Destackification
and Motivic Classes of Stacks

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Abstract

This thesis consists of three articles treating topics in the theory of algebraic stacks. The first two papers deal with motivic invariants. In the first, we show that the class of the classifying stack $\mathrm{B}\,\mathrm{PGL}_n$ is the inverse of the class of PGL_n in the Grothendieck ring of stacks for $n \leq 3$. This shows that the multiplicativity relation holds for the universal torsors, although it is known not to hold for torsors in general for the groups PGL_2 and PGL_3.

In the second paper, we introduce an exponential function which can be viewed as a generalisation of Kapranov's motivic zeta function. We use this to derive a binomial theorem for a power operation defined on the Grothendieck ring of varieties. As an application, we give an explicit expression for the motivic class of a universal quasi-split torus, which generalises a result by Rökaeus.

The last paper treats destackification. We give an algorithm for removing stackiness from smooth, tame stacks with abelian stabilisers by repeatedly applying stacky blow-ups. As applications, we indicate how the result can be used for destackifying general Deligne–Mumford stacks in characteristic zero, and to obtain a weak factorisation theorem for such stacks.

Sammanfattning

Den här avhandlingen består av tre artiklar, vilka alla innehåller resultat inom teorin för algebraiska stackar. De två första artiklarna handlar om motiviska invarianter. I den första visar vi att klassen för den klassificerande stacken $B\,\mathrm{PGL}_n$ är invers till klassen för PGL_n i Grothendieckringen för stackar för $n \leq 3$. Detta visar att multiplikativitetsrelationen gäller för de universella torsorerna, trots att det är känt att den inte gäller för godtyckliga torsorer för grupperna PGL_2 och PGL_3.

I den andra artikeln härleder vi ett explicit uttryck för den motiviska klassen av en universell kvasisplittrad torus. Detta generaliserar en sats av Rökaeus. I vårt bevis introducerar vi en exponentialfunktion, som kan ses som en generaliserig av Kapranovs motiviska zeta-funktion.

Den sista artikeln handlar om destackifiering. Vi konstruerar en algoritm som avlägsnar stackighet från glatta, tama stackar med abelska stabilisatorer genom upprepade stackiga uppblåsningar. Vi beskriver också skissartat hur detta kan användas för att destackifiera allmänna Deligne–Mumford-stackar i karaktäristik noll. Detta medför existens av svag faktorisering av birationella avbildningar mellan dylika stackar.

List of Papers

Paper A
Motivic classes of some classifying stacks.
Preprint (29 pages).

Paper B
The Binomial Theorem and motivic classes of universal quasi-split tori.
Preprint (12 pages).

Paper C
Functorial destackification of tame stacks with abelian stabilisers.
Preprint (50 pages).

Contents

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1. Introduction

Algebraic stacks were introduced since they provide a natural means to study moduli problems in algebraic geometry. This thesis contains three papers which all study various aspects of algebraic stacks, but not so much for their connection to moduli problems, but rather as interesting objects in their own right.

The first two papers treat the computation of motivic invariants for certain algebraic stacks. More precisely, they contain computations in the Grothendieck ring of stacks, which is closely related to the Grothendieck ring of varieties. I will give an introduction to these motivic rings in the next chapter, with the aim of putting the results from the papers into a broader context.

The last paper regards removal of stackiness from an algebraic stack in a process which we call *destackification*. The methods used have much in common with classic methods for resolving singularities of varieties. In the last chapter of this introduction, I will describe this connection.

Since algebraic stacks play a central role in all three papers, I start by giving a short introduction to these objects.

1.1 Algebraic Stacks

Stacks were first introduced by Giraud in his description of non-abelian cohomology [Gir71]. Deligne and Mumford used stacks to describe moduli of curves, and they defined what is meant for a stack to be algebraic [DM69]. This definition was later generalised by Artin [Art74].

The notion of algebraic stacks is a generalisation of the notions of varieties or schemes. Algebraic stacks are geometric objects, and most geometric properties and concepts which we use to describe schemes are equally relevant when describing algebraic stacks. For instance, we can talk about smoothness, properness and separatedness. Just as for schemes, the set of points of an algebraic stack is endowed with a natural topology, which is called the Zariski topology, and we can talk about closed and open substacks.

One way to obtain algebraic stacks is as *stack quotients*. Let U be a smooth variety over the complex numbers, and assume that G is a finite group acting on U. Then we have a sequence of natural maps

$$U \xrightarrow{f} [U/G] \xrightarrow{g} U/G,$$

where U/G denotes the usual, *coarse* quotient[1] and $[U/G]$ denotes the stack quotient. If the action of G is free, then g is an isomorphism and the two different concepts of quotient coincide. In this case, the object U/G is smooth, and the natural map $g \circ f \colon U \to U/G$ is unramified. However, if the action is not free, then U/G might be singular, and $g \circ f$ is usually ramified. In contrast, the stack quotient $[U/G]$ will always be smooth, and the map f will always be unramified. In other words, a stack quotient always behaves as if the action were free. We give two examples.

Example 1.1.1. Let k be a field of characteristic $\neq 2$. Assume that the group $\mu_2 = \{1, -1\}$ acts on the affine plane $\mathbb{A}^2 = \operatorname{Spec} k[x, y]$ diagonally. In other words, the action is given by $\zeta(x, y) = (\zeta x, \zeta y)$. Denote the stack quotient $[\mathbb{A}^2/\mu_2]$ by X. The coarse quotient is the spectrum $X_{\mathrm{cs}} = \operatorname{Spec} k[x^2, xy, y^2]$ of the invariant ring. Note that X_{cs} has a singularity at the origin and that the natural map $\mathbb{A}^2 \to X_{\mathrm{cs}}$ is ramified over the singularity.

Example 1.1.2. Let μ_2 be as above, but let its action on $\mathbb{A}^2$ be defined by $\zeta(x, y) = (\zeta x, y)$. Denote the stack quotient by Y. The coarse quotient is given by $Y_{\mathrm{cs}} = \operatorname{Spec} k[x^2, y]$. This time, the space Y_{cs} is smooth since $k[x^2, y]$ is a polynomial ring in the symbols x^2 and y, but the map $\mathbb{A}^2 \to Y_{\mathrm{cs}}$ is ramified over the divisor $x^2 = 0$.

It is useful to think of the stack $[U/G]$ as some kind of formal quotient of U by G. It is, however, important to note that the scheme U and the group G is not part of the structure of the stack $[U/G]$, but rather a *presentation* for $[U/G]$. Just as there are many ways to express a scheme as a free quotient[2] by a group, there are many ways to present the stack $[U/G]$ as a stack quotient. However, the *stabilisers* of the group action are independent of the presentation. For instance, the stack X in Example 1.1.1 has trivial stabilisers everywhere except at the origin where the stabiliser is μ_2. The stack Y has non-trivial stabilisers precisely along the divisor $x = 0$. An often used heuristic when describing algebraic stacks is thinking of them as schemes with stabilisers attached to points. This is, however, usually the wrong way to view stacks. Note that in Example 1.1.1, the stack X has more in common with the *atlas* $\mathbb{A}^2$ than the coarse space X_{cs}. For instance, the smoothness property of $\mathbb{A}^2$ *descends* to X.

[1]The coarse quotient might not exist as a scheme, but it exists as an analytic variety or an algebraic space.

[2]A silly example would be to express the variety U as the quotient of $U \times G$ by G.

Algebraic stacks for which all stabilisers are trivial are called *algebraic spaces*. Algebraic spaces are quite mild generalisations of schemes. Sometimes, it is of interest to approximate an algebraic stack X by an algebraic space in the best way possible. Let X be an algebraic stack with finite stabilisers. If $\pi\colon X \to X_{\mathrm{cs}}$ is a map which is initial among maps to algebraic spaces and the induced map $|X| \to |X_{\mathrm{cs}}|$ between topological spaces is a homeomorphism, we say that X_{cs} is the *coarse space*[1] of X. We have already seen examples of this. Indeed, the coarse space of a stack quotient by a finite group is simply the coarse quotient.

Not every algebraic stack can be obtained as a stack quotient, but in many interesting cases they have associated coarse spaces anyway. There is a nice characterisation of when an algebraic stack has a coarse space in terms of the so called *inertia stack*. Every stack X has an inertia stack $I_X \to X$, which parametrises the stabilisers of the stack. More precisely, the stabiliser at a k-point $x\colon \operatorname{Spec} k \to X$ is the pullback of I_X along x. An algebraic stack has a coarse space precisely when its inertia I_X is finite over X. This was first proven by Keel–Mori [KM97] in the noetherian case, and in more general settings by Conrad and Rydh [Con05, Ryd13].

In the two examples above, we assumed that the characteristic of the base field was different from 2. If we instead assume that the characteristic of k is 2, then $\mu_2 = \operatorname{Spec} k[x]/(x^2-1)$ is a non-reduced group scheme. Both the stack quotients and the coarse quotients still exist and the stack quotients are still smooth. However, the stack quotients will not be algebraic in the sense of Deligne–Mumford, but they are in the sense of Artin. They belong to a class of stacks which are called *tame stacks*. Tame Artin stacks were introduced by Abramovich–Vistoli–Olsson, who also gave a nice structure theorem characterising them [AOV08]. In an appendix to Paper C, we sharpen this structure theorem in the case of smooth stacks, and we simplify a rather technical step in the original proof.

In general, an algebraic space can be the coarse space of several different non-isomorphic smooth stacks. An algebraic stack X is called *canonical* for its coarse space X_{cs} provided that it is terminal among smooth stacks which have X_{cs} as coarse space. This condition is equivalent to the coarse map $\pi\colon X \to X_{\mathrm{cs}}$ being an isomorphism away from a locus of codimension 2. In the two examples above, the stack X is canonical over X_{cs}, but Y is not canonical over Y_{cs}. Indeed, since Y_{cs} is smooth, it is its own canonical stack. Canonical stacks were first mentioned by Vistoli in [Vis89]. He observed that for a variety Z over a field of characteristic zero to have a canonical stack, it is enough for

[1]Many use the term *coarse moduli space*, but we drop the modifier *moduli* when talking about coarse spaces without having a particular moduli problem in mind.

it to be locally[1] a quotient of a smooth variety by a finite group. In positive characteristic, the situation is less clear, but a result in the same direction has been obtained by [Sat12] in the tame case.

One way of constructing stacks is via the *root construction*. Given a smooth stack X and an effective Cartier divisor $E \subset X$, we can form the d-th *root stack* $\pi\colon X_{d^{-1}E} \to X$. The morphism π is an isomorphism over the locus $X \setminus E$, and the induced map π_{cs} between the coarse spaces is an isomorphism. There is a canonically defined divisor $d^{-1}E$ on $X_{d^{-1}E}$ with the property that $d \cdot d^{-1}E \simeq \pi^* E$. We think of the the construction as formally adjoining a d-th root of E. The map $Y \to Y_{\mathrm{cs}}$ in Example 1.1.2 is an example of the root construction. Here the divisor $x = 0$ in Y is the square root of the divisor $x^2 = 0$ in Y_{cs}. More detailed descriptions of the root constructions are given in [AGV08, Cad07] and [FMN10, §1.3.b].

So far, we have mostly discussed stacks with finite stabilisers. Much of the theory is identical if we allow stabilisers of positive dimension, but there is no corresponding theory of coarse spaces[2].

Perhaps the simplest examples of stacks are the *classifying stacks* $\mathrm{B}G$ where G is an algebraic group. They are obtained as stack quotients of trivial group actions. The stack $\mathrm{B}G$ parametrises G-torsors in the sense that the morphisms $T \to \mathrm{B}G$ from a scheme T correspond to G-torsors over T. As an example, the stack $\mathrm{B}\,\mathrm{GL}_n$ parametrises GL_n-bundles, or, equivalently, locally free sheaves of rank n. Classifying stacks play a prominent role in Paper A.

The standard references for algebraic spaces and algebraic stacks are the text books [Knu71] and [LMB00] respectively. A nice and thorough introduction to stacks and descent theory is given in [Vis05]. There is also an upcoming text book on algebraic spaces and stacks by Martin Olsson. Another useful resource is the Stacks Project [SP], which is an open source reference work for algebraic stacks.

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[1] By locally, we mean étale locally in this situation.

[2] There is, however, a notion of *good moduli spaces* introduced by Alper [Alp13], which can be seen as a partial generalisation.

2. Motivic Invariants

The first two articles treat computation of motivic classes in the Grothendieck group of stacks. In the first article, the class of the classifying stacks $\mathrm{B}\,\mathrm{PGL}_2$ and $\mathrm{B}\,\mathrm{PGL}_3$ are computed. The motivation behind this was a question regarding a multiplicativity relation in the Grothendieck ring, as explained in Section 2.4. In the second article, the explicit formulae for classes of universal quasi-split are derived. These are stated in Section 2.7. The results are used in the computations in the first article.

We start with some background material on the Grothendieck rings of varieties and stacks, and explain some of the techniques that can be used to make computations in them.

2.1 The Grothendieck Ring of Varieties

Let Var_k denote the category of varieties over a field k. The *Grothendieck ring of varieties*, denoted $\mathrm{K}_0(\mathrm{Var}_k)$, is defined as the free abelian group on the set of isomorphism classes $\{X\}$ of varieties X subject to the *scissors relations*

$$\{X\} = \{X \setminus Z\} + \{Z\}$$

for closed subvarieties Z of X, and the *bundle relations*[1]

$$\{E\} = \{\mathbb{A}^n \times X\}$$

for rank n vector bundles $E \to X$. The multiplication in $\mathrm{K}_0(\mathrm{Var}_k)$ is induced by the categorical product. In other words, we have

$$\{X\} \cdot \{Y\} = \{X \times Y\}$$

for varieties X and Y. The class of the affine line is called the *Lefschetz class* and is denoted by $\mathbb{L}$. Although the ring $\mathrm{K}_0(\mathrm{Var}_k)$ actually was considered by Grothendieck [CS01, (Letter of August 16, 1964)], the modern interest in it was initiated by a talk by Kontsevich on motivic integration [Kon95].

[1] For varieties, this relation is redundant, but we include it for later purposes.

A priori, from the definition of $\mathrm{K}_0(\mathrm{Var}_k)$, it is not clear that the ring does not collapse to the trivial ring. If $k = \mathbb{F}_q$ is a finite field, non-triviality follows from the existence of a *point counting measure*

$$\psi_q\colon \mathrm{K}_0(\mathrm{Var}_k) \to \mathbb{Z}, \qquad \psi_q\colon \{X\} \mapsto \#X(\mathbb{F}_q).$$

By considering ψ_{q^n} for varying n, it is easy to show that in fact the subring $\mathbb{Z}[\mathbb{L}]$ generated by the Lefschetz class is a polynomial ring. By invoking a more refined point counting argument, one can show that the subring $\mathbb{Z}[\mathbb{L}]$ is polynomial also for arbitrary fields k. This is slightly more technical, and involves spreading out the variety X over schemes S of finite type over $\operatorname{Spec}\mathbb{Z}$ and counting points in the fibres over finite type points in S.

In some special cases, it is easy to use the scissors relations directly to determine the class of a variety in terms of $\mathbb{L}$. For instance, we have the classes

$$\{\mathrm{Gr}(m,n)\} = \prod_{i=0}^{m-1} \frac{\mathbb{L}^n - \mathbb{L}^i}{\mathbb{L}^m - \mathbb{L}^i}, \qquad \{\mathrm{GL}_n\} = \prod_{i=0}^{n-1} \left(\mathbb{L}^n - \mathbb{L}^i\right).$$

for the Grassmannian and the general linear group respectively. Of particular importance for Paper A, is the class $\{\mathrm{PGL}_n\} = \{\mathrm{GL}_n\}/(\mathbb{L}-1)$ of the projective general linear group.

The structure of the ring $\mathrm{K}_0(\mathrm{Var}_k)$ is quite mysterious, but the quotient ring $\mathrm{K}_0(\mathrm{Var}_k)/(\mathbb{L})$ can be described in terms of *stably birational geometry*. Recall that two varieties X, Y are *stably birationally equivalent* provided that $X \times \mathbb{P}^n$ is birationally equivalent to $Y \times \mathbb{P}^m$ for some n and m. The equivalence classes of varieties under stable birationality form a commutative monoid SB_k with multiplication induced by the categorical product. Due to a result by Larsen and Lunts [LL03], there is a natural isomorphism $\mathrm{K}_0(\mathrm{Var}_k)/(\mathbb{L}) \to \mathbb{Z}[\mathrm{SB}_k]$ taking a smooth projective variety to its stably birational equivalence class. The hardest part of establishing this isomorphism is to show that the map extends to all varieties. This step uses Nagata compactification, resolution of singularities and weak factorisation, and the result is therefore currently only known to hold in characteristic zero. This step has also been conceptualised by Heinloth-Bittner [Bit04], who constructed a presentation of $\mathrm{K}_0(\mathrm{Var}_k)$ with generators being smooth and projective varieties and relations in terms of blowing up smooth centres. As an application, Poonen demonstrated that $\mathrm{K}_0(\mathrm{Var}_k)$ has zero-divisors, using that the stably birational class of an abelian variety is uniquely determined by its isomorphism class [Poo02].

For the purpose of this thesis, the connection with stably birational geometry is less interesting, since we shall usually formally invert the Lefschetz class. In this setting, much less is known. It is also unknown whether $\mathbb{L}$ is a zero-divisor in $\mathrm{K}_0(\mathrm{Var}_k)$, so it is unclear how much information is lost by inverting $\mathbb{L}$.

There are several generalisations of the Grothendieck ring of varieties. For instance, it is sometimes convenient to work with the *relative Grothendieck ring of varieties* $\mathrm{K}_0(\mathrm{Var}_S)$, where S is an arbitrary scheme. In [Bit04] a formalism similar to Grothendieck's six operations is developed for these rings. In the same article, the *equivariant Grothendieck ring of varieties* $\mathrm{K}_0(G\text{-}\mathrm{Var}_k)$, where G is a group, is studied. Here G-Var_k denotes the category of varieties endowed with an action by a group G. In fact, this is a special case of the relative Grothendieck ring, if we allow S to be an algebraic stack. Indeed, the categories G-Var_k and $\mathrm{Var}_{\mathrm{B}G}$ are equivalent. Finally, we will also consider the Grothendieck ring $\mathrm{K}_0(\mathrm{Stack}_k)$ of stacks, and its relative version. This ring was studied by Ekedahl in a series of preprints [Eke09a, Eke09b, Eke08]. Similar constructions have been studied independently by Behrend–Dhillon in [BD07] and also by Toën in the context of higher stacks [Toë05]. The main structure result is that $\mathrm{K}_0(\mathrm{Stack}_k)$ is the localisation of $\mathrm{K}_0(\mathrm{Var}_k)$ in the Lefschetz class $\mathbb{L}$ and the cyclotomic polynomials in $\mathbb{L}$. It should be noted that the bundle relation makes a difference in the definition of $\mathrm{K}_0(G\text{-}\mathrm{Var}_k)$ and $\mathrm{K}_0(\mathrm{Stack}_k)$, whereas it is redundant in the case of $\mathrm{K}_0(\mathrm{Var}_S)$ when S is a scheme or an algebraic space. There are also authors who omit the bundle relations when they study the Grothendieck ring of stacks.

2.2 Generalised Euler Characteristics

A ring homomorphism $\mathrm{K}_0(\mathrm{Var}_k) \to R$, for some ring R, is called a *motivic measure*. We have already seen one example in the point counting measure ψ_q defined in Section 2.1. The prototypical example is the Euler characteristic with compact supports. If k is the field of complex numbers, the classical Euler characteristic with compact supports respects the scissors relations for closed subvarieties, and therefore induces a ring homomorphism $\chi_{\mathrm{c}}\colon \mathrm{K}_0(\mathrm{Var}_k) \to \mathbb{Z}$.

The modern definition of Euler characteristic is in terms of cohomology. If we use ℓ-adic cohomology, this makes sense for any base field k. This approach can be generalised to get a much more fine-grained invariant. Recall that a closed subvariety Z of X gives rise to a long exact sequence

$$H^i_{\mathrm{c}}(X \setminus Z, \mathbb{Z}_\ell) \to H^i_{\mathrm{c}}(X, \mathbb{Z}_\ell) \to H^i_{\mathrm{c}}(Z, \mathbb{Z}_\ell) \to H^{i+1}_{\mathrm{c}}(X \setminus Z, \mathbb{Z}_\ell)$$

of cohomology with compact supports. The Euler characteristic for X is obtained by taking the alternating sum of the ranks of $H^i_{\mathrm{c}}(X, \mathbb{Z}_\ell)$. A fancy way of thinking about this is to instead take the alternating sum of the classes of the cohomology groups in the Grothendieck ring $\mathrm{K}_0(\mathrm{Ab})$ of finitely generated abelian groups. The ring $\mathrm{K}_0(\mathrm{Ab})$ is defined as the free abelian group on the isomorphism classes $[A]$ of finitely generated abelian groups A subject to the relations $[A] + [C] = [B]$ for short exact sequences

$$0 \to A \to B \to C \to 0.$$

Since the map $\mathrm{K}_0(\mathrm{Ab}) \to \mathbb{Z}$ induced by taking the rank is an isomorphism, we get nothing new, but the approach indicates how this can be generalised. We outline an approach to this, which is described in more detail in [Eke09a, §2].

Typically, the cohomology groups of an algebraic variety have more structure than just the group structure. If we work over $\mathbb{C}$ or $\mathbb{R}$, they are endowed with mixed Hodge structures [Del71a, Del71b]. There is also a similar notion for finite fields provided by the Weil conjectures. This can be utilised for arbitrary fields by spreading out over a finite type scheme over $\operatorname{Spec}\mathbb{Z}$, similarly to what was outlined for the point counting measure. Finally, for an arbitrary field k, we can also consider actions by the Galois groups of finitely generated subfields of k. The upshot is that the cohomology groups land in an abelian category, which we somewhat vaguely denote Coh_k, and we can consider the generalised Euler characteristic, which we also denote by χ_{c}, taking values in $\mathrm{K}_0(\mathrm{Coh}_k)$. This motivic measure gives much more information about the variety than the classical Euler characteristic.

2.3 Fibre Bundles and Torsors

In geometry, a *fibre bundle* with fibre F is a map of spaces $E \to S$ which locally on S looks like the projection of a Cartesian product $F \times S$ to the second factor. It is common to restrict transformations gluing the bundle together over different coordinate patches to lie in a group G acting as automorphisms on the fibre F. The group G is called the *structure group* of the bundle. Fibre bundles with structure group G are classified by so called *G-torsors*. Torsors, also called *principal homogeneous spaces*, are fibre bundles in their own right. They are fibred by the group G viewed as a G-space by translation.

The rank n vector bundles constitute the prototypical example of fibre bundles. The fibre is an n-dimensional vector space, and the structure group is the general linear group GL_n. Given a rank n-vector bundle, we obtain the corresponding GL_n-torsor as the frame bundle of the vector bundle.

In differential or complex analytic geometry it is fairly clear what we mean by a fibre bundle looking like a product *locally*. In algebraic geometry the question is more subtle. In many cases, the Zariski topology is to coarse to capture the geometry of fibre bundles. As an illustration of this, we consider the following example:

Example 2.3.1. Choose coordinates s and t for the affine plane $\mathbb{A}^2_{\mathbb{C}}$ and consider the Zariski open subset S defined by removing the coordinate axes. Let

C be the plane projective curve over S defined by the equation

$$s \cdot x^2 + t \cdot y^2 + z^2 = 0.$$

Since s and t do not vanish on S, the fibre over each point in S is a non-singular conic. Such a curve is isomorphic to the projective line provided that it has a rational point, which is true over each closed point of S. This gives a hint that we might want to view C as a $\mathbb{P}^1$-fibred bundle over S. But over the generic point of S we have no rational points since the defining equation of C has no solutions in the function field $\mathbb{C}(s,t)$. As a consequence, there can be no Zariski open subset $U \subset S$ over which C is isomorphic to $\mathbb{P}^1 \times U$. Hence C is not a fibre bundle over S if we require bundles to trivialise in the Zariski topology.

Instead, we can formally adjoin the square roots of s and t to the coordinate ring of S. This corresponds to a variety S' surjecting onto S. Over S' the defining equation of C does have solutions and hence C is isomorphic to $\mathbb{P}^1 \times S'$ over S'. The surjection $S' \to S$ is called an *étale covering*. Note that in the classical topology the space S' is a degree 4 covering space over S. In particular, this implies that C is a fibre bundle over S in the complex analytic sense.

When referring to fibre bundles in algebraic geometry, we usually mean with respect to the étale topology[1]. That is, fibre bundles should trivialise over étale coverings as in the example above.

For some structure groups, most notably GL_n, being a fibre bundle in the generalised sense described above is equivalent to being a fibre bundle in the Zariski topology. Such groups are called *special* and have been classified by Serre and Grothendieck [CGS58]. The example above shows that the automorphism group of $\mathbb{P}^1$, namely PGL_2, is *non-special*. The same is true for all projective linear groups PGL_n.

2.4 Multiplicativity Relations for Fibre Bundles

Let $T \to S$ be a torsor for an algebraic group G over the field k. By the *multiplicativity relation* for the torsor, we mean the relation

$$\{T\} = \{G\}\{S\},$$

which might, or might not, hold in $\mathrm{K}_0(\mathrm{Var}_k)$. If T is the trivial torsor, this relation holds by the definition of multiplication in the Grothendieck ring. It is also easy to see that it holds if T trivialises in the Zariski topology. In

[1]Or the even finer *fppf* topology.

particular, it holds when G is a special group. If G is connected, and $k = \mathbb{C}$, then we have the relation

$$\chi_c(T) = \chi_c(G) \cdot \chi_c(S)$$

for the classical Euler characteristic. The same is true over an arbitrary field if we use the generalised Euler characteristic taking values in $\mathrm{K}_0(\mathrm{Coh}_k)$ [Eke09a, p. 6]. In [BD07, A.9], a similar result is obtained by Dhillon, using a generalised Euler characteristic based on Voevodsky's motivic cohomology. This made Behrend and Dhillon raise the question whether multiplicativity actually holds already in $\mathrm{K}_0(\mathrm{Var}_k)$ [BD07, Remark 3.3] for all connected groups. Ekedahl gave a negative answer to this question in [Eke08]. He showed that for any affine, connected, *non-special* group G over the complex numbers, there exists a G-torsor for which the multiplicativity relation does not hold.

When working in the context of stacks, it is natural to ask whether the multiplicativity relation holds in $\mathrm{K}_0(\mathrm{Stack}_k)$ for the universal G-torsor over the stack $\mathrm{B}G$, where G is a connected affine group. In this case, since the universal G-torsor is just the one point space, the multiplicativity relation states that the class of $\mathrm{B}G$ is the inverse of the class of G. In Paper A, we investigate this for the groups PGL_n. It turns out that the class $\{\mathrm{B}\,\mathrm{PGL}_n\}$ in fact is the inverse of $\{\mathrm{PGL}_n\}$ for $n \leq 3$ under mild hypotheses on the base field.

2.5 Étale Classes and the Burnside Ring

When computing classes in $\mathrm{K}_0(\mathrm{Var}_k)$ and the related rings, it is sometimes useful to consider the subring of *étale classes*. This is the subring generated by varieties which are finite étale over the base k, i.e., spectra of finite separable k-algebras. Let K/k be a finite[1] Galois extension with Galois group G. Then Galois descent induces a functor from the category of finite G-sets to Var_k. This induces a ring homomorphism

$$\mathrm{A}(G) \to \mathrm{K}_0(\mathrm{Var}_k),$$

where $\mathrm{A}(G)$ denotes the *Burnside ring* for G. The Burnside ring for G is defined as the Grothendieck ring for the semi-ring of isomorphism classes of finite G-sets. The multiplication and addition operations on $\mathrm{A}(G)$ are induced by the categorical product and coproduct respectively.

The ring homomorphism described above allows us to use the powerful tools for making computations in the Burnside ring to obtain results in the

[1] One could also be more ambitious and consider the functor from the category of finite sets with a continuous action of the absolute Galois group. This functor induces a surjection onto the ring of étale classes.

Grothendieck ring. We give a brief overview of two methods used in Paper A and Paper B.

The first method is classic, and was used already by Burnside himself. For an introduction, see [Knu73]. There is an embedding of the Burnside ring for a group G into the ring of *super central functions* $\mathrm{SCF}(G)$. A super central function for G is simply an integer-valued function on the set of subgroups of G which is constant on conjugacy classes. The ring homomorphism $\mathrm{A}(G) \to \mathrm{SCF}(G)$ is induced by taking a G-set S to the super central function

$$H \mapsto \#S^H, \qquad H \subset G$$

where $\#S^H$ denotes the number of fixed points for S under the action of H.

The other method is newer and originates from Quillen. An introduction is given in [Bou00]. The method depends on the so called *Lefschetz invariant* defined on the category of G-posets. By a G-poset, we mean a finite G-set endowed with a partial ordering respecting the G-action. The Lefschetz invariant of a G-poset P is defined as the alternating sum

$$\Lambda_P = \sum_{i \geq 0} (-1)^i \{\mathrm{Sd}_i P\}$$

where $\mathrm{Sd}_i P$ denotes the G-set of chains $x_0 < \cdots < x_i$ of length i in P. A fundamental fact is that every element in $\mathrm{A}(G)$ can be obtained as Λ_P for some G-poset P. The *Lefschetz invariant* also satisfies several functorial properties. This gives us a convenient method to make computations involving non-effective classes in $\mathrm{A}(G)$.

2.6 The Motivic Zeta Function

The ring $\mathrm{K}_0(\mathrm{Var}_k)$ and its relatives are endowed with a (non-special) lambda ring structure, which is defined in terms of Kapranov's motivic zeta function. Let $[n] = \{1, \ldots, n\}$ denote the set of n symbols, and let Σ_n be the symmetric group acting on it. Given a variety X, we use the notation $X^{[n]}$ for the n-fold product of X with itself considered as a Σ_n-variety with the action given by permuting the factors. The n-th *symmetric power* of X is the quotient $X^{[n]}/\Sigma_n$, and we denote its class in the Grothendieck ring by $\sigma^n(X)$. Consider the element

$$\sigma_t(X) = \sum_{i \geq 0} \sigma^i(X) t^i$$

in the power series ring $\mathrm{K}_0(\mathrm{Var}_k)[[t]]$. The *motivic zeta function* is the extension of $\sigma_t(X)$ to the Grothendieck group of varieties[1]. The lambda ring

[1] A common notation for the zeta function is $Z(X,t)$, but we use $\sigma_t(X)$ since this is standard when discussing lambda rings.

structure on $\mathrm{K}_0(\mathrm{Var}_k)$ is defined via the power series relation

$$\sigma_t(x)\lambda_{-t}(x) = 1, \qquad x \in \mathrm{K}_0(\mathrm{Var}_k).$$

In Paper B, we introduce an exponential function defined on $\mathrm{K}_0(\mathrm{Var}_k)$, which can be seen as a generalisation of the motivic zeta function. The construction was first used by Bouc in the context of Burnside rings [Bou92]. It takes its values in a different ring, which we denote by $\mathscr{K}_0(\mathrm{Var}_k)$. The elements in $\mathscr{K}_0(\mathrm{Var}_k)$ are denoted as formal power series $\sum_{i\geq 0} a_i t^i$ in the symbol t, but each of the coefficients a_i lies in a different group $\mathrm{K}_0(\mathrm{Var}_{\mathrm{B}\Sigma_i})$. The exponential function is defined on effective elements by

$$\{X\} \mapsto \sum_{i\geq 0} \{[X^{[n]}/\Sigma_n]\} t^i.$$

We use the exponential function to extend the power operation $(-)^{[n]}$ to non-effective elements, and to derive a binomial theorem for this power operation. This is used to compute the motivic class of a universal quasi-split torus, as described in next section.

Although not central to this thesis, it should be mentioned that much of the study of the motivic zeta function has been centred around rationality questions. Let X be a smooth, projective scheme over a finite field. If one applies the counting measure to $\sigma_t(X)$, one obtains the local zeta function at X, which was proven to be rational by Deligne[1] [Del74, Del80]. It is therefore natural to ask whether also the motivic zeta function is rational at smooth projective X for arbitrary k. This question was raised by Kapranov [Kap00], and given a negative answer by Larsen and Lunts [LL03, LL04]. However, their counterexample uses the relation to the ring $\mathbb{Z}[\mathrm{SB}_k]$, so the argument is not valid if we invert the Lefschetz class. In the setting where $\mathbb{L}$ is inverted, the question is still open, but rationality has been proven for X a curve if k is a field of characteristic zero [Kap00, Lit14].

2.7 Motivic Classes of Tori

Recall that a torus of rank n is a group scheme which étale locally is isomorphic to a product of n copies of the group $\mathbb{G}_\mathrm{m}$. Such a torus is called *quasi-split* if it is a Weil restriction of the group $\mathbb{G}_\mathrm{m}$ along a map $\operatorname{Spec} L \to \operatorname{Spec} k$, where L is a separable algebra of degree n over k. The group of k-points of such a torus is simply the group $L^\times$ of units in L. In [Rök11], Rökaeus gives an explicit formula for the class of $L^\times$ in $\mathrm{K}_0(\mathrm{Var}_k)$ in terms of the lambda operations on the ring $\mathrm{K}_0(\mathrm{Var}_k)$.

[1] This is the Riemann hypothesis part of the Weil conjectures.

We extend this result to a universal setting in Paper B. By taking the Weil restriction of $\mathbb{G}_{\mathrm{m}}$ along the morphism $\mathrm{B}\Sigma_{n-1} \to \mathrm{B}\Sigma_n$ of stacks, we obtain a *universal* quasi-split torus of rank n over $\mathrm{B}\Sigma_n$. This torus is universal in the sense that every quasi-split torus over any base can be obtained from it via a base change. We show that the rank n universal quasi-split torus has the class

$$\sum_{i=0}^{n}(-1)^i\lambda^i([\Sigma_{n-1}/\Sigma_n])\mathbb{L}^{n-i}$$

in $\mathrm{K}_0(\mathrm{Var}_{\mathrm{B}\Sigma_n})$. In the proof, we use the binomial theorem mentioned in the previous section to reduce the computation to one in the Burnside ring using the methods described in Section 2.5. The same technique could be used to compute the class of any variety which is the Weil restriction of a variety X along a finite étale map, provided that the class of X lies in $\mathbb{Z}[\mathbb{L}]$.

3. Destackification

Let X be a stack which is smooth over a base scheme S, and assume that X has a coarse space X_{cs}. Recall that the coarse space X_{cs} need not be smooth. Consider the following problem: is it possible to find a proper birational morphism $X' \to X$, such that the coarse space X'_{cs} is smooth? We call such a map a *destackification* for X.

The term destackification is motivated as follows. Assume, for simplicity, that X has trivial generic stabilisers. Then the the destackification gives rise to a roof-shaped diagram

$$
\begin{array}{ccc} & X' & \\ {}^{\pi}\swarrow & & \searrow \\ X'_{\text{cs}} & & X \end{array}
$$

of smooth stacks with the morphisms being proper and birational. Since X'_{cs} has trivial stackiness, the diagram can be viewed as a removal of the stackiness from X. In the case when X has generic stabilisers, the canonical map $X' \to X'_{\text{cs}}$ will only be birational up to rigidification.

In Paper C, we show smooth a tame stack X with abelian stabilisers always admits a destackifications $X' \to X$ with very nice properties. We do this by giving an algorithm which produces an explicit construction of the destackification as a sequence of so called *stacky blow-ups*. By this we mean a mix of usual blow-ups and root stacks. At the terminating stage of the process, the coarse map $X' \to X'_{\text{cs}}$ has a canonical factorisation as a gerbe followed by a root stack.

The construction has several functoriality properties. Given a morphism $f\colon Y \to X$ we say that the construction is functorial with respect to f if the destackification obtained by applying the algorithm to Y equals the pull-back of the destackification of X along f. Our algorithm is functorial with respect to the following types of maps:

1. Maps obtained by base change; that is, if X is smooth over the base scheme S, and $Y \to X$ is obtained as the base change of an arbitrary map $S' \to S$ of schemes.

2. Smooth, stabiliser preserving maps. In other words: the sequence of

stacky blow-ups only depends on the stackiness.

3. Gerbes. In other words: the algorithm ignores generic stackiness.

Note that a destackification $X' \to X$ induces a proper, birational map on coarse spaces $X'_{\mathrm{cs}} \to X_{\mathrm{cs}}$. Since X'_{cs} is smooth, this is a desingularisation. Our approach to destackification has many similarities with traditional methods for obtaining desingularisations. On the local scale, the methods are essentially toric. On the global scale, the methods are similar to modern approaches to functorial desingularisation. We explain these connections in more detail in the next two sections.

3.1 Toric and Toroidal Destackification

Toric stacks were introduced by Borisov–Chen–Smith [BCS05], and contributions to the general structure theory have been made by Iwanari [Iwa09b, Iwa09a] and Fantechi–Mann–Nironi [FMN10]. Toric stacks are the prototypical examples of tame stacks with abelian stabilisers. In Paper C, we show that any smooth tame stack with abelian stabilisers étale stabiliser-preserving locally looks like a toric stack. We say that the stack has a *toric chart* at each point.

Toric stacks can be described using combinatorics which is very similar to that of toric varieties. Instead of a usual fan, we have a *stacky fan*. For our purposes, it is enough to consider the case when the stack has trivial generic stabiliser, and in this case a stacky fan is just a usual fan with marked lattice points on each of its rays. The coarse space of a toric stack is the toric variety[1] corresponding to the fan obtained by forgetting these markings. Both examples given in Section 1.1 are examples of toric stacks.

We have the same orbit–cone correspondence for toric stacks as for usual toric varieties. In particular, the rays correspond to toric divisors. The markings on the rays determine the generic stackiness along the divisors. Taking a root of a divisor corresponds to moving the marking on the corresponding ray, and blowing up an intersection of toric divisors corresponds to subdividing the corresponding cone at the ray going through the sum of the marked lattice points.

Recall that the usual desingularisation algorithm for simplicial toric varieties works by repeatedly subdividing cones with high multiplicity [CLS11, §11]. From the description above, it is quite easy to see how this can be adapted to obtain a destackification algorithm for toric stacks. However, the combinatorics gets somewhat more involved if we want our algorithm to have the right

[1] In the relative case it is a flat family of toric varieties.

functorial properties. This has to do with the non-local behaviour of taking roots of divisors.

A *toroidal variety* X is a variety which is endowed with a *toroidal structure* in the form of a divisor $D \subset X$, see [KKMSD73]. Étale locally at each point, the toroidal variety is required to be toric and the toroidal structure is required to match the toric divisors. The toroidal structure makes it possible to desingularise X using a generalisation of the combinatorial device used in the toric case.

The concept of toroidal variety generalises to stacks, and we can talk about toroidal stacks. Moreover, the toroidal desingularisation algorithm generalises more or less directly to a toroidal destackification algorithm. In Paper C, however, we take an approach to destackification which does not require a toroidal structure.

3.2 Comparison with Functorial Desingularisation

In this section, we compare our approach to destackification with the classical desingularisation algorithm by Hironaka [Hir64] in its more modern, functorial formulation by Bierstone–Milman [BM97].

In Hironaka's desingularisation algorithm, which works over fields of characteristic 0, the variety X is assumed to be embedded in a smooth ambient variety M_0. The variety M_0 is successively modified by blowing up carefully chosen smooth centres $Z_i \subset M_i$, recursively producing a sequence

$$\Pi \colon M_n \to \cdots \to M_0$$

of blow-ups. The strict transform of X under the composition $M_i \to M_0$ is denoted by X_i. At each step, the exceptional locus is recorded as a simple normal crossings divisor E_i on M_i. The divisor E_i is defined recursively as the union of the strict transform of E_{i-1} and the exceptional divisor of the blow-up. When the algorithm ends, the variety X_n is smooth. In particular, the induced map $X_n \to X$ is a desingularisation of X.

In the formulation by Bierstone–Milman, each blow-up is determined as the maximal locus of an upper semi-continuous invariant inv_X. Functoriality is ensured by the fact that inv_X only depends on the completed local ring $\widehat{\mathcal{O}}_{X,x}$ at each point. It is, however, important to understand that the invariant inv_X is recursively defined, and the choice of centre at each step depends on the history of the blow-up process. As described in for instance [Kol07, §3.6], it is not possible to construct a functorial desingularisation algorithm by successive blow-ups in smooth centres, that works one blow-up at a time.

The destackification algorithm described in Paper C operates in a similar way. In our case, we produce a sequence of stacky blow-ups

$$\Pi\colon X_n \to \cdots \to X_0$$

where $X_0 = X$. We also keep record of a set E_i of smooth divisors, ordered by age, which only have simple normal crossings. At each step, the set E_i is defined as the set of strict transforms of divisors in E_{i-1} with the exceptional divisor of the blow-up added as the youngest divisor. The centre at each blow-up is smooth and transversal to the divisors in the ordered set. This ensures that the blow-up and the new divisors are smooth, and that the new divisors have simple normal crossings, which allows for a recursive set-up. Note that, in contrast to the case with desingularisation, each stack X_i is smooth in its own right, so there is no need for an embedding. Instead of the process being guided by the singularities of X_i, it is guided by the stackiness.

Just as in the case with the algorithm of Bierstone–Milman, we blow up the maximal locus of an upper semi-continuous invariant at each step in the destackification algorithm. The invariant is based on what we call the conormal representation at each point of the stack. On closed points, this is, more or less, the conormal bundle of the residual gerbe at the point[1]. Recall that the stacks we are working with have toric charts at each point. The conormal representation at a point completely determines the isomorphism class of the toric chart at the same point.

3.3 Applications and Possible Generalisations

At first sight, the assumption that our tame stack X must have abelian stabilisers in order for our destackification algorithm to work seems to be quite restrictive. But at least if we work over a field of characteristic zero, this can be overcome. By first using functorial embedded desingularisation on the stacky locus of X with the Bierstone–Milman variant of Hironaka's method [BM97], we reduce to the case where the stacky locus is contained in a simple normal crossings divisor. But this implies that the stabilisers are in fact diagonalisable [RY00, Thm. 4.1]. Hence we have reduced the problem to a case where our algorithm can be applied directly. I am currently working on a more detailed description of this jointly with David Rydh.

One application to the destackification algorithm is to obtain a functorial desingularisation algorithm for varieties X with simplicial toric quotient singularities. In this situation, there exists a canonical stack X_{can} which is smooth

[1] In characteristic zero, we could have worked with the canonical representation of the stabiliser on the tangent space, but in positive characteristic, the tangent space is not well-behaved and is best described as a stack in its own right.

and has X as coarse space [Vis89, Sat12]. The canonical stack X_{can} will also be tame with abelian stabilisers, so we obtain a destackification $Y \to X_{\text{can}}$ by applying our algorithm. The corresponding map $Y_{\text{cs}} \to X$ on coarse spaces will be a desingularisation of X, and the construction is clearly functorial. Note that this works regardless of the characteristic of the base field, and no toroidal structure on X is needed. This makes the method more general than the toroidal methods described in [KKMSD73].

Destackification can be used to obtain a version of the weak factorisation theorem by Włodarczyk [Wło00] for Deligne–Mumford stacks in characteristic zero. The corollary is obtained by applying Włodarczyk's result on the algebraic space X'_{cs} from the roof-shaped diagram in the beginning of the chapter.

A possible application to weak factorisation would be to obtain a Heinloth-Bittner type presentation for the Grothendieck group of Deligne–Mumford stacks in characteristic zero. For this, one would also need the stacky compactification result developed by Rydh [Ryd11].

An obvious future direction is to try to extend the results of Paper C to stacks with not necessarily finite diagonalisable stabilisers. In the toric case, similar questions have been explored by Edidin–More [EM12]. Recent results by Alper–Hall–Rydh [AHR14] suggest that it is possible to find a toric chart at each point of an algebraic stack with diagonalisable stabilisers, provided that the stack has a good moduli space in the sense of Alper [Alp13].

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