 0; \\ F_{W_t}\left(w_0 d_U\right) - F_{W_t}\left(w_0 \max\left\{c_0, c_U\right\}\right), & \text{if } j_L = 0 \text{ and } j_U \ne 0; \\ F_{W_t}\left(w_0 \min\left\{c_L, d_0\right\}\right) - F_{W_t}\left(w_0 d_L\right), & \text{if } j_L \ne 0 \text{ and } j_U = 0; \\ F_{W_t}\left(w_0 \min\left\{c_L, d_0\right\}\right) - F_{W_t}\left(w_0 \max\left\{c_0, c_U\right\}\right), & \text{if } j_L \ne 0 \text{ and } j_U \ne 0; \end{cases}
\end{aligned}
$$

where

$$
\begin{aligned}
c_0 &= \nu_{L,0} - b_1 \xi_{L,i_L} - b_3,\; d_0 = \nu_{U,0} - a_1 \xi_{U,i_U} - a_3, \\
c_L &= \nu_{L,j_L} - b_1 \xi_{L,i_L} - b_3,\; d_L = \nu_{L,j_L-1} - b_1 \xi_{L,i_L} - b_3, \\
c_U &= \nu_{U,j_U-1} - a_1 \xi_{U,i_U} - a_3,\; d_U = \nu_{U,j_U} - a_1 \xi_{U,i_U} - a_3,
\end{aligned}
$$

and $i = i_L\left(\eta_U + 1\right) + i_U$, $j = j_L\left(\eta_U + 1\right) + j_U$ with $i_L,\ j_L \in \{0, 1, \ldots, \eta_L\}$, $i_U,\ j_U \in \{0, 1, \ldots, \eta_U\}$.

For the $\left[\left(\eta_L + 2\right) \times \left(\eta_U + 2\right)\right]^2$ transition matrix $\mathbf{P}_{W_0}$ whose (i, j)th component is the conditional probability $P_{i,j|W_0}$, all the information contained in $\mathbf{P}_{W_0}$ can be obtained in the sub-matrix $\mathbf{Q}_{W_0}$ by excluding the rows and columns contained absorbing states. Let random variable $T_{k|W_0}$ be the conditional ARL of the two sided-chart that starts at one of the nonabsorbing state k. For convenience, we define the $\left(\eta_L + 1\right)\left(\eta_U + 1\right) \times 1$ vectors

$\mathbf{T}$ and $\mathbf{t}$ by

$$\mathbf{T} = \begin{bmatrix} T_{0|W_0} \\ T_{1|W_0} \\ \vdots \\ T_{\eta|W_0} \end{bmatrix} \text{ and } \mathbf{t} = \begin{bmatrix} t \\ t \\ \vdots \\ t \end{bmatrix} = t\mathbf{1},$$

where $\mathbf{1}$ is an $(\eta_L + 1)(\eta_U + 1) \times 1$ vector of ones. Then the conditional probability mass function of $\mathbf{T}$ has the form

$$P(\mathbf{T} = \mathbf{t} \,|W_0 = w_0) = \mathbf{Q}_{W_0}^{t-1} (\mathbf{I} - \mathbf{Q}_{W_0}) \mathbf{1},$$

for $t = 1, 2, 3, \ldots$. The unconditional transition matrix is given by

$$\mathbf{Q} = \int_0^{\infty} \int_{-\infty}^{\infty} \mathbf{Q}_{W_0} f_{W_0}(w_0)\, dw_0.$$

Thus, the unconditional probability mass function of $\mathbf{T}$ is given by

$$P(\mathbf{T} = \mathbf{t}) = \mathbf{Q}^{t-1} (\mathbf{I} - \mathbf{Q}) \mathbf{1},$$

The conditional expectations of these $\eta_L + 1$ random variables is determined by

$$\begin{aligned} E(\mathbf{T}_L \,|W_0 = w_0) &= \begin{bmatrix} E(T_{L,0} \,|W_0 = w_0) \\ E(T_{L,1} \,|W_0 = w_0) \\ \vdots \\ E(T_{L,\eta_L} \,|W_0 = w_0) \end{bmatrix} \\ &= \sum_{t=1}^{\infty} t\mathbf{Q}_{W_0}^{t-1} (\mathbf{I} - \mathbf{Q}_{W_0}) \mathbf{1} = (\mathbf{I} - \mathbf{Q}_{W_0}) \mathbf{1}. \end{aligned}$$

The unconditional expectations are given by

$$E(\mathbf{T}) = (\mathbf{I} - \mathbf{Q}) \mathbf{1}.$$

4.4 Gamma Distributed Data

In this section, we will assume that the quality measurement X has a gamma distribution. The gamma distribution can be defined by its density function

$$f_X(x) = \frac{1}{\Gamma(\theta)\,\theta^{\kappa}} x^{\kappa-1} e^{-x/\theta} I_{(0,\infty)}(x),$$

where $\theta > 0$ and $\kappa > 0$. The parameters θ and κ are known as the scale and shape parameters, respectively. We will assume that the process is in a state of statistical in control if $\theta = \theta_0$ and $\kappa = \kappa_0$ for fixed constants θ_0 and κ_0. Typically, these in-control process parameters are unknown and must be estimated. In this case, we assume that we will have from a Phase I study m samples each of size n of items from the output of the process believed to come from an in-control process. Further, assume that the mn quality measurements $X_{i,1}, \ldots, X_{i,n}$ for $i = 1, \ldots, m$ are independent and identically distributed with common distribution a gamma distribution with scale and shape parameters θ_0 and κ_0, respectively. From these data, we will obtain the estimates $\widehat{\theta}_0$ and $\widehat{\kappa}_0$ of θ_0 and κ_0, respectively. These estimates are then used to define the meaning of the process being in a state of statistical in control.

Three methods that are commonly used for estimating process parameters are the methods of least squares, maximum likelihood, and moments. Since

$$E(X) = \kappa_0\theta_0 \text{ and } E\left(X^2\right) = \kappa_0(\kappa_0+1)\,\theta_0^2,$$

the method of moment estimates $\widehat{\theta}_0$ and $\widehat{\kappa}_0$ for θ_0 and κ_0 are the solutions to the system of equations

$$\widehat{\kappa}_0\widehat{\theta}_0 = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij} \text{ and } \widehat{\kappa}_0(\widehat{\kappa}_0+1)\,\widehat{\theta}_0^2 = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}^2.$$

The solutions $\widehat{\theta}_0$ and $\widehat{\kappa}_0$ can be expressed by

$$\begin{aligned}\widehat{\theta}_0 &= \frac{\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}^2-\left(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}\right)^2}{\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}} \text{ and}\\ \widehat{\kappa}_0 &= \frac{\left(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}\right)^2}{\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}^2-\left(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}\right)^2}.\end{aligned}$$

In the special case of κ_0 being a positive integer (Erlang distribution) and κ_0 known, then the method of moments gives

$$\widehat{\theta}_0 = \frac{1}{mn\kappa_0}\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}.$$

It is not difficult to show that in this case that $\widehat{\theta}_0$ has a gamma distribution with scale parameter $\theta_0/(mn\kappa_0)$ and shape parameter $mn\kappa_0$. Further, we see that

$$W_0 = \frac{mn\kappa_0\widehat{\theta}_0}{\theta_0} \sim GAMMA\left(\frac{1}{mn}, mn\right).$$

In the Phase II, information about the quality of the process comes in the form of the quality measurements $X_{t,1},\ldots,X_{t,n}$ on n items from the process output produced at time t. We take as our estimators for θ and κ the statistics

$$\begin{aligned}\widehat{\theta}_t &= \frac{\frac{1}{n}\sum_{j=1}^{n}X_{t,j}^2-\left(\frac{1}{n}\sum_{j=1}^{n}X_{t,j}\right)^2}{\frac{1}{n}\sum_{j=1}^{n}X_{tj}} \text{ and}\\ \widehat{\kappa}_t &= \frac{\left(\frac{1}{n}\sum_{j=1}^{n}X_{t,j}\right)^2}{\frac{1}{n}\sum_{j=1}^{n}X_{t,j}^2-\left(\frac{1}{n}\sum_{j=1}^{n}X_{t,j}\right)^2}.\end{aligned}$$

For the case in which κ_0 is a known positive integer that is not affected when the process changes from an in-control state to an out-of-control state, then we have that our estimate $\widehat{\theta}_t$ for θ at time t is

$$\widehat{\theta}_t = \frac{1}{n}\sum_{j=1}^{n}X_{t,j}^2.$$

Defining the random variable $W_t = \widehat{\theta}_t$, then we can write

$$\frac{\widehat{\theta}_t}{\widehat{\theta}_0} = \lambda \frac{n\kappa_0 \left(\widehat{\theta}_t/\theta\right)}{W_0},$$

where $\lambda = \theta/\theta_0$. It is not difficult to show that the conditional distribution of W_t given $W_0 = w_0$ is a gamma distribution with scale parameter λ/n and shape parameter n. Hence, the results in Sections 2 and 3 of this chapter can be applied to one- and two-sided generalized cumulative sum type charts based on the statistic

4.5 Monitoring for a Change in the Process Variance

Here we return to the case in which the quality measurement X has a normal distribution with mean μ and variance σ^2. The mean is a special type of parameter known as a location parameter. The variance is a scale parameter. We assume that when the process is in-control, the process parameters μ and σ^2 have the values μ_0 and σ_0^2, respectively. Our interest in this section is to examine the use of the generalized family of cumulative sum type charts for monitoring the process variance for a change from σ_0^2 to σ^2. The function of the lower one-sided chart is to monitor for a decrease (process improvement) in the variance and the upper one-sided chart to monitor for an increase in the variance.

Typically, the values μ_0 and σ_0^2 will not be known and will need to be estimated from data measured on the output of the process when it is believed to be in-control. These data can be obtained from a Phase I study. We assume the practitioner will have available m sets $\{X_{i,1}, \ldots, X_{i,n}\}$ for $n > 1$ and $i = 1, \ldots, m$ of measurements that can be taken as independent random samples. The statistic

$$\widehat{\sigma}_0^2 = \frac{1}{m\,(n-1)} \sum_{i=1}^{m} \sum_{j=1}^{n} \left(X_{i,j} - \overline{X}_i\right)^2 = \frac{1}{m} \sum_{i=1}^{m} S_i^2$$

is an unbiased estimator of σ_0^2, where $\overline{X}_i$ and S_i^2 are the mean and variance of the ith sample. In the monitoring phase (Phase II), we assume that periodically the practitioner will have available a sequence $\{X_{t,1}, \ldots, X_{t,n}\}$ of measurements for $n > 1$ and $t = 1, 2, 3, \ldots$ on the output of the process to make a decision about the quality of the process. We assume that these samples are independent random samples and independent of the measurements from Phase I. We define the statistic Y_t to be used to define our chart statistics by

$$Y_t = \frac{\widehat{\sigma}_t^2}{\widehat{\sigma}_0^2},$$

where $\widehat{\sigma}_t^2 = S_t^2$ is variance of the sample collected at time t defined by

$$S_t^2 = \frac{1}{n-1} \sum_{j=1}^{n} \left(X_{t,j} - \overline{X}_t\right)^2.$$

We can write Y_t as W_t/W_0, where

$$W_0 = \frac{m\,(n-1)\,\widehat{\sigma}_0^2}{\sigma_0^2} \text{ and } W_t = m\lambda^2 \frac{(n-1)\,\widehat{\sigma}_t^2}{\sigma^2},$$

where $\lambda^2 = \sigma^2/\sigma_0^2$. It can be shown that

$$\begin{aligned} W_0 &\sim GAMMA\left(2, \frac{m\,(n-1)}{2}\right) \text{ and} \\ \frac{W_t}{W_0}\,|W_0 = w_0 &\sim GAMMA\left(\frac{2\lambda^2}{w_0/m}, \frac{n-1}{2}\right). \end{aligned}$$

Thus, our previous results can directly be used to analyze the run length performance of the charts based on the statistic $Y_t = \widehat{\sigma}_t^2/\widehat{\sigma}_0^2$.

4.6 Conclusion

Our interest in this chapter was the use of the generalized cumulative sum type control charting procedures to monitor for a change in the scale parameter. Useful integral equations were derived for determine the run length performance of the one-sided charts. The Markov chain method was presented as a way to analyze the performance of the two-sided charts. For both the integral equations and Markov chain approach, it was only required that the statistic $Y_t = W_t/W_0$ is defined such that the distributions of W_t and W_t given $W_0 = w_0$ are both gamma distributions. Two special cases discussed were in monitoring for a change in scale parameter of an Erlang distribution with positive integer shape parameter that is known and monitoring for a change in the variance of a normal distribution.

CHAPTER 5

CONCLUSION

5.1 General Conclusions

Control charts are used in two phases of production process. In Phase I, the chart is used to estimate what is meant by the process being in control. In Phase II, a control chart is used to compare the data to detect whether the process changes from a in control process to an out-of-control process. As an aid to the practitioner, various Phase II quality control charts have been developed for monitoring for increases as well as decreases of the mean or standard deviation of a continuous quality measurement.

The generalized family of cumulative sum type control charts proposed by Champ, Woodall, Moshen (1991) include several most commonly used charts, such as Shewhart, CUSUM, and EWMA charts. Equivalent forms of the generalized control chart was presented that require fewer chart parameters to be specified by the practitioner. The run length performance of the generalized control charts were studied integral equations and a Markov chain approximation. We have given integral equations useful in determining the run length distribution of the lower and upper one-sided charts. The Markov chain methods for the one- and two-sided charts are given. We discussed the use of generalized cumulative sum type control charts in monitoring for a change in the mean of a normal distribution in which a performance analysis is given. We also designed and analyzed a chart for monitoring the scale parameter when the quality measurement follows a gamma distribution which includes the design and analysis of a chart for monitoring for a change in the variance of a normal distribution. As special cases, we discussed monitoring for a change in scale parameter of an Erlang distribution with positive integer shape parameter that is known and monitoring for a change in the variance of a normal distribution.

5.2 Areas for Further Research

We'll continue to work on this topic as we still have plenty to do.There are many interesting areas in the analysis of the family of generalized cumulative sum type control charts for further research. In our following work, we are intereted in designing efficient programs for determining the run length properties of a chart. Also, we have interest in examing the performance of those control charts that take $Y_{U,t}$ and $Y_{L,t}$ as the following statistics:

$$Y_{U,t} = \max\left\{\mu_0, \overline{X_t}\right\}, Y_{L,t} = \min\left\{\mu_0, \overline{X_t}\right\}$$

$$Y_{U,t} = \max\left\{\sigma_0^2, S_t^2\right\}, Y_{L,t} = \min\left\{\sigma_0^2, S_t^2\right\}.$$

Adaptive control charts were discussed by Champ (1986). He suggested using more stringent runs rules for detecting a shift in the process if there were evidence the process may be out-of-control and less stringent runs rules otherwise. Since then, adaptive versions of most of the popular control charts found in the literature have been proposed. It would be useful to develop adaptive versions of the cumulative sum type control charts.

As we discussed, the performance of the generalized cumulative sum type control charts depends on six parameters. For the equivalent versions, there are four parameters need to be selected by the practitioners. The selection of these parameters can be posed as an optimization problem. Aparisi, Lluch and de Luna (2008) showed how the optimum values of this chart found by employing Genetic Algorithm. We are intereted in developing better methods to solve the optimzation problem to improve the performance of our charts.

In many production processes, the quality measurement of interest is the lifetime X of the product. In the literature, one can find a variety of lifetime distributions that could be used to model the distribution of X. For example, the Weibull distribution is the commonly studied. However, in the production process, one must be able to obtain information about the quality of the process in a relatively short period of time. One way to obtain information about the lifetime of the product can be done in some cases using accelerated life testing along with censored sampling. It is our interest to develop a design procedure for generalized cumulative sum type control charts when the quality measurement is a lifetime variable.

Moreover, it is would be very interesting to use generalized cumulative sum type control charts in medical surveillance and industrial surveillance. Many methods have been developed for industrial statistical process control. Woodall (2006) showed that there are many applications of control charts in health-care monitoring and in public-health surveilance and there can be a connection between the two areas.

BIBLIOGRAPHY

[1] Aparisi, F., Lluch, L., and de Luna, M.A., *The Optimum Design of the General Control Chart*, EngOpt 2008 - International Conference on Engineering Optimization, Rio de Janeiro, Brazil, June 1-5.

[2] L. Bain and M. Engelhardt, *Introduction to Probability and Mathematical Statistics*, 2nd ed., Duxbury (1992).

[3] D. Brook and D.A. Evans, *An Approach to The Probability Distribution of Cusum Run Length*, Biometrika **59**(**3**) 1972), 539-549.

Champ1987 C.W. Champ and W.H. Woodall, *Exact Results for Shewhart Charts with Supplementary Run Rules*, Technometrics **29** (1987), 393-399.

[4] C.W. Champ, W.H. Woodall, H.A. Mohsen, *A Generalized Quality Control Procedure*, Statistics and Probability Letters **11** (1991), 211-218.

[5] C.W. Champ and S-P. Chou, *Comparison of Standard and Individual Limits Phase I Shewhart, R, and S Charts*, Journal of Quality and Reliability Engineering International **19** (2003), 161-170.

[6] C.W. Champ, S.E. Rigdon, *Comparison of The Markov Chain and The Integral Equation Approaches for Evaluating The Run Length Distribution of Quality Control Charts*, Communications in Statistics - Simulation and Computation **20**(**1**) (1991), 191-204.

[7] C.W. Champ, S.E. Rigdon, and K.A. Scharnagl, *Method for Deriving Integral Equations Useful in Control Chart Performance Analysis*, Nonlinear Analysis: Theory, Methods, and Applications **47**(3) (2001), 2089-2101.

[8] S. Knoth, *Accurate ARL Computation for EWMA-S^2 Control Charts*, Statistics and Computing, **15** (2005), 341-352.

[9] D.C. Montgomery, *Introduction to Statistical Quality Control*, Wiley (New York) (2008).

[10] P.B. Newton and C.W. Champ, *Probability Limits for Shewhart Phase I -Charts*, Proceedings of the Southeast Decision Sciences Institute Annual Conference, February 26-28 (1997), Atlanta, Georgia, 234-236.

[11] E.S. Page, *Cumulative Sum Charts*, Echnometrics **3**(**1**) (1961), 1-9.

[12] S.W. Roberts, *Control Chart Tests Based on Geometric Moving Averages*, Technometrics **1** (1959), 239-250.

[13] W.A. Shewhart, *Some Applications of Statistical Methods to the Analysis of Physical and Engineering Data*, Bell System Technical Journal, 1924.

[14] W.A. Shewhart, *The Application of Statistics as an Aid in Maintaining Quality of a Manufactured Product*, Journal of the American Statistical Association **20** (1925), 546-548.

[15] W.A. Shewhart, *Economic Control of Quality of Manufactured Product*, Van Nostrand, (New York) (1931).

[16] L. Shu, W. Jiang and S. Wu, *A One-sided EWMA Control Chart for Monitoring Process Means*, Communications in Statistics, Simulation and Computation **36** (2007), 901-920.

[17] W.H. Woodall, *The Distribution of the Run Length of One-Sided CUSUM Procedure for Continuous Random Variables*, Technometrics **3**) (1983), 295-301.

[18] W.H. Woodall, *On the Markov Chain Approach to the Two-Sided CUSUM Procedure*, Technometrics **26**(**1**) (1984), 41-46.

[19] W.H. Woodall, *The Use of Control Charts in Health-Care and Public-Health Surveilance*, Journal of Quality Technology **38**(**2**) (2006), 89-104.