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Algorithmic randomness and analysis

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ABSTRACT

In this thesis we study the interaction between algorithmic randomness and mathematical analysis. In particular, we focus on the connection between analysis and the fields of effective dimension and resource bounded randomness.

We begin with the effective dimension of Euclidean points. We show that the techniques from algorithmic information can be used successfully to study problems in fractal geometry. Specifically, we investigate the Hausdorff of projections of Euclidean subsets. Using Kolmogorov complexity, we give a new proof of the celebrated Marstrand projection theorem. We also prove, using similar methods, two new lower bounds on projections. The first shows that Marstrand's theorem holds for more general subsets of $\mathbb{R}^n$. The second gives a lower bound on the packing dimension of projections for arbitrary sets.

Our next work is on the algorithmic dimension spectra of lines in the Euclidean plane. Given any line L with slope a and vertical intercept b, the dimension spectrum $\mathrm{sp}(L)$ is the set of all effective Hausdorff dimensions of individual points on L. We use Kolmogorov complexity and geometrical arguments to show that, if the effective Hausdorff dimension $\dim(a, b)$ is equal to the effective packing dimension $\mathrm{Dim}(a, b)$, then $\mathrm{sp}(L)$ contains a unit interval. We also show that, if the dimension $\dim(a, b)$ is at least one, then $\mathrm{sp}(L)$ is infinite. Together with previous work, this implies that the dimension spectrum of any line is infinite.

Our last topic is on the connection between polynomial space randomness and a fundamental result of analysis, the Lebesgue differentiation theorem. We generalize

Ko's framework for polynomial space computability in $\mathbb{R}^n$ to define *weakly pspace-random* points, a new variant of polynomial space randomness. We show that the Lebesgue differentiation theorem characterizes weakly pspace random points. That is, a point x is weakly pspace random if and only if the Lebesgue differentiation theorem holds for a point x for every pspace L_1-computable function.

CHAPTER 1. INTRODUCTION

What does it mean for a mathematical object to be *intrinsically* random? Until the middle of the 20th century, the notion of an object having randomness would seem paradoxical. However, the theory of computing enables a mathematically meaningful way of measuring the randomness of an object. Kolmogorov [22], and, independently Solomonoff [48] and Chaitin [9], gave the first measure of the intrinsic randomness of finite binary strings, now known as Kolmogorov complexity. Under this definition, the randomness inherent to a finite string x is the length of the shortest algorithm which outputs x (for a formal definition see Chapter 2). Martin-Löf [36] used computability theory to give an *effective* version of measure theory. We say that a sequence A is (Martin-Löf) random if the singleton $\{A\}$ is not of effective measure zero. Since the work of Kolmogorov and Martin-Löf, the field of algorithmic randomness has expanded to include a hierarchy of notions of randomness, all of which make essential use of the theory of computing. In this dissertation, we will focus on two areas of algorithmic randomness, *effective (algorithmic) dimension* and *resource bounded randomness*, and their connection with mathematical analysis.

Algorithmic dimension was developed by J. Lutz [26, 27] as an effectivization of Hausdorff dimension, a fundamental tool of fractal geometry (see Chapter 2 for preliminary definitions). Although originally used to study complexity classes [26, 2, 40], effective dimension has proven to give geometrically meaningful information about Euclidean points [12, 18, 30, 31]. The connection between algorithmic dimension and analysis was further deepened in the recent work of J. Lutz and N. Lutz. They proved that the Hausdorff and packing dimensions of a set can be characterized by the effective dimension of its points [29]. This allows techniques of algorithmic randomness to be applied to problems in analysis. Although this method has only recently been established, there have been several results showing the usefulness of this approach. In the same paper that introduced the point-to-set principle, J. Lutz and N. Lutz applied it to give a new proof of Davies' theorem [10] settling the Kakeya conjecture in the plane. Subsequently, N. Lutz and Stull [33] applied the principle to the dimensions of points on lines in the plane to give improved bounds on Furstenberg sets. The point-to-set principle also allowed N. Lutz [32] to show that the fundamental product inequality for Hausdorff dimension holds for *arbitrary* sets.

In Chapter 3, we use the point-to-set principle to study the fractal dimensions of projections, a fundamental problem in fractal geometry. We show that algorithmic randomness can be successfully applied to this problem. Our first result gives a new, entirely information theoretic, proof of the celebrated Marstrand projection theorem. Our second main theorem shows that the conclusion of Marstrand's theorem holds for a different class of sets than those Marstrand considered. As a corollary, we strengthen Marstrand's projection theorem. In our third main theorem of Chapter 3, we prove a new bound on the packing dimension of projections of arbitrary sets.

We continue our investigation of effective dimension and analysis in Chapter 4. Given the pointwise nature of effective Hausdorff dimension, it is natural to investigate the dimension spectrum of a set $E \subseteq \mathbb{R}^n$, i.e., the set of all dimensions of points in E. In this chapter, motivated by questions of fractal geometry, we study the (effective) dimension of points on a given line in the Euclidean plane. Our first main result gives sufficient criterion for the dimension spectrum of a line to contain a unit interval. Our second main theorem shows that, for any line in the plane, the dimension spectrum of that line is infinite.

The second part of this thesis studies resource bounded randomness through the lens of analysis. With the prominence of complexity theory in computation, a natural step is to impose resource bounds on the computation in algorithmic randomness. Resource bounded randomness studies the different notions of what it means for an object to be random" relative to a *resource bounded* observer. Investigation into resource bounded randomness began in earnest with Lutz's development of *resource bounded measure*, a resource bounded effectivization of Lebesgue measure theory [25].

Recent research has used computable analysis to study the connection between randomness and classical analysis [3, 15, 16, 41, 42, 46, 47]. With the rise of measure theory, many fundamental theorems of analysis have been "almost everywhere" results. Theorems of this type state that a certain property holds for almost every point; i.e., the set of points that does not satisfy the property is of measure zero. However, almost everywhere theorems typically give no information about which points satisfy the stated property. By adding computability restrictions, tools from algorithmic randomness are able to strengthen a theorem from a property simply holding almost everywhere, to one that holds for all random points. In this thesis, we are interested in the connection between *resource bounded* randomness and analysis. While there has been work on this interaction [6, 28, 44], the interplay of resource bounded randomness and analysis is still poorly understood.

In Chapter 5, we define *weak resource bounded randomness*, a new notion of resource bounded randomness. Using ideas from Ko's framework [21] for computational complexity in $\mathbb{R}^n$, we define weak randomness using resource bounded null covers. We show that, in the polynomial space setting, weak randomness is strictly weaker notion of randomness than that of Lutz. In Chapter 6 we investigate weak randomness in the context of measure theoretic analysis. We show that the Lebesgue differentiation theorem, a fundamental theorem of analysis, characterizes weak polynomial space randomness. With this result, we generalize the work of Pathak, Rojas and Simpson [46], who previously characterized Schnorr randomness with Lebesgue's theorem.

CHAPTER 2. EFFECTIVE DIMENSION IN EUCLIDEAN SPACE

In this chapter we review the key definitions and theorems of effective dimension that will be used in Chapters 3 and 4. We begin in Section 2.1 with the definition of Kolmogorov complexity for discrete objects. We then leverage this definition to give the Kolmogorov complexity of Euclidean points. In Section 2.2 we review Lutz's notions of effective dimension, and their characterization using Kolmogorov complexity. In Section 2.3 we state the point-to-set principles of J. Lutz and Hitchcock, and J. Lutz and N. Lutz.

2.1 Kolmogorov Complexity in Discrete and Continuous Domains

The *conditional Kolmogorov complexity* of binary string $\sigma \in \{0,1\}^*$ given a binary string $\tau \in \{0,1\}^*$ is the length of the shortest program π that will output σ given τ as input. Formally, the conditional Kolmogorov complexity of σ given τ is

$$K(\sigma|\tau) = \min_{\pi \in \{0,1\}^*} \{\ell(\pi) : U(\pi, \tau) = \sigma\} ,$$

where U is a fixed universal prefix-free Turing machine and $\ell(\pi)$ is the length of π. Any π that achieves this minimum is said to *testify* to, or be a *witness* to, the value $K(\sigma|\tau)$. The *Kolmogorov complexity* of a binary string σ is $K(\sigma) = K(\sigma|\lambda)$, where λ

is the empty string. These definitions extend naturally to other finite data objects, e.g., vectors in $\mathbb{Q}^n$, via standard binary encodings; see [24] for details.

One of the most useful properties of Kolmogorov complexity is that it obeys the *symmetry of information.* That is, for every $\sigma, \tau \in \sigma \in \{0,1\}^*$,

$$K(\sigma, \tau) = K(\sigma) + K(\tau \mid \sigma, K(\sigma)) + O(1) \,,$$

Kolmogorov complexity can be naturally extended to points in Euclidean space, as we now describe. The *Kolmogorov complexity* of a point $x \in \mathbb{R}^m$ at *precision* $r \in \mathbb{N}$ is the length of the shortest program π that outputs a *precision-r* rational estimate for x. Formally, this is

$$K_r(x) = \min\left\{K(p) \,:\, p \in B_{2^{-r}}(x) \cap \mathbb{Q}^m\right\} \,,$$

where $B_\varepsilon(x)$ denotes the open ball of radius ε centered on x. The *conditional Kolmogorov complexity of x at precision r given $y \in \mathbb{R}^n$ at precision $s \in \mathbb{R}^n$* is

$$K_{r,s}(x|y) = \max\Big\{\min\{K_r(p|q) \,:\, p \in B_{2^{-r}}(x) \cap \mathbb{Q}^m\} \,:\, q \in B_{2^{-s}}(y) \cap \mathbb{Q}^n\Big\} \,.$$

When the precisions r and s are equal, we abbreviate $K_{r,r}(x|y)$ by $K_r(x|y)$. As a matter of notational convenience, if we are given a nonintegral positive real as a precision parameter, we will always round up to the next integer. For example, $K_r(x)$ denotes $K_{\lceil r \rceil}(x)$ whenever $r \in (0, \infty)$.

The following lemma, due to Case and J. Lutz and J. Lutz and N. Lutz, shows that the Kolmogorov complexity of a point is linearly sensitive to its inputs.

Lemma 2.1 (Case and J. Lutz [8], J. Lutz and N. Lutz [29])**.** *Let $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. For all $r, s, r', s' \in \mathbb{N}$,*

1. $K_{r'}(x) = K_r(x) + O(|r' - r|) + O(\log r)$.

2. $K_{r',s'}(x|y) = K_{r,s}(x|y) + O(|r'-r| + |s'-s|) + O(\log rs)$.

We will often use the following result which shows that symmetry of information holds for Kolmogorov complexity in $\mathbb{R}^n$. The proof may be found in [33].

Lemma 2.2 (J. Lutz and N. Lutz [29], N. Lutz and Stull [33])**.** *Let $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. For all $r, s \in \mathbb{N}$ with $r \geq s$,*

1. $K_r(x, y) = K_r(x|y) + K_r(y) + O(\log r)$.

2. $K_r(x) = K_{r,s}(x|x) + K_s(x) + O(\log r)$.

The following lemma states that, if at some precision r, a point x gives little information about a point z, then x gives little information about z for all precisions $s \leq r$. The proof is deferred to Appendix .

Lemma 2.3. *Let $m, n \in \mathbb{N}$, $x \in \mathbb{R}^m$, $z \in \mathbb{R}^n$, $\varepsilon > 0$ and $r \in \mathbb{N}$. If $K^x_r(z) \geq K_r(z) - \varepsilon r$, then the following hold for all $s \leq r$.*

(i) $|K^x_s(z) - K_s(z)| \leq \varepsilon r - O(\log r)\,.$

(ii) $|K_{s,r}(x \mid z) - K_s(x)| \leq \varepsilon r - O(\log r)\,.$

2.2 Effective Hausdorff and Packing Dimensions

J. Lutz [26] initiated the study of algorithmic dimensions by effectivizing Hausdorff dimension using betting strategies called *gales*, which generalize martingales. Subsequently, Athreya, et al., defined effective packing dimension, also using gales [2]. Mayordomo showed that effective Hausdorff dimension can be characterized using Kolmogorov complexity [39]. Mayordomo and J. Lutz then showed that effective packing dimension can also be characterized in this way [30]. In this paper, we use these characterizations as definitions. The *effective Hausdorff dimension* and *effective packing dimension* of a point $x \in \mathbb{R}^n$ are

$$\dim(x) = \liminf_{r\to\infty} \frac{K_r(x)}{r} \quad \text{and} \quad \mathrm{Dim}(x) = \limsup_{r\to\infty} \frac{K_r(x)}{r}\,.$$

Intuitively, these dimensions measure the density of algorithmic information in the point x. J. Lutz and N. Lutz [29] generalized these definitions by defining the *lower* and *upper conditional dimension* of $x \in \mathbb{R}^m$ given $y \in \mathbb{R}^n$ as

$$\dim(x|y) = \liminf_{r\to\infty} \frac{K_r(x|y)}{r} \quad \text{and} \quad \mathrm{Dim}(x|y) = \limsup_{r\to\infty} \frac{K_r(x|y)}{r}\,.$$

2.3 The Point-to-set Principle

By letting the underlying fixed prefix-free Turing machine U be a universal *oracle* machine, we may *relativize* the definitions in this section to an arbitrary oracle set $A \subseteq \mathbb{N}$. The definitions of $K^A(\sigma|\tau)$, $K^A(\sigma)$, $K^A_r(x)$, $K^A_r(x|y)$, $\dim^A(x)$, $\mathrm{Dim}^A(x)$ $\dim^A(x|y)$, and $\mathrm{Dim}^A(x|y)$ are then all identical to their unrelativized versions, except that U is given oracle access to A. We will frequently consider the complexity of a point $x \in \mathbb{R}^n$ *relative to a point* $y \in \mathbb{R}^m$, i.e., relative to a set A_y that encodes the binary expansion of y is a standard way. We then write $K^y_r(x)$ for $K^{A_y}_r(x)$.

The following *point-to-set principles* show that the classical notions of Haudorff and packing dimension of a set can be characterized by the *effective* dimension of its points. The first point-to-set principle, for a restricted class of sets, was implicitly proven by Lutz [26] and Hitchcock [19].

Theorem 2.4. *Let $E \subseteq \mathbb{R}^n$ be a a F_σ set, and $A \subseteq \mathbb{N}$ be an oracle such that E is a Σ^0_2 set relative to A. Then,*

$$\dim_H(E) = \sup_{x\in E} \dim^A(x).$$

Recently, J. Lutz and N. Lutz [29] improved this result to show that the Hausdorff and packing dimension of *any* set is characterized by their corresponding (relativized) effective dimensions.

Theorem 2.5 (Point-to-set principle)**.** *Let $n \in \mathbb{N}$ and $E \subseteq \mathbb{R}^n$. Then*

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x), \quad \textit{and}$$

$$\dim_P(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \operatorname{Dim}^A(x),$$

CHAPTER 3. PROJECTION THEOREMS AND EFFECTIVE DIMENSION

In this chapter we use the point-to-set principles stated in Section 2.3 to study the Hausdorff and packing dimension of projections. This chapter is joint work with Neil Lutz.

Given a set $E \subseteq \mathbb{R}^n$ and $e \in S^{n-1}$, the *projection along* e is the function $P_e : \mathbb{R}^n \to \mathbb{R}$ by

$$P_e(x) = e \cdot x.$$

Determining how the dimension of a set is changed by a projection is an important problem in fractal geometry[14, 38]. As a projection is Lipschitz continuous, the Hausdorff dimension of the projection $P_e(E)$ is at most the Hausdorff dimension of E. A natural question is whether the Hausdorff dimension of a projection is *equal* to the dimension of E. Basic examples from fractal geometry show that this is not true in general [14]. However, a fundamental theorem due to Marstrand [35] shows that, if E is analytic, then for *most* $e \in S^{n-1}$, the projection of any set along e is maximal.

Theorem 3.1. *[Marstrand's Projection Theorem] Let $E \subseteq \mathbb{R}^2$ be an analytic set with $\dim_H(E) = s \leq 1$. Then for almost every $e \in S^1$,*

$$\dim_H(P_e(E) = s.$$

Subsequently, Mattila [37] showed that Marstrand's theorem holds for all $n \geq 2$.

In this chapter, we use techniques from algorithmic information to study the Hausdorff and packing dimension of projections. Our first main theorem gives a new proof of Theorem 3.1. In addition, we prove two new generalizations of Marstrand's projection theorem. An immediate question is whether we can remove the requirement that the set be analytic. Unfortunately, Davies [11] showed that, assuming the continuum hypothesis, there are nonanalytic sets for which Marstrand's theorem fails[1]. However, our second main theorem shows that we can remove the assumption that E is analytic, assuming the Hausdorff and packing dimensions of E agree.

Theorem 3.2. *Let $E \subseteq \mathbb{R}^n$ be any set with $\dim_H(E) = \dim_P(E) = s \leq 1$. Then for almost every $e \in S^{n-1}$,*

$$\dim_H(P_e(E) = s.$$

This result therefore shows that the conclusion of Marstrand's theorem holds for a broader class of sets.

Our final main theorem is on the "size" of the projection of *arbitrary* sets. Due to Davies' construction, we cannot give a lower bound on the Hausdorff dimension of the projection. However, we are able to give a lower bound on the *packing* dimension of the projection of arbitrary sets.

Theorem 3.3. *Let $E \subseteq \mathbb{R}^n$ be any set with $\dim_H(E) = s \leq 1$. Then for almost every $e \in S^{n-1}$,*

$$\dim_P(P_e(E) \geq s.$$

[1] Indeed, Davies constructed a set in the plane of Hausdorff dimension 2 whose projection has dimension 0 for almost every $e \in S^1$

3.1 Bounding the Complexity of Projections

In this section, we will focus on bounding the Kolmogorov complexity of a projected point *at a given precision*. In Sections 3.2 and 3.3, we will use these results in conjunction with the point-to-set principle to prove our main theorems.

We begin by giving intuition of the main idea behind this lower bound. We will show that under certain conditions, given (an approximation of) the projection $P_e(z)$ and e, we can compute an approximation of the original point z. Informally, these conditions are the following.

1. The complexity, $K_r(z)$, of the original point is small.
2. If $P_e(w) = P_e(z)$, then either $K_r(w)$ is large, or w is close to z.

Assuming that both conditions are satisfied, we can recover z from $P_e(z)$ by enumerating over all points u of low complexity such that $P_e(u) = P_e(z)$. By our assumption, any such point must be close z; i.e., u is a good approximation of z. We now formalize this intuition.

Lemma 3.4. *Suppose that* $z \in \mathbb{R}^n$ $e \in S^{n-1}$, $r \in \mathbb{N}$, $\delta \in \mathbb{R}_+$, *and* $\varepsilon, \eta \in \mathbb{Q}_+$ *satisfy* $r \geq \log(2\|z\| + 5) + 1$ *and the following conditions.*

(i) $K_r(z) \leq (\eta + \varepsilon)\, r$.

(ii) *For every* $w \in B_1(z)$ *such that* $P_e(w) = P_e(z)$,

$$K_r(w) \geq (\eta - \varepsilon)\, r + \delta \cdot (r - t)\,,$$

whenever $t = -\log\|z - w\| \in (0, r]$.

Then for every oracle set $A \subseteq \mathbb{N}$,

$$K_r^{A,e}(P_e(z)) \geq K_r^{A,e}(z) - \frac{n\varepsilon}{\delta} r - K(\varepsilon) - K(\eta) - O_z(\log r)\,.$$

Proof. Suppose z, e, r, δ, ε, η, and A satisfy the hypothesis.

Define an oracle Turing machine M that does the following given oracle (A, e) and input $\pi = \pi_1\pi_2\pi_3\pi_4\pi_5$ such that $U^A(\pi_1) = q \in \mathbb{Q}$, $U(\pi_2) = h \in \mathbb{Q}^n$, $U(\pi_3) = s \in \mathbb{N}$, $U(\pi_4) = \zeta \in \mathbb{Q}$, and $U(\pi_5) = \iota \in \mathbb{Q}$.

For every program $\sigma \in \{0,1\}^*$ with $\ell(\sigma) \leq (\iota+\zeta)s$, in parallel, M simulates $U(\sigma)$. If one of the simulations halts with some output $p = (p_1, \ldots, p_n) \in \mathbb{Q}^n \cap B_{2^{-1}}(h)$ such that $|P_e(p) - q| < 2^{-s}$, then $M^{A,e}$ halts with output p. Let c_M be a constant for the description of M.

Let π_1, π_2, π_3, π_4, and π_5 testify to $K^{A,e}_r(P_e(z))$, $K_1(z)$, $K(r)$, $K(\varepsilon)$, and $K(\eta)$, respectively, and let $\pi = \pi_1\pi_2\pi_3\pi_4\pi_5$. Let σ be a program of length at most $(\eta+\varepsilon)r$ such that $\|p - z\| \leq 2^{-r}$, where $U(\sigma) = p$. Note that such a program must exist by condition (i) of our hypothesis. Then it is easily verified that

$$|P_e(z) - P_e(p)| \leq 2^{-r+c_z},$$

for some fixed constant c_z depending only on z. Therefore $M^{A,e}$ is guaranteed to halt on π.

Let $M^{A,e}(\pi) = p = (p_1, \ldots, p_n) \in \mathbb{Q}^n$. Another routine calculation (Observation A.6) shows that there is some

$$w \in B_{2^{\gamma-r}}(p) \subseteq B_{2^{-1}}(p) \subseteq B_{2^0}(a, b)$$

such that $P_e(w) = P_w(z)$, where γ is a constant depending only on z, e. Then,

$$\begin{aligned} K^{A_e}_r(w) &\leq |\pi| \\ &\leq K^{A,e}_r(P_e(z)) + K_1(z) + K(r) + K(\varepsilon) + K(\eta) + c_M \\ &= K^{A,e}_r(P_e(z)) + K(\varepsilon) + K(\eta) + O(\log r), \end{aligned}$$

Rearranging this yields

$$K^{A,e}_r(P_e(z)) \geq K^{A,e}_r(w) - K(\varepsilon) - K(\eta) - O(\log r). \tag{3.1}$$

Let $t = -\log \|z - w\|$. If $t \geq r$, then the proof is complete. If $t < r$, then $B_{2^{-r}}(p) \subseteq B_{2^{1-t}}(z)$, which implies that $K^{A,e}_r(p) \geq K^{A,e}_{t-1}(z)$. Therefore,,

$$K^{A,e}_r(w) \geq K^{A,e}_r(z) - n(r - t) - O(\log r). \tag{3.2}$$

We now bound $r - t$. By our construction of M,

$$\begin{aligned}(\eta + \varepsilon)r &\geq K(p)\\ &\geq K_r(w) - O(\log r).\end{aligned}$$

By condition (ii) of our hypothesis, then,

$$(\eta + \varepsilon)r \geq (\eta - \varepsilon)r + \delta(r - t),$$

which implies that

$$r - t \leq \tfrac{n\varepsilon}{\delta} r + O(\log r).$$

Combining this with inequalities (3.1) and (3.2) concludes the proof. □

With the above lemma in mind, we wish to give a lower bound on the complexity of points w such that $P_e(w) = P_e(z)$. Our next lemma gives a bound based on the complexity, relative to z, of the direction $e \in S^{n-1}$. This is based on the observation that we can solve for $e = (e_1, \ldots, e_n)$ given w, z and $e_3, \ldots, e_n$. This follows from solving the system of two equations

$$\begin{aligned}(z - w) \cdot e &= 0\\ e_1^2 + \ldots e_n^2 &= 1.\end{aligned}$$

This suggests that

$$K^{z,e_3,\ldots,e_n}_r(e) \leq K^{z,e_3,\ldots,e_n}_r(w).$$

However, for our purposes, we must be able to recover (an approximation of) e given *approximations* of w and z. Intuitively, the following lemma shows that we can algorithmically compute an approximation of e whose error is linearly correlated with distance between w and z. We can then bound the complexity of w using a symmetry of information argument.

Lemma 3.5. *Let $z \in \mathbb{R}^n$, $e \in S^{n-1}$, and $r \in \mathbb{N}$. Let $w \in \mathbb{R}^n$ such that $P_e(z) = P_e(w)$. Then there are numbers $i, j \in \{1, \ldots, n\}$ such that*

$$K_r(w) \geq K_t(z) + K^{e-\{e_i,e_j\}}_{r-t,r}(e \,|\, z) + O(\log r),$$

where $t = -\log \|z - w\|$.

Proof. Let z, w, e, and r be as in the statement of the lemma. We first choose i so that $|z_i - w_i|$ is maximal. We then choose j so that

$$\begin{aligned} &sgn((z_i - w_i)e_i) \neq sgn((z_j - w_j)e_j), \text{ and} \\ &|z_j - w_j| > 0, \end{aligned}$$

where sgn denotes the sign. Note that such a j must exist since $(z - w) \cdot e = 0$. For the sake of removing notational clutter, we will assume, without loss of generality, that $i = 1$ and $j = 2$.

We first show that

$$K^{e_3,\ldots,e_n}_{r-t,r}(e_2 \,|\, z) \leq K_r(w \,|\, z) + O(1). \tag{3.3}$$

As mentioned in the informal discussion preceding this lemma, note that

$$e_2 = \frac{-b + (-1)^h\sqrt{b^2 - 4ac}}{2a}, \tag{3.4}$$

where

- $h \in \{0, 1\}$,
- $a = (z_1 - w_1)^2 + (w_2 - z_2)^2$,
- $b = 2(w_2 - z_2) \sum_{i=3}^{n} (w_i - z_i)e_i$, and
- $c = (\sum_{i=3}^{n} (w_i - z_i)e_i)^2 + (z_1 - w_1)^2 \sum_{i=3}^{n} e_i^2 - 1$.

With this in mind, let M be the Turing machine such that, whenever $q = (q_1, \ldots, q_n) \in \mathbb{Q}^n$ and $U(\pi, q) = p = (p_1, \ldots, p_n) \in Q^2$ with $p_1 \neq q_1$,

$$M^{e_3,\ldots,e_n}(\pi, q, j) = \tfrac{-b' + (-1)^h \sqrt{b'^{,2} - 4a'c'}}{2a'},$$

where

- $h \in \{0, 1\}$,
- $a' = (q_1 - p_1)^2 + (p_2 - q_2)^2$,
- $b' = 2(p_2 - q_2) \sum_{i=3}^{n} (p_i - q_i) d_i$, and
- $c' = (\sum_{i=3}^{n} (p_i - q_i) d_i)^2 + (q_1 - p_1)^2 \sum_{i=3}^{n} d_i^2 - 1.$, and
- $d = (d_3, \ldots, d_n) \in \mathbb{Q}^{n-2}$ is an nr-approximation of $(e_3, \ldots, e_n)$.

Let $q \in B_{2^{-r}}(z) \cap \mathbb{Q}^n$, π_q testify to $\hat{K}_r(w \mid q)$. It tedious, but straightforward (Lemma A.5), to verify that

$$|M^{e_3,\ldots,e_n}(\pi_q, q, h) - e_2| \leq 2^{\alpha + t - r},$$

where α is a constant depending only on e. Hence, inequality (3.3) holds. Since

$$K_s^{e_3,\ldots,e_n}(e_2) = K_s^{e_3,\ldots,e_n}(e) + O(1)$$

holds for every s, we see that

$$K_{r-t,r}^{e_3,\ldots,e_n}(e \mid z) \leq K_r(w \mid z) + O(1). \tag{3.5}$$

To complete the proof, we note that

$$\begin{aligned} K_r(w \mid z) &\leq K_{r,t}(w \mid z) + O(\log r) \\ &= K_{r,t}(w \mid w) + O(\log r) \\ &= K_r(w) - K_t(w) + O(\log r) \\ &= K_r(w) - K_t(z) + O(\log r). \end{aligned}$$

The lemma follows from rearranging the above inequality, and combining inequality (3.5). □

Finally, to satisfy the condition that $K_r(z)$ is small, we will "artificially" decrease the complexity of z at precision r. We will achieve this by using the following lemma due to N. Lutz and Stull [33]. For completeness, we provide of proof in Appendix .

Lemma 3.6. *Let $n, r \in \mathbb{N}$, $z \in \mathbb{R}^n$, and $\eta \in \mathbb{Q} \cap [0, \dim(z)]$. Then there is an oracle $D = D(n, r, z, \eta)$ and a constant $k \in \mathbb{N}$ depending only on n, z and η satisfying*

(i) *For every $t \leq r$, $K_t^D(z) = \min\{\eta r, K_t(z)\} + k \log r$.*

(ii) *For every $m, t \in \mathbb{N}$ and $y \in \mathbb{R}^m$, $K_{t,r}^D(y|z) = K_{t,r}(y|z) + O(\log r)$ and $K_t^{z,D}(y) = K_t^z(y) + k \log r$.*

The previous results gave us sufficient conditions for strong lower bounds on the complexity of $P_e(z)$ at a given precision, and methods to ensure that the conditions are satisfied. The following theorem encapsulates these results so that we may apply them in the proof of our main theorems. Informally, it states that if

- e has high complexity, and
- (A, e) does not significantly change the complexity of z,

then the complexity of $P_e^{A,e}(z)$ is roughly $K_r(z)$.

Theorem 3.7. *Let $z \in \mathbb{R}^n$, $e \in S^{n-1}$, $A \subseteq \mathbb{N}$, $\eta' \in \mathbb{Q} \cap (0,1) \cap (0, \dim(z))$, $\varepsilon' > 0$, and $r \in \mathbb{N}$. Assume the following are satisfied.*

1. *For every $s \leq r$, $K_s^{e_3,\ldots,e_n}(e) \geq s - \log(s)$.*
2. *$K_r^{A,e}(z) \geq K_r(z) - \varepsilon' r$.*

Then,

$$K_r^{A,e}(P_e(z)) \geq \eta' r - \varepsilon' r - \tfrac{(n+1)\varepsilon'}{1-\eta'} r - K(2\varepsilon') - K(\eta') - O_z(\log r).$$

Proof. Assume the hypothesis, and let $\eta = \eta'$, $\varepsilon = 2\varepsilon'$ and $\delta = 1 - \eta'$. Let $D_r = D(n, r, z, \eta)$ be the oracle as defined in Lemma 3.6, relative to A.

First assume that the conditions of Lemma 3.4, relative to (A, D_r), hold for z, e, r, η, ε and δ. Then we may apply Lemma 3.4, which, combined item (2) and Lemma 3.6, yields

$$\begin{aligned} K_r^{A,D_r,e}(P_e(z)) &\geq K_r^{A,D_r,e}(z) - \frac{4\varepsilon}{\delta} r - K(\varepsilon) - K(\eta) - O_z(\log r) \\ &\geq K_r^{D_r}(z) - \varepsilon' r - \frac{4\varepsilon}{\delta} r - K(\varepsilon) - K(\eta) - O_z(\log r) \\ &= \eta' r - \varepsilon' r - \frac{(n+1)\varepsilon'}{1-\eta'} r - K(\varepsilon') - K(\eta') - O_z(\log r). \end{aligned}$$

Therefore, to complete the proof, it suffices to show that the conditions of Lemma 3.4 hold.

Item (i) of Lemma 3.4 holds by our construction of D_r. To see that condition (ii) holds, let $w \in B_1(z)$ such that $P_e(w) = P_e(z)$. By Lemma 3.5 and condition (1) of the present lemma,

$$K_r^{D_r}(w) \geq K_t^{D_r}(z) + K_{r-t,r}^{D_r,e_3,\ldots,e_n}(e \mid z) + O(\log r), \tag{3.6}$$

where $t = -\log \|z - w\|$. By Lemma 3.6,

$$\begin{aligned} K_t^{D_r}(z) + K_{r-t,r}^{D_r,e_3,\ldots,e_n}(e \mid z) &\geq \eta' t + r - t - \varepsilon' r - O(\log r) \\ &= t(\eta' - 1) + r(1 - \varepsilon') - O(\log r) \\ &\geq (\eta - \varepsilon) r + \delta(r - t), \end{aligned}$$

Hence, the conditions of Lemma 3.4 are satisfied and the proof is complete. □

3.2 Marstrand's Projection Theorem

We begin with a new proof of Marstrand's projection theorem. Recall that

Theorem 3.1. *Let $E \subseteq \mathbb{R}^n$ be analytic with $\dim_H(E) = s \leq 1$. Then for almost every $e \in S^{n-1}$, $\dim_H(P_e(E)) = s$.*

Note the order of the quantifiers. To use the point-to-set principle, we must first choose a direction $e \in S^{n-1}$. We then must show that for every oracle A and $\epsilon > 0$, there is some $z \in E$ such that

$$\dim^A(P_e(z)) \geq \dim_H(E) - \epsilon \ .$$

In order to apply Theorem 3.7, we must guarantee that (A, e) does not significantly change the complexity of z. To ensure this, we will first use the point-to-set principle of Lutz and Hitchcok (Theorem 2.4). While less general than the principle of J. Lutz and N. Lutz, it has be nice property that it *specifies* the oracle characterizing the dimension of a F_σ set.

To take advantage of this, we use the following lemmas.

Lemma 3.8. *Let $E \subseteq \mathbb{R}^n$ be analytic with $\dim_H(E) = s$. Then there is a F_σ set $F \subseteq E$ such that $\dim_H(F) = \dim_H(E)$.*

Proof. It is well known that if $E \subseteq \mathbb{R}^n$ is analytic, then for every $s' < \dim_H(E)$ there is a compact subset $F \subseteq E$ such that $\dim_H(F) = s'$ (see e.g. Bishop and Peres [4]). □

Lemma 3.9. *Let $E \subseteq \mathbb{R}^n$ be a F_σ set, and $A \subseteq \mathbb{N}$ be an oracle such that E is a Σ^0_2 set relative to A. Then for every $e \in S^{n-1}$, $P_e(E)$ is a Σ^0_2 set relative to (A, e).*

Finally, we must ensure that e does not significantly change the complexity of z. For this, we will use the following definition and theorem due to Calude and Zimand [7]. We rephrase their work in terms of points in Euclidean space. Let $n \in N$, $z \in \mathbb{R}^n$ and $e \in S^{n-1}$. We say that z and e are *independent* if, for every $r \in \mathbb{N}$, $K^e_r(z) \geq K_r(z) - O(\log r)$ and $K^z_r(e) \geq K_r(e) - O(\log r)$.

Theorem 3.10. *For every $x \in \mathbb{R}^n$, the set of all $e \in S^{n-1}$ such that z and e are independent is of measure* 1*.*

With these ingredients we can formally reprove Marstrand's projection theorem using algorithmic information theory.

Proof of Theorem 3.1. Let $E \subseteq \mathbb{R}^n$ be analytic with $\dim_H(E) = s \leq 1$. By Lemma 3.8, there is a F_σ set $F \subseteq E$ such that $\dim_H(F) = s$. Let $A \subseteq \mathbb{N}$ be an oracle such that F is Σ^0_2 relative to A. Using the F_σ point-to-set principle (Theorem 2.4), for every $k \in \mathbb{N}$ we may choose a point $z_k \in F$ such that

$$\dim^A(z_k) \geq s - 1/k.$$

Let $e \in S^{n-1}$ be a point such that, for every $k \in \mathbb{N}$, the following hold.

- For every r and $s < r$, $K^{A,z_k,e_3\ldots,e_n}_s(e) \geq s - O(1)$.
- For every r, $K^{A,e}_r(z_k) \geq K^A_r(z_k) - O(\log r)$.

Note that the set of points e satisfying the first item is of measure one. By Theorem 3.10, the set of points satisfying the second item is also of measure one. So almost every e satisfies these requirements.

Fix $k \in \mathbb{N}$. Let $\eta' \in \mathbb{Q} \cap (0, \dim^A(z_k))$ and $\varepsilon' > 0$. We claim that, so long as r is sufficiently large, the conditions of Theorem 3.7, relativized to oracle A, are satisfied by these choices of z_k, e, ε', and η'.

Let $r \in \mathbb{N}$ and $s \leq r$. Then, by our choice of e and z_k, we have

$$\begin{aligned} K^{A,e_3\ldots,e_n}_{s,r}(e \mid z_k) &\geq K^{A,z_k,e_3\ldots,e_n}_s(e) - O(\log r) \\ &\geq s - 1/k - O(\log r) \\ &\geq s - \varepsilon r, \end{aligned}$$

for sufficiently large r, and so condition (1) is satisfied. By our choice of e, condition (2) of Theorem 3.7 is also satisfied.

We may therefore apply Theorem 3.7, resulting in

$$K_r^{A,e,D_r}(P_e(z_k)) \geq \eta r - \varepsilon r - \tfrac{n\varepsilon}{\delta} r - K(\varepsilon) - K(\eta) - O_z(\log r).$$

Hence,

$$\begin{aligned}
\dim^{A,e}(P_e(z_k)) &= \liminf_{r\to\infty} \frac{K_r^{A,e}(P_e(z_k))}{r} \\
&\geq \liminf_{r\to\infty} \frac{K_r^{A,e,D_r}(P_e(z_k))}{r} \\
&\geq \liminf_{r\to\infty} \frac{\eta r - \varepsilon r - \frac{n\varepsilon}{\delta} r - K(\varepsilon) - K(\eta) - O_{z_k}(\log r))}{r} \\
&= \eta - \varepsilon - \frac{n\varepsilon}{\delta}.
\end{aligned}$$

Since both η and ε were arbitrary, we see that

$$\begin{aligned}
\dim^{A,e}(P_e(z_k)) &\geq \dim^{A,e}(z_k) \\
&\geq s - 1/k.
\end{aligned}$$

As k was chosen arbitrarily,

$$\sup_{z\in F} \dim^{A,e}(P_e(z)) = s.$$

Therefore, by the F_σ point-to-set principle, the proof is complete. □

3.3 Projection Theorems For Non-Analytic Sets

In the proof of Theorem 3.1 of the previous section, we took advantage of the fact that E was analytic by using the weaker point-to-set principle to get a *specific* oracle characterizing the dimension of the projected set $P_e(E)$. We would like to prove similar results about projections of more general sets.

Our second main theorem shows that if the Hausdorff and packing dimensions of E are equal, the conclusion of Marstrand's theorem holds. Essentially, this assumption guarantees, for every oracle, direction pair (A, e), the existence of a point $z \in E$ such that $\dim^{A,e}(z) \geq \dim_H(E) - \epsilon$; that is, (A, e) does not change the complexity of z. This allows us to use Theorem 3.7 in a similar manner as before.

Theorem 3.2. *Let $E \subseteq \mathbb{R}^n$ be any set with $\dim_H(E) = \dim_P(E) = s \leq 1$. Then for almost every $e \in S^{n-1}$,*

$$\dim_H(P_e(E) \geq s.$$

Proof. Let $E \subseteq \mathbb{R}^n$ be any set with $\dim_H(E) = \dim_P(E) = s \leq 1$. By the point-to-set principle, there is an oracle $B \subseteq \mathbb{N}$ testifying to $\dim_H(E)$ and $\dim_P(E)$. Let $e \in S^{n-1}$ be any point which is random relative to B. Note that the points satisfying this requirement is of measure 1. Let $A \subseteq \mathbb{N}$ be the oracle testifying to $\dim_H(P_e(E))$. Then, by the point-to-set principle, it suffices to show that for every $\epsilon > 0$ there is a $z \in E$ such that

$$\dim^A(P_e(z)) \geq s - \epsilon.$$

To that end, let $\eta \in \mathbb{Q} \cap (0, s)$ and $\varepsilon > 0$. By the point-to-set principle, there is a $z \in E$ such that

$$s - \frac{\varepsilon}{4} \leq \dim^{A,B,e}(z) \leq \dim^B(z) = \operatorname{Dim}^B(z) \leq s. \tag{3.7}$$

We now show that the conditions of Theorem 3.7 are satisfied for these choices, relative to B, for all sufficiently large $r \in \mathbb{N}$. We first note that, by inequality (3.7), and the definition of effective dimension,

$$\begin{aligned} sr - \frac{\varepsilon}{4}r - \frac{\varepsilon}{4}r &\leq K_r^{A,B,e}(z) \\ &\leq K_r^B(z) + O(1) \\ &\leq sr + \frac{\varepsilon}{2}r, \end{aligned}$$

for all sufficiently large r. Therefore, for all such r,

$$K_r^{A,B,e}(z) \geq K_r^B(z) - \varepsilon r. \tag{3.8}$$

By inequality (3.8),

$$K_r^{A,B,e}(z) \geq K_r^B(z) - \varepsilon r,$$

and so, by Lemma 2.3(ii), property (1) is satisfied. Property (2) follows from inequality (3.8).

Therefore, we may apply Theorem 3.7, resulting in

$$K_r^{A,B,e}(P_e(z)) \geq \eta r - \varepsilon r - \tfrac{4\varepsilon}{\delta} r - K(\varepsilon) - K(\eta) - O_z(\log r).$$

Hence,

$$\begin{aligned} \dim^A(P_e(z)) &\geq \dim^{A,B,e}(P_e(z)) \\ &= \liminf_{r\to\infty} \frac{K_r^{A,B,e}(P_e(z))}{r} \\ &\geq \liminf_{r\to\infty} \frac{\eta r - \varepsilon r - \frac{4\varepsilon}{\delta} r - K(\varepsilon) - K(\eta) - O_z(\log r))}{r} \\ &= \eta - \varepsilon - \frac{4\varepsilon}{\delta}. \end{aligned}$$

Since both η and ε were arbitrary, we see that

$$\sup_{z\in E} \dim^A(P_e(z)) = s.$$

By the point-to-set principle, the conclusion follows. □

Our last main theorem gives a lower bound for the *packing* dimension of a projection for *general* sets. The proof of this theorem again relies on the ability to choose, for every (A, e), a point z whose complexity is unaffected relative to (A, e). This cannot be assumed to hold for every precision r. However, by the point-to-set principle, we can show that this can be done for infinitely many precision parameters r.

Theorem 3.3. *Let $E \subseteq \mathbb{R}^n$ be any set with $\dim_H(E) = s \leq 1$. Then for almost every $e \in S^{n-1}$,*

$$\dim_P(P_e(E) \geq s.$$

Proof. Let $E \subseteq \mathbb{R}^n$ be any set with $\dim_H(E) = s \leq 1$. By the point-to-set principle, there is an oracle $B \subseteq \mathbb{N}$ testifying to $\dim_H(E)$ and $\dim_P(E)$. Let $e \in S^{n-1}$ be any point which is random relative to B. Note that the points satisfying this requirement is of measure 1. Let $A \subseteq \mathbb{N}$ be the oracle testifying to $\dim_P(P_e(E))$. Then, by the point-to-set principle, it suffices to show that for every $\epsilon > 0$ there is a $z \in E$ such that

$$\mathrm{Dim}^A(P_e(z)) \geq s - \epsilon.$$

To that end, let $\eta \in \mathbb{Q} \cap (0, s)$ and $\varepsilon > 0$. By the point to set principle, there is a $z \in E$ such that

$$s - \frac{\varepsilon}{4} \leq \dim^{A,B,e}(z) \leq \dim^B(z) \leq s. \tag{3.9}$$

We now show that the conditions of Theorem 3.7 are satisfied for these choices, relative to B, for infinitely many $r \in \mathbb{N}$. We first note that, by equation (3.9),

$$\begin{aligned} sr - \frac{\varepsilon}{4}r - \frac{\varepsilon}{4}r &\leq K_r^{A,B,e}(z) \\ &\leq K_r^B(z) + O(1) \\ &\leq sr + \frac{\varepsilon}{2}r, \end{aligned}$$

for infinitely many r. Hence, for all such r,

$$K^{A,B,e}(z) \geq K^{B}(z) - \varepsilon r. \tag{3.10}$$

By inequality (3.10),

$$K_r^{A,B,e}(z) \geq K_r^{B}(z) - \varepsilon r,$$

and therefore property (1) holds by Lemma 2.3(ii). Property (2) follows from inequality (3.8).

Therefore, we may apply Theorem 3.7, resulting in

$$K_r^{A,B,e}(P_e(z)) \geq \eta r - \varepsilon r - \tfrac{4\varepsilon}{\delta} r - K(\varepsilon) - K(\eta) - O_z(\log r),$$

for infinitely many $r \in \mathbb{N}$. Hence,

$$\begin{aligned}
\operatorname{Dim}^A(P_e(z)) &\geq \operatorname{Dim}^{A,B,e}(P_e(z)) \\
&= \limsup_{r\to\infty} \frac{K_r^{A,B,e}(P_e(z))}{r} \\
&\geq \limsup_{r\to\infty} \frac{\eta r - \varepsilon r - \frac{4\varepsilon}{\delta} r - K(\varepsilon) - K(\eta) - O_z(\log r))}{r} \\
&= \eta - \varepsilon - \frac{4\varepsilon}{\delta}.
\end{aligned}$$

Since both η and ε were arbitrary, we see that

$$\sup\nolimits_{z\in E} \operatorname{Dim}^{A,e}(P_e(z)) \geq s.$$

By the point-to-set principle, the conclusion follows. $\square$

CHAPTER 4. DIMENSION SPECTRA OF LINES IN THE PLANE

In this chapter, we study the spectra of possible dimensions of points on a line in the Euclidean plane. This chapter is joint work with Neil Lutz and some portion of it have appeared in [33] and [34].

Given the pointwise nature of effective Hausdorff dimension, it is natural to investigate not only the supremum $\sup_{x\in E}\dim(x)$ but the entire *(effective Hausdorff) dimension spectrum* of a set $E \subseteq \mathbb{R}^n$, i.e., the set

$$\mathrm{sp}(E) = \{\dim(x) : x \in E\}\,.$$

The dimension spectra of several classes of sets have been previously investigated. Gu, et al. studied the dimension spectra of randomly selected subfractals of self-similar fractals [17]. Dougherty, et al. focused on the dimension spectra of random translations of Cantor sets [12]. In the context of symbolic dynamics, Westrick has studied the dimension spectra of subshifts [50].

This work concerns the dimension spectra of lines in the Euclidean plane $\mathbb{R}^2$. Given a line $L_{a,b}$ with slope a and vertical intercept b, we ask what $\mathrm{sp}(L_{a,b})$ might be. It was shown by Turetsky [49] that, for every $n \geq 2$, the set of all points in $\mathbb{R}^n$ with effective Hausdorff 1 is connected, guaranteeing that $1 \in \mathrm{sp}(L_{a,b})$.

In recent work [33], N. Lutz and Stull showed that the dimension spectrum of a line in $\mathbb{R}^2$ cannot be a singleton. By proving a general lower bound on $\dim(x, ax+b)$, which is presented as Theorem 4.3 here, we demonstrated that

$$\min\{1, \dim(a,b)\} + 1 \in \operatorname{sp}(L_{a,b})\,.$$

Together with the fact that $\dim(a,b) = \dim(a, a^2+b) \in \operatorname{sp}(L_{a,b})$ and Turetsky's result, this implies that the dimension spectrum of $L_{a,b}$ contains both endpoints of the unit interval $[\min\{1, \dim(a,b)\}, \min\{1, \dim(a,b)\} + 1]$.

Here we build on that work with two main theorems on the dimension spectrum of a line. Our first theorem gives conditions under which the entire unit interval must be contained in the spectrum. We refine the techniques of [33] to show in our main theorem (Theorem 4.7) that, whenever $\dim(a,b) = \operatorname{Dim}(a,b)$, we have

$$[\min\{1, \dim(a,b)\}, \min\{1, \dim(a,b)\} + 1] \subseteq \operatorname{sp}(L_{a,b})\,.$$

Given any value $s \in [0,1]$, we construct, by padding a random binary sequence, a value $x \in \mathbb{R}$ such that $\dim(x, ax+b) = s + \min\{\dim(a,b), 1\}$. Our second main theorem shows that the dimension spectrum $\operatorname{sp}(L_{a,b})$ is infinite for every line such that $\dim(a,b)$ is at least one. Together with (a corollary of) Theorem 4.3, this shows that the dimension spectrum of *any* line has infinite cardinality.

4.1 Background and Approach

In this section we describe the basic ideas behind our investigation of dimension spectra of lines. We briefly discuss some of our earlier work on this subject, and we present two technical lemmas needed for the proof our main theorems.

The dimension of a point on a line in $\mathbb{R}^2$ has the following trivial bound.

Observation 4.1. *For all $a, b, x \in \mathbb{R}$, $\dim(x, ax+b) \leq \dim(x, a, b)$.*

In this work, our goal is to find values of x for which the approximate converse

$$\dim(x, ax+b) \geq \dim^{a,b}(x) + \dim(a,b) \tag{4.1}$$

holds. There exist oracles, at least, relative to which (4.1) does not always hold. This follows from the point-to-set principle of J. Lutz and N. Lutz [29] and the existence of Furstenberg sets with parameter α and Hausdorff dimension less than $1+\alpha$ (attributed by Wolff [51] to Furstenberg and Katznelson "in all probability"). The argument is simple and very similar to our proof in [33] of a lower bound on the dimension of generalized Furstenberg sets.

Specifically, for every $s \in [0,1]$, we want to find an x of effective Hausdorff dimension s such that (4.1) holds. Note that equality in Observation 4.1 implies (4.1).

Observation 4.2. *Suppose $ax+b = ux+v$ and $u \neq a$. Then*

$$\dim(u,v) \geq \dim^{a,b}(u,v) \geq \dim^{a,b}\left(\frac{b-v}{u-a}\right) = \dim^{a,b}(x)\,.$$

In our previous work [33], we used an argument of this type to prove a general lower bound on the dimension of points on lines in $\mathbb{R}^2$:

Theorem 4.3. *For all $a, b, x \in \mathbb{R}$,*

$$\dim(x, ax+b) \geq \dim^{a,b}(x) + \min\{\dim(a,b),\, \dim^{a,b}(x)\}\,.$$

The strategy in that work is to use oracles to artificially lower $K_r(a,b)$ when necessary, to essentially force $\dim(a,b) < \dim^{a,b}(x)$. This enables the above argument structure to be used, but lowering the complexity of (a,b) also weakens the conclusion, leading to the minimum in Theorem 4.3.

4.1.1 Technical Lemmas

In the present work, we circumvent this limitation and achieve inequality (4.1) by controlling the choice of x and placing a condition on (a,b). Adapting the above argument to the case where $\dim(a,b) > \dim^{a,b}(x)$ requires refining the techniques of [33]. In particular, we use the following two technical lemmas, which strengthen results from that work. Lemma 4.4 weakens the conditions needed to compute an estimate of (x,a,b) from an estimate of $(x,ax+b)$.

Lemma 4.4. *Let $a,b,x\in\mathbb{R}$, $k\in\mathbb{N}$, and $r_0=1$. Suppose that $r_1,\ldots,r_k\in\mathbb{N}$, $\delta\in\mathbb{R}_+$, and $\varepsilon,\eta\in\mathbb{Q}_+$ satisfy the following conditions for every $1\leq i\leq k$.*

1. $r_i\geq \log(2|a|+|x|+6)+r_{i-1}$.
2. $K_{r_i}(a,b)\leq(\eta+\varepsilon)\,r_i$.
3. *For every $(u,v)\in\mathbb{R}^2$ such that $t=-\log\|(a,b)-(u,v)\|\in(r_{i-1},r_i]$ and $ux+v=ax+b$, $K_{r_i}(u,v)\geq(\eta-\varepsilon)\,r_i+\delta\cdot(r_i-t)$.*

Then for every oracle set $A\subseteq\mathbb{N}$,

$$K^A_{r_k}(a,b,x\,|\,x,ax+b)\leq 2^k\left(K(\varepsilon)+K(\eta)+\frac{4\varepsilon}{\delta}r_k+O(\log r_k)\right).$$

Proof. Let $a,b,x\in\mathbb{R}$. We proceed by induction on k. By Corollary A.8, the conclusion holds for $k=1$. Assume the conclusion holds for all $i<k$. Let $r_1,\ldots,r_k$, δ, ε, η, and A be as described in the lemma statement.

Define an oracle Turing machine M that does the following given oracle A and input $\pi=\pi_1\pi_2\pi_3\pi_4\pi_5$ such that $U^A(\pi_1)=(q_1,q_2)\in\mathbb{Q}^2$, $U(\pi_2)=(s_1,\ldots,s_k)\in\mathbb{N}^k$, $U(\pi_3)=\zeta\in\mathbb{Q}$, $U(\pi_4)=\iota\in\mathbb{Q}$ and $U^A(\pi_5,q_1,q_2)=h\in\mathbb{Q}^2$

For every program $\sigma\in\{0,1\}^*$ with $\ell(\sigma)\leq(\iota+\zeta)s_k$, in parallel, M simulates $U(\sigma)$. If one of the simulations halts with some output $(p_1,p_2)\in\mathbb{Q}^2\cap B_{2^{-r_{k-1}}}(h)$ such that

$$|p_1q_1+p_2-q_2|<2^{-s_2}(|p_1|+|q_1|+3)\,,$$

then M halts with output (p_1,p_2,q_1). Let c_M be a constant for the description of M.

Now let π_1, π_2, π_3, π_4, and π_5 testify to $K^A_r(x, ax+b)$, $K(r_1, \ldots, r_k)$, $K(\varepsilon)$, $K(\eta)$, and $K_{r_{k-1},r_k}(a, b \mid x, ax+b)$ respectively, and let $\pi = \pi_1\pi_2\pi_3\pi_4\pi_5$.

By condition 2, there is some $(\hat{p}_1, \hat{p}_2) \in B_{2^{-r_k}}(a, b)$ such that $K(\hat{p}_1, \hat{p}_2) \leq (\eta+\varepsilon)r_k$, meaning that there is some $\hat{\sigma} \in \{0,1\}^*$ with $\ell(\hat{\sigma}) \leq (\eta+\varepsilon)r_k$ and $U(\hat{\sigma}) = (\hat{p}_1, \hat{p}_2)$. By Observation A.9(1),

$$|\hat{p}_1 q_1 + \hat{p}_2 - q_2| < 2^{-r_k}(|\hat{p}_1| + |q_1| + 3)\,,$$

for every $(q_1, q_2) \in B_{2^{-r_k}}(x, ax+b)$, so M is guaranteed to halt on input π.

Hence, let $(p_1, p_2, q_1) = M(\pi)$. By Observation A.9(2), there is some

$$(u, v) \in B_{2^{\gamma-r_k}}(p_1, p_2) \subseteq B_{2^{-r_{k-1}}}(a, b)$$

such that $ux + v = ax + b$, where $\gamma = \log(2|a| + |x| + 5)$. We have

$$\|(p_1, p_2) - (u, v)\| < 2^{\gamma - r_k}$$

and $|q_1 - x| < 2^{-r_k}$, so

$$(p_1, p_2, q_1) \in B_{2^{\gamma+1-r_k}}(u, v, x)\,.$$

It therefore follows that

$$\begin{aligned}
K^A_{r_k-\gamma-1,r_k}(u, v, x \mid x, ax+b) &\leq K(p_1, p_2, q_1)\\
&\leq \ell(\pi_1\pi_2\pi_3\pi_4\pi_5) + c_M\\
&\leq \ell(\pi_5) + K(r_1, \ldots, r_k) + K(\varepsilon) + K(\eta) + c_M\\
&= \ell(\pi_5) + K(\varepsilon) + K(\eta) + O(\log r_k)\,.
\end{aligned}$$

Applying Lemma 2.1 yields

$$K^A_{r_k}(u, v, x \mid x, ax+b) \leq \ell(\pi_5) + K(\varepsilon) + K(\eta) + O(\log r_k). \tag{4.2}$$

By our inductive hypothesis, we have that

$$\begin{aligned}
\ell(\pi_5) &= K_{r_{k-1},r_k}(a,b \mid x, ax+b) \\
&= K_{r_{k-1}}(a,b \mid x, ax+b) + O(\log r_{k-1}) \\
&\leq 2^{k-1}\left(K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{\delta} r_{k-1} + O(\log r_{k-1})\right) . \qquad (4.3)
\end{aligned}$$

To complete the proof, we bound $K^A_{r_k}(a,b,x \mid u,v,x)$. If $t > r_k$, then

$$K^A_{r_k}(a,b,x \mid u,v,x) \leq \log(r_k) \, .$$

Otherwise, when $t \leq r_k$, by our construction of M and Lemma 2.1,

$$\begin{aligned}
(\eta+\varepsilon) r_k &\geq K(p_1,p_2) \\
&\geq K_{r_k-\gamma}(u,v) \\
&\geq K_{r_k}(u,v) - O(\log r_k) \, .
\end{aligned}$$

Combining this with condition 3 in the lemma statement and simplifying yields

$$r_k - t \leq \frac{2\varepsilon}{\delta} r_k + O(\log r_k) \, .$$

Therefore, by Lemma 2.1, we have

$$\begin{aligned}
K_{r_k}(a,b,x \mid u,v,x) &\leq 2(r_k - t) + O(\log r_k) \\
&\leq \frac{4\varepsilon}{\delta} r_k + O(\log r_k) \, , \qquad (4.4)
\end{aligned}$$

for every $t \in \mathbb{N}$.

Combining inequalities (4.2), (4.3) and (4.4) gives

$$\begin{aligned}
K_{r_k}(a,b,x \mid x, ax+b) &\leq K_{r_k}(u,v,x \mid x, ax+b) + K_{r_k}(a,b,x \mid u,v,x) \\
&\leq K_{r_k}(u,v,x \mid x, ax+b) + \frac{4\varepsilon}{\delta} r_k + O(\log r_k) \\
&\leq \ell(\pi_5) + K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{\delta} r_k + O(\log r_k) \\
&\leq 2^k \left(K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{\delta} r_k + O(\log r_k)\right) .
\end{aligned}$$

□

The following lemma, first proven in [33], provides a lower bound on the complexity of any line (u,v) intersecting (a,b) at x. Intuitively, this shows that, when x is chosen to be of high complexity, $K_r(u,v) > K_r(a,b)$ unless (u,v) is very close to (a,b). As $K_r(u,v)$ is upper semicomputable, this is algorithmically useful: We can enumerate all pairs (u,v) whose precision-r complexity falls below a certain threshold. If one of these pairs satisfies, approximately, $ux+v = ax+b$, then we know that (u,v) is close to (a,b). Thus, an estimate for $(x, ax+b)$ algorithmically yields an estimate for (x,a,b).

Lemma 4.5 (N. Lutz and Stull [33])**.** *Let $a,b,x \in \mathbb{R}$. For all $(u,v) \in \mathbb{R}^2$ such that $ux+v = ax+b$ and $t = -\log \|(a,b)-(u,v)\| \in (0,r]$,*

$$K_r(u,v) \geq K_t(a,b) + K^{a,b}_{r-t}(x) - O(\log r)\,.$$

Proof. Fix $a,b,x \in \mathbb{R}$. By Lemma 2.2(i), for all $(u,v) \in B_1(a,b)$ and every $r \in \mathbb{N}$,

$$K_r(u,v) \geq K_r(u,v|a,b) + K_r(a,b) - K_r(a,b|u,v) - O_{a,b}(\log r)\,. \tag{4.5}$$

We bound $K_r(a,b) - K_r(a,b|u,v)$ first. Since $(u,v) \in B_{2^{-t}}(a,b)$, for every $r \geq t$ we have $B_r(u,v) \subseteq B_{2^{1-t}}(a,b)$, so

$$K_r(a,b|u,v) \leq K_{r,t-1}(a,b|a,b)\,.$$

By Lemma 2.2(ii), then,

$$\begin{aligned} K_r(a,b) - K_r(a,b|u,v) &\geq K_r(a,b) - K_{r,t-1}(a,b|a,b) \\ &\geq K_{t-1}(a,b) - O_{a,b}(\log r)\,. \end{aligned}$$

Lemma 2.1 tells us that

$$K_{t-1}(a,b) \geq K_t(a,b) - O(\log t)\,.$$

Therefore we have, for every $u, v \in B_1(a, b)$ and every $r \geq t$,

$$K_r(a, b) - K_r(a, b|u, v) \geq K_t(a, b) - O_{a,b}(\log r)\,. \tag{4.6}$$

We now bound the term $K_r(u, v|a, b)$. Let $(u, v) \in \mathbb{R}^2$ be such that $ux+v = ax+b$. If $t \leq r < t+|x|+2$, then $r - t = O_x(1)$, so by Lemma 2.1, $K_{r-t,r}(x|a, b) = O_x(1)$. In this case, $K_r(u, v|a, b) \geq K_{r-t,r}(x|a, b) - O_{a,b,x}(\log r)$ holds trivially. Hence, assume $r \geq t + |x| + 2$.

Let M be a Turing machine such that, whenever $q = (q_1, q_2) \in \mathbb{Q}^2$ and $U(\pi, q) = p = (p_1, p_2) \in \mathbb{Q}^2$, with $p_1 \neq q_1$,

$$M(\pi, q) = \frac{p_2 - q_2}{p_1 - q_1}\,.$$

For each $q \in B_{2^{-r}}(a, b) \cap \mathbb{Q}^2$, let π_q testify to $\hat{K}_r(u, v|q)$. Then

$$U(\pi_q, q) \in B_{2^{-r}}(u, v) \cap \mathbb{Q}^2\,.$$

It follows by a routine calculation that

$$|M(\pi_q, q) - x| = \left|\frac{p_2 - q_2}{p_1 - q_1} - \frac{b - v}{a - u}\right| < 2^{4+2|x|+t-r}\,.$$

Thus, $M(\pi_q, q) \in B_{2^{4+2|x|+t-r}}(x) \cap \mathbb{Q}^2$. For some constant c_M, then,

$$\begin{aligned}\hat{K}_{r-4-2|x|-t}(x|q) &\leq \ell(\pi_q) + c_M \\ &= \hat{K}_r(u, v|q) + c_M\,.\end{aligned}$$

Taking the maximum of each side over $q \in B_{2^{-r}}(a, b) \cap \mathbb{Q}^2$ and rearranging,

$$K_r(u, v|a, b) \geq K_{r-4-2|x|-t,r}(x|a, b) - c_M\,.$$

Then since Lemma 2.1 implies that

$$K_{r-4-2|x|-t,r}(x|a, b) \geq K_{r-t,r}(x|a, b) - O_x(\log r)\,,$$

we have shown, for every (u, v) satisfying $ux + v = ax + b$ and every $r \geq t$,

$$K_r(u, v|a, b) \geq K_{r-t,r}(x|a, b) - O_{a,b,x}(\log r)\,. \tag{4.7}$$

The lemma follows immediately from (4.5), (4.6), and (4.7). □

Lemma 4.6 strengthens the oracle construction of [33], allowing us to control complexity at multiple levels of precision.

Lemma 4.6. *Let $z \in \mathbb{R}^n$, $\eta \in \mathbb{Q} \cap [0, \dim(z)]$, and $k \in \mathbb{N}$. For all $r_1, \ldots, r_k \in \mathbb{N}$, there is an oracle $D = D(r_1, \ldots, r_k, z, \eta)$ such that*

1. *For every $t \leq r_1$, $K^D_t(z) = \min\{\eta r_1, K_t(z)\} + O(\log r_k)$*
2. *For every $1 \leq i \leq k$,*
$$K^D_{r_i}(z) = \eta r_1 + \sum_{j=2}^{i} \min\{\eta(r_j - r_{j-1}), K_{r_j,r_{j-1}}(z \mid z)\} + O(\log r_k)\,.$$
3. *For every $t \in \mathbb{N}$ and $x \in \mathbb{R}$, $K^{z,D}_t(x) = K^z_t(x) + O(\log r_k)$.*

Proof. We define the sequence of oracles recursively. Let $D_1 = A(r_1, z, \eta)$, as defined in Lemma 3.6, and for every $1 < i \leq k$, let

$$D_i = \begin{cases} D_{i-1} & \text{if } K^{D_{i-1}}_{r_i}(z) < \eta r_i \\ \langle D_{i-1}, A^{D_{i-1}}(r_i, z, \eta)\rangle & \text{otherwise}\,. \end{cases}$$

Notice that, for every $1 \leq i \leq k$, D_i is a finite oracle, so $\dim^{D_i}(z) = \dim(z)$ and $\eta \in [0, \dim^{D_k}(z)]$.

We now show via induction on k that the lemma holds for all $k \in \mathbb{N}$. For $k = 1$, all three properties hold by Lemma 3.6. Fix $j > 1$, assume the properties hold for $k = j - 1$.

We first show that property 1 holds for $k = j$. Let $t \leq r_1$. It follows from the definition of the oracle D_j and Lemma 3.6, relative to D_{j-1}, that

$$K_t^{D_j}(z) = \min\{\eta r_j, K_t^{D_{j-1}}(z)\} + O(\log r_j)\,.$$

By the induction hypothesis, $K_t^{D_{j-1}}(z) = \min\{\eta r_1, K_t(z)\} + O(\log r_{j-1})$. Thus,

$$\begin{aligned} K_t^{D_j}(z) &= \min\{\eta r_j, \min\{\eta r_1, K_t(z)\} + O(\log r_{j-1})\} + O(\log r_j) \\ &= \min\{\eta r_1, K_t(z)\} + O(\log r_j)\,. \end{aligned}$$

We now show the property 2 holds for $k = j$. Suppose that $i < j$. Then by the definition of D_j,

$$K_{r_i}^{D_j}(z) = \min\{\eta r_j, K_{r_i}^{D_{j-1}}(z)\} + O(\log r_j)\,,$$

and by the induction hypothesis,

$$K_{r_i}^{D_{j-1}}(z) = \eta r_1 + \sum_{l=2}^{i} \min\{\eta(r_l - r_{l-1}), K_{r_l, r_{l-1}}(z \mid z)\} + O(\log r_{j-1})\,.$$

Since

$$\eta r_1 + \sum_{l=2}^{i} \min\{\eta(r_l - r_{l-1}), K_{r_l, r_{l-1}}(z \mid z)\} \leq \eta r_i\,,$$

we have

$$K_{r_i}^{D_j}(z) = \eta r_1 + \sum_{l=2}^{i} \min\{\eta(r_l - r_{l-1}), K_{r_l, r_{l-1}}(z \mid z)\} + O(\log r_j)$$

Now suppose that $i = j$. If $K^{D_{j-1}}_{r_j}(z) < \eta r_j$, then, by our induction hypothesis and Lemma 3.6,

$$\begin{aligned}
K^{D_j}_{r_i}(z) =& K^{D_{j-1}}_{r_i}(z)\\
=& K^{D_{j-1}}_{r_{i-1}}(z) + K^{D_{j-1}}_{r_i,r_{i-1}}(z \mid z) - O(\log r_j)\\
=& \eta r_1 + \sum_{l=2}^{i-1} \min\{\eta(r_l - r_{l-1}), K_{r_l,r_{l-1}}(z \mid z)\} + O(\log r_j)\\
&+ K_{r_i,r_{i-1}}(z \mid z) + O(\log r_{j-1}\\
=& \eta r_1 + \sum_{l=2}^{i} \min\{\eta(r_l - r_{l-1}), K_{r_l,r_{l-1}}(z \mid z)\} + O(\log r_j)\,.
\end{aligned}$$

If instead $K^{D_{j-1}}_{r_i}(z) \geq \eta r_i$, then $K^{D_j}_{r_i}(z) = \eta r_i - O(\log r_i)$ by Lemma 3.6, relative to D_{j-1}. Since $K^{D_{j-1}}_{r_i}(z) \geq \eta r_i$ implies that $K_{r_i,r_{i-1}}(z \mid z) \geq \eta(r_i - r_{i-1})$,

$$K^{D_j}_{r_i}(z) = \eta r_1 + \textstyle\sum_{l=2}^{i} \min\{\eta(r_l - r_{l-1}), K_{r_l,r_{l-1}}(z \mid z)\} + O(\log r_i)$$

Therefore property 2 holds for all $1 \leq i \leq k$.

To complete the proof we show that property 3 is satisfied for $k = j$. Let $t \in \mathbb{N}$ and $y \in \mathbb{R}^m$. By Lemma 3.6, relativized to D_{j-1}, and our induction hypothesis,

$$\begin{aligned}
K^{z,D_j}_t(y) &= K^{z,D_{j-1}}_t(y) + O(\log r_j)\\
&= K^z_t(y) + O(\log r_{j-1}) + O(\log r_j)\\
&= K^z_t(y) + O(\log r_j)\,.
\end{aligned}$$

Thus, by mathematical induction, the lemma holds for all $k \in \mathbb{N}$. □ □

4.2 Main Theorems

We are now prepared to prove our two main theorems. We first show that, for lines $L_{a,b}$ such that $\dim(a, b) = \mathrm{Dim}(a, b)$, the dimension spectrum $\mathrm{sp}(L_{a,b})$ contains the unit interval.

Theorem 4.7. *Let $a, b \in \mathbb{R}$ satisfy* $\dim(a, b) = \operatorname{Dim}(a, b)$*. Then for every $s \in [0, 1]$ there is a point $x \in \mathbb{R}$ such that* $\dim(x, ax + b) = s + \min\{\dim(a, b), 1\}$.

Proof. Every line contains a point of effective Hausdorff dimension 1 [49], and by the preservation of effective dimensions under computable bi-Lipschitz functions, $\dim(a, a^2 + b) = \dim(a, b)$, so the theorem holds for $s = 0$. For $s = 1$, we may choose an $x \in \mathbb{R}$ that is random relative to (a, b). That is, there is some constant $c \in \mathbb{N}$ such that for all $r \in \mathbb{N}$, $K_r^{a,b}(x) \geq r - c$. By Theorem 4.3,

$$\begin{aligned}\dim(x, ax + b) &\geq \dim^{\{} a, b\}(x) + \min\{\dim(a, b), 1\} \\ &= \min\{\dim(a, b), 1\} + \liminf_{r \to \infty} \frac{K_r(x)}{r} \\ &= \min\{\dim(a, b), 1\} + 1,\end{aligned}$$

and the conclusion holds.

Now let $s \in (0, 1)$ and $d = \dim(a, b) = \operatorname{Dim}(a, b)$. Let $y \in \mathbb{R}$ be random relative to (a, b). Define the sequence of natural numbers $\{h_j\}_{j \in \mathbb{N}}$ inductively as follows. Define $h_0 = 1$. For every $j > 0$, let

$$h_j = \min\left\{h \geq 2^{h_{j-1}} : K_h(a, b) \leq \left(d + \frac{1}{j}\right) h\right\}.$$

Note that h_j always exists. For every $r \in \mathbb{N}$, let

$$x[r] = \begin{cases} 0 & \text{if } \frac{r}{h_j} \in (s, 1] \text{ for some } j \in \mathbb{N} \\ y[r] & \text{otherwise} \end{cases}$$

where $x[r]$ is the rth bit of x. Define $x \in \mathbb{R}$ to be the real number with this binary expansion. Then $K_{sh_j}(x) = sh_j + O(\log sh_j)$.

We first show that $\dim(x, ax+b) \leq s + \min\{d, 1\}$. For every $j \in \mathbb{N}$,

$$\begin{aligned}
K_{h_j}(x, ax+b) &= K_{h_j}(x) + K_{h_j}(ax+b \,|\, x) + O(\log h_j) \\
&= K_{sh_j}(x) + K_{h_j}(ax+b \,|\, x) + O(\log h_j) \\
&= K_{sh_j}(y) + K_{h_j}(ax+b \,|\, x) + O(\log h_j) \\
&\leq sh_j + \min\{d, 1\} \cdot h_j + o(h_j)\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\dim(x, ax+b) &= \liminf_{r\to\infty} \frac{K_r(x, ax+b)}{r} \\
&\leq \liminf_{j\to\infty} \frac{K_{h_j}(x, ax+b)}{h_j} \\
&\leq \liminf_{j\to\infty} \frac{sh_j + \min\{d,1\}h_j + o(h_j)}{h_j} \\
&= s + \min\{d, 1\}\,.
\end{aligned}$$

If $1 > s \geq d$, then by Theorem 4.3 we also have

$$\begin{aligned}
\dim(x, ax+b) &\geq \dim^{\{} a, b\}(x) + \dim(a, b) \\
&= \dim(x) + d \\
&= \liminf_{r\to\infty} \frac{K_r(x)}{r} + d \\
&= \liminf_{j\to\infty} \frac{K_{h_j}(x)}{h_j} + d \\
&= s + \min\{d, 1\}\,.
\end{aligned}$$

Hence, we may assume that $s < d$.

Let $H = \mathbb{Q} \cap (s, \min\{d, 1\})$. Let $\eta \in H$, $\delta = 1 - \eta > 0$, and $\varepsilon \in \mathbb{Q}_+$. We now show that $\dim(x, ax+b) \geq s + \eta - \frac{\alpha\varepsilon}{\delta}$, where α is some constant independent of η and ε. Let $j \in \mathbb{N}$ and $m = \frac{s-1}{\eta-1}$. We first show that

$$K_r(x, ax+b) \geq K_r(x) + \eta r - c\frac{\varepsilon}{\delta}r - o(r)\,, \tag{4.8}$$

for every $r \in (sh_j, mh_j]$. Let $r \in (sh_j, mh_j]$. Set $k = \frac{r}{sh_j}$, and define $r_i = ish_j$ for all $1 \leq i \leq k$. Note that k is bounded by a constant depending only on s and η. Therefore a $o(r_k) = o(r_i)$ for all r_i. Let $D_r = D(r_1, \ldots, r_k, (a, b), \eta)$ be the oracle defined in Lemma 4.6. We first note that, since $\dim(a, b) = \mathrm{Dim}(a, b)$,

$$\begin{aligned}
K_{r_i, r_{i-1}}(a, b \,|\, a, b) &= K_{r_i}(a, b) - K_{r_{i-1}}(a, b) - O(\log r_i) \\
&= \dim(a, b) r_i - o(r_i) - \dim(a, b) r_{i-1} - o(r_{i-1}) - O(\log r_i) \\
&= \dim(a, b)(r_i - r_{i-1}) - o(r_i) \\
&\geq \eta(r_i - r_{i-1}) - o(r_i).
\end{aligned}$$

Hence, by property 2 of Lemma 4.6, for every $1 \leq i \leq k$,

$$|K_{r_i}^{D_r}(a, b) - \eta r_i| \leq o(r_k). \tag{4.9}$$

We now show that the conditions of Lemma 4.4 are satisfied. By inequality (4.9), for every $1 \leq i \leq k$,

$$K_{r_i}^{D_r}(a, b) \leq \eta r_i + o(r_k)\,,$$

and so $K_{r_i}^{D_r}(a, b) \leq (\eta + \varepsilon) r_i$, for sufficiently large j. Hence, condition 2 of Lemma 4.4 is satisfied.

To see that condition 3 is satisfied for $i = 1$, let $(u, v) \in B_1(a, b)$ such that $ux + v = ax + b$ and $t = -\log \|(a, b) - (u, v)\| \leq r_1$. Then, by Lemmas 4.5 and 4.6, and our construction of x,

$$\begin{aligned}
K_{r_1}^{D_r}(u, v) &\geq K_t^{D_r}(a, b) + K_{r_1 - t, r_1}^{D_r}(x | a, b) - O(\log r_1) \\
&\geq \min\{\eta r_1, K_t(a, b)\} + K_{r_1 - t}(x) - o(r_k) \\
&\geq \min\{\eta r_1, dt - o(t)\} + (\eta + \delta)(r_1 - t) - o(r_k) \\
&\geq \min\{\eta r_1, \eta t - o(t)\} + (\eta + \delta)(r_1 - t) - o(r_k) \\
&\geq \eta t - o(t) + (\eta + \delta)(r_1 - t) - o(r_k)\,.
\end{aligned}$$

We conclude that $K^{D_r}_{r_1}(u,v) \geq (\eta - \varepsilon) r_1 + \delta(r_1 - t)$, for all sufficiently large j.

To see that that condition 3 is satisfied for $1 < i \leq k$, let $(u,v) \in B_{2^{-r_{i-1}}}(a,b)$ such that $ux + v = ax + b$ and $t = -\log \|(a,b) - (u,v)\| \leq r_i$. Since $(u,v) \in B_{2^{-r_{i-1}}}(a,b)$,

$$r_i - t \leq r_i - r_{i-1} = ish_j - (i-1)sh_j \leq sh_j + 1 \leq r_1 + 1\,.$$

Therefore, by Lemma 4.5, inequality (4.9), and our construction of x,

$$\begin{aligned}
K^{D_r}_{r_i}(u,v) &\geq K^{D_r}_t(a,b) + K^{D_r}_{r_i - t, r_i}(x|a,b) - O(\log r_i)\\
&\geq \min\{\eta r_i, K_t(a,b)\} + K_{r_i - t}(x) - o(r_i)\\
&\geq \min\{\eta r_i, dt - o(t)\} + (\eta + \delta)(r_i - t) - o(r_i)\\
&\geq \min\{\eta r_i, \eta t - o(t)\} + (\eta + \delta)(r_i - t) - o(r_i)\\
&\geq \eta t - o(t) + (\eta + \delta)(r_i - t) - o(r_i)\,,
\end{aligned}$$

We conclude that $K^{D_r}_{r_i}(u,v) \geq (\eta - \varepsilon) r_i + \delta(r_i - t)$, for all sufficiently large j. Hence the conditions of Lemma 4.4 are satisfied, and we have

$$\begin{aligned}
K_r(x, ax+b) &\geq K^{D_r}_r(x, ax+b) - O(1)\\
&\geq K^{D_r}_r(a,b,x) - 2^k \left(K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{\delta} r + O(\log r) \right)\\
&= K^{D_r}_r(a,b) + K^{D_r}_r(x \,|\, a,b)\\
&\qquad - 2^k \left(K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{\delta} r + O(\log r) \right)\\
&\geq sr + \eta r - 2^k \left(K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{\delta} r + O(\log r) \right).
\end{aligned}$$

Thus, for every $r \in (sh_j, mh_j]$,

$$K_r(x, ax+b) \geq sr + \eta r - \frac{\alpha\varepsilon}{\delta} r - o(r)\,,$$

where α is a fixed constant, not depending on η and ε.

To complete the proof, we show that (4.8) holds for every $r \in [mh_j, sh_{j+1})$. By Lemma 2.2 and our construction of x,

$$\begin{aligned} K_r(x) &= K_{r,h_j}(x \mid x) + K_{h_j}(x) + o(r) \\ &= r - h_j + sh_j + o(r) \\ &\geq \eta r + o(r)\,. \end{aligned}$$

The proof of Theorem 4.3 gives $K_r(x, ax+b) \geq K_r(x) + \dim(x)r - o(r)$, and so $K_r(x, ax+b) \geq r(s+\eta)$.

Therefore, equation (4.8) holds for every $r \in [sh_j, sh_{j+1})$, for all sufficiently large j. Hence,

$$\begin{aligned} \dim(x, ax+b) &= \liminf_{r\to\infty} \frac{K_r(x, ax+b)}{r} \\ &\geq \liminf_{r\to\infty} \frac{K_r(x) + \eta r - \frac{\alpha\varepsilon}{\delta} r - o(r)}{r} \\ &\geq \liminf_{r\to\infty} \frac{K_r(x)}{r} + \eta - \frac{\alpha\varepsilon}{\delta} \\ &= s + \eta - \frac{\alpha\varepsilon}{\delta}\,. \end{aligned}$$

Since η and ε were chosen arbitrarily, the conclusion follows. □ □

Theorem 4.8. *Let $a, b \in \mathbb{R}$ such that $\dim(a,b) \geq 1$. Then for every $s \in [\frac{1}{2}, 1]$ there is a point $x \in \mathbb{R}$ such that $\dim(x, ax+b) \in \left[\frac{3}{2} + s - \frac{1}{2s}, s+1\right]$.*

Proof. Let $s \in [\frac{1}{2}, 1]$ and $y \in \mathbb{R}$ be random relative to (a,b). That is, there is some constant $c \in \mathbb{N}$ such that for all $r \in \mathbb{N}$,

$$K_r^{a,b}(y) \geq r - c.$$

Define sequence of natural numbers $\{h_j\}_{j\in\mathbb{N}}$ inductively as follows. Define $h_0 = 1$. For every $j > 0$, define

$$h_j = \min\left\{ h \geq 2^{h_{j-1}} : K_h(a,b) \leq \left(\dim(a,b) + \frac{1}{j}\right) h \right\}.$$

Note that h_j always exists. For every $r \in \mathbb{N}$, let

$$x[r] = \begin{cases} 0 & \text{if } \frac{r}{h_j} \in (1, \frac{1}{s}] \text{ for some } j \in \mathbb{N} \\ y[r] & \text{otherwise} \end{cases}$$

Define $x \in \mathbb{R}$ to be the real number with this binary expansion. Then,

$$K_{h_j}(x) = h_j + O(\log h_j)\,.$$

We first show that $\dim(x, ax + b) \leq s + 1$. For every $j \in \mathbb{N}$,

$$\begin{aligned} K_{h_j/s}(x, ax+b) &= K_{h_j/s}(x) + K_{h_j/s}(ax+b \mid x) + O(\log h_j/s) \\ &= K_{h_j}(x) + K_{h_j/s}(ax+b \mid x) + O(\log h_j) \\ &\leq h_j + 1 \cdot h_j/s + o(h_j). \end{aligned}$$

Therefore,

$$\begin{aligned} \dim(x, ax+b) &= \liminf_{r\to\infty} \frac{K_r(x, ax+b)}{r} \\ &\leq \liminf_{j\to\infty} \frac{K_{h_j/s}(x, ax+b)}{h_j/s} \\ &\leq \liminf_{j\to\infty} \frac{sh_j + h_j + o(h_j)}{h_j} \\ &= s + 1. \end{aligned}$$

Let $H = \mathbb{Q} \cap (s, 1)$, and $\eta \in H$. Let $\eta' \in \mathbb{Q} \cap (0, s]$, $\delta = 1 - \eta > 0$, and $\varepsilon \in \mathbb{Q}_+$. Let $j \in \mathbb{N}$. We first show that

$$K_r(x, ax+b) \geq sr + \eta r - c\frac{\varepsilon}{\delta} r - o(r), \tag{4.10}$$

for every $r \in (h_j, 2h_j]$. Let $r \in (h_j, 2h_j]$. Let $r_1 = h_j$, $r_2 = r$, and $D_r = D(r_1, r_2, (a, b), \eta)$ be the oracle defined in Lemma 4.6. We first note that, by our construction of x,

$$\begin{aligned}
K_{r,r_1}(a,b\,|\,a,b) &= K_r(a,b) - K_r(a,b) + O(\log r)\\
&\geq K_r(a,b) - \dim(a,b)r_1 - h_j/j + O(\log r)\\
&\geq \dim(a,b)r - \dim(a,b)r_1 - h_j/j + O(\log r)\\
&\geq \dim(a,b)(r-r_1) - h_j/j + O(\log r)\\
&> \eta(r-r_1) - h_j/j + O(\log r).
\end{aligned}$$

Hence, by property 2 of Lemma 4.6

$$\eta r - h_j/j - O(\log r) \leq K^{D_r}_r(a,b) \leq \eta r + O(\log r). \tag{4.11}$$

We now show that the conditions of Lemma 4.4 are satisfied. By Lemma 4.6, for each $i \in \{1,2\}$,

$$K^{D_r}_{r_i}(a,b) \leq \eta r_i + O(\log r_2)\,.$$

Hence, condition 2 of Lemma 4.4 is satisfied.

To see that condition 3 is satisfied for $i = 1$, let $(u,v) \in B_1(a,b)$ such that $ux + v = ax + b$ and $t = -\log\|(a,b)-(u,v)\| \leq r_1$. Then, by Lemmas 4.5 and 4.6, and our construction of x,

$$\begin{aligned}
K^{D_r}_{r_1}(u,v) &\geq K^{D_r}_t(a,b) + K^{D_r}_{r_1-t,r_1}(x|a,b) - O(\log r_1)\\
&\geq \min\{\eta r_1, K_t(a,b)\} + K_{r_1-t}(x) - o(r_k)\\
&\geq \min\{\eta r_1, \dim(a,b)t - o(t)\} + (\eta+\delta)(r_1-t) - o(r_k)\\
&\geq \min\{\eta r_1, \eta t - o(t)\} + (\eta+\delta)(r_1-t) - o(r_k)\\
&\geq \eta t - o(t) + (\eta+\delta)(r_1-t) - o(r_k)\\
&\geq (\eta-\varepsilon)r_1 + \delta(r_1-t)
\end{aligned}$$

for all sufficiently large j.

To see that that condition 3 is satisfied for $i = 2$, let $(u, v) \in B_{2^{-r_1}}(a, b)$ such that $ux + v = ax + b$ and $t = -\log \|(a, b) - (u, v)\| \le r_2$. Since $(u, v) \in B_{2^{-r_1}}(a, b)$,

$$\begin{aligned} r_2 - t &\le r_2 - r_1 \\ &\le 2r_1 - r_1 \\ &= r_1. \end{aligned}$$

Therefore, by Lemmas 4.5 and 4.6, inequality (4.11) and our construction of x,

$$\begin{aligned} K^{D_r}_{r_2}(u, v) &\ge K^{D_r}_t(a, b) + K^{D_r}_{r_2-t,r_2}(x|a, b) - O(\log r_2) \\ &\ge \min\{\eta r_2, K_t(a, b)\} + K_{r_2-t}(x) - o(r_2) \\ &\ge \min\{\eta r_2, \eta t - h_j/j - o(t)\} + (\eta + \delta)(r_2 - t) - o(r_2) \\ &\ge \eta t - h_j/j - o(t) + (\eta + \delta)(r_2 - t) - o(r_2) \\ &= \eta r_2 - h_j/j - o(t) + \delta(r_2 - t) - o(r_2) \\ &\ge \eta r_2 - r_2/j - o(t) + \delta(r_2 - t) - o(r_2) \\ &\ge (\eta - \varepsilon)r_2 + \delta(r_2 - t), \end{aligned}$$

for all sufficiently large j. Hence the conditions of Lemma 4.4 are satisfied, and we have

$$\begin{aligned} K_r(x, ax + b) &\ge K^{D_r}_r(x, ax + b) - O(1) \\ &\ge K^{D_r}_r(a, b, x) - 4\left(K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{\delta}r + O(\log r)\right) \\ &= K^{D_r}_r(a, b) + K^{D_r}_r(x \mid a, b) \\ &\quad - 4\left(K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{\delta}r + O(\log r)\right) \\ &\ge sr + \eta r - 4\left(K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{\delta}r + O(\log r)\right). \end{aligned}$$

Hence, for every $r \in (h_j, 2h_j]$,

$$\begin{aligned} K_r(x, ax+b) &\geq sr + \eta r - \frac{\alpha\varepsilon}{\delta} r - o(r) \\ &\geq sr + \eta r - \frac{\alpha\varepsilon}{\delta} r - o(r) \end{aligned}$$

where α is a fixed constant, not depending on η and ε.

To complete the proof, it suffices to show that $K_r(x, ax+b) \geq r(\frac{3}{2} + s - \frac{1}{2s} - \varepsilon)$, for every $r \in (2h_j, h_{j+1}]$. Let $r \in (2h_j, h_{j+1}]$. Then by Lemma 2.2 and our construction of x,

$$\begin{aligned} K_r(x) &= K_{r,h_j/s}(x \mid x) + K_{h_j/s}(x) + O(\log r) \\ &= r - h_j/s + h_j + O(\log r). \end{aligned}$$

The proof of Theorem 4.3 shows that

$$\begin{aligned} K_r(x, ax+b) &\geq K_r(x) + \eta' r - o(r) \\ &\geq r - h_j/s + h_j + \eta' r - o(r) \\ &\geq r(\frac{3}{2} + s - \frac{1}{2s} - \varepsilon) \end{aligned}$$

for sufficiently large j.

Since η, η' and ε were chosen arbitrarily, the conclusion follows. □

Corollary 4.9. *Let $L_{a,b}$ be any line in $\mathbb{R}^2$. Then the dimension spectrum $\mathrm{sp}(L_{a,b})$ is infinite.*

Proof. Let $(a, b) \in R^2$. If $\dim(a, b) < 1$, then by Theorem 4.3 and Observation 4.1, the spectrum $\mathrm{sp}(L_{a,b})$ contains the interval $[\dim(a, b), 1]$. Assume that $\dim(a, b) \geq 1$. By Theorem 4.8, for every $s \in [\frac{1}{2}, 1]$, there is a point x such that $\dim(x, ax + b) \in [\frac{3}{2} + s - \frac{1}{2s}, s + 1]$. Since these intervals are disjoint for $s_n = \frac{2n-1}{2n}$, the dimension spectrum $\mathrm{sp}(L_{a,b})$ is infinite. □

CHAPTER 5. WEAK RANDOMNESS IN EUCLIDEAN SPACE

In this chapter, we extend the notion of weak randomness to Euclidean space. This chapter is joint work with Xiang Huang and some portions of it have appeared in [20].

Martin-Löf's original notion of randomness was defined using effective null covers; that is, a descending sequence of uniformly c.e. open sets whose intersection is of measure zero. Null covers are now fundamental in the theory of algorithmic randomness, and almost every significant notion has a null cover characterization.

Null covers have not had the same prominence in resource bounded randomness. One of the main obstacles in generalizing null cover definitions from the computable setting is the difficulty in imposing resource bounds on the concept of enumerability. A much more natural concept for resource bounded computation is decidability. In this chapter, we use concepts from computable analysis to give a new notion of resource bounded randomness, *weak polynomial space randomness*.

5.1 Preliminaries

5.1.1 Resource Bounded Randomness Using Martingales

A *martingale* is a function $d : \{0,1\}^* \to [0,\infty)$ satisfying

$$d(w) = \frac{d(w0) + d(w1)}{2}, \tag{5.1}$$

for every finite string $w \in \{0,1\}^*$. A martingale can be thought of as a strategy for betting on successive bits of an infinite binary sequence. The quantity $d(w)$ is, then, the amount of "money" the martingale has after betting on the first $|w|$ bits of the sequence with prefix w. The martingale condition (eq. (5.1)) ensures that the payoffs are fair. We say that a martingale d *succeeds* on an infinite binary sequence A if

$$\limsup_{n\to\infty} d(A[0 \ldots n-1]) = \infty.$$

The *success set* of a martingale d is the set

$$S^\infty(d) = \{A \in \mathbf{C} \mid d \text{ succeeds on } A\}.$$

Lutz [25] used resource bounded martingales to define an effective notion of Lebesgue measure theory, and showed this could be used to study complexity theory. As noted by Ambos-Spies, et al. [1], Lutz's resource bounded measure implicitly defines a notion of resource bounded randomness.

For any function $t : \mathbb{N} \to \mathbb{N}$[1], we say that a martingale d is *computable in time (resp. space)* $t(n)$ if there is a $t(n)$-time (resp. space) computable function $f : \Sigma^* \times \mathbb{N} \to \mathbb{Q}$ such that

$$|d(w) - f(w,r)| \leq 2^{-r}.$$

[1]We will only consider resource bounds t which are *time constructible*. That is, functions t such that $t(n) \geq n$ and the function $f : \Sigma^* \to \mathbb{N}$, defined by $f(w) = t(|w|)$, is computable.

Definition 5.1. *For a function $t : \mathbb{N} \to \mathbb{N}$, we say that an infinite sequence A is $t(n)$-time (-space) random if no $t(n)$-time (-space) martingale succeeds on A. We denote the set of $t(n)$-time and -space random sequences by $\mathsf{RAND}_{\mathsf{t(n)}}$ and $\mathsf{RAND}_{\mathsf{t(n)-space}}$, respectively.*

While specific time and space bounds provide a fine grain definition of randomness, we are also interested in sequences which are random with respect to classes of functions. The two most prominent are the classes of polynomial time and polynomial space functions.

Definition 5.2. *An infinite sequence A is polynomial time (space) random if no n^k-time (resp. space) martingale succeeds on A, for any $k \in \mathbb{N}$. We denote the set of polynomial time and polynomial space random sequences by $\mathsf{RAND}_{\mathsf{p}}$ and $\mathsf{RAND}_{\mathsf{pspace}}$, respectively.*

It is often the case that defining a single martingale which "bets" on all of the conditions we care about becomes too technical. In this case, we may simplify the analysis by defining a sequence of martingales, each betting on a single condition. As long as the sequence is uniformly computable, there is a single martingale, computable in almost the same time, which succeeds on the union of the component success sets.

Lemma 5.3. *If a set of $t(n)$-time computable martingales $\{d_n\}$ is $t(n)$-time uniformly computable, then there is a $(nt(n))$-time computable martingale d such that*

$$S^\infty(d) = \cup_n S^\infty(d_n).$$

5.1.2 Resource Bounded Randomness in Euclidean Space

Lutz and Lutz recently adapted resource-bounded randomness using martingales to Euclidean space [28]. In this section, we review their definition of polynomial space randomness in $\mathbb{R}^n$.

A *dyadic rational number* d is a rational number that has a finite binary expansion; that is $d = \frac{m}{2^r}$ for some integers m, r with $r \geq 0$. We denote the set of all dyadic rational numbers by $\mathbf{D}$. We denote the set of all dyadic rationals d of precision r by $\mathbf{D}_r$. Formally,

$$\mathbf{D}_r = \{\frac{m}{2^r} \,|\, m \in \mathbb{Z}\}.$$

We denote the set of dyadic rationals in the interval $[0,1]$ by $\mathbf{D}[0,1]$. We denote the set of dyadic rationals of precision r in the interval $[0,1]$ by $\mathbf{D}_r[0,1]$. An *open dyadic cube of precision* r is a subset $Q \subseteq \mathbb{R}^n$ such that

$$Q = (\frac{a_1}{2^r}, \frac{a_1+1}{2^r}) \times \ldots \times (\frac{a_n}{2^r}, \frac{a_n+1}{2^r}),$$

where $a_i \in \mathbb{Z}$, and $r \in \mathbb{N}$. We say that the points $\{\frac{a_1}{2^r}, \frac{a_1+1}{2^r}, \ldots \frac{a_n}{2^r}, \frac{a_n+1}{2^r}\}$ are the *endpoints of* Q. In the same manner, we define closed dyadic cubes, and half-open dyadic cubes. Define the family

$$\mathcal{Q}_r = \{Q_r(\mathbf{u}) \mid \mathbf{u} \in \{0, \ldots, 2^r - 1\}^n\}.$$

So then $\mathcal{Q}_r$ is a partition of the unit cube $[0,1)^n$. The family

$$\mathcal{Q} = \bigcup_{r=0}^{\infty} \mathcal{Q}_r,$$

is the set of all half-open dyadic cubes in $[0,1)^n$.

A *(dyadic) martingale* on $[0,1)^n$ is a function $d : \mathcal{Q} \to [0,\infty)$ satisfying

$$d(Q_r(\mathbf{u})) = 2^{-n} \sum_{\mathbf{a} \in \{0,1\}^n} d(Q_{r+1}(2\mathbf{u} + \mathbf{a})), \tag{5.2}$$

for all $Q_r(\mathbf{u}) \in \mathcal{Q}$. We may think of a martingale d as a strategy for placing successive bets on which cube contains x. After r bets have been placed, the bettor's capital is

$$d^{(r)}(\mathbf{x}) = d(Q_r(\mathbf{u})),$$

where $\mathbf{u}$ is the unique element of $\{0, \ldots, 2^r - 1\}^n$ such that $\mathbf{x} \in Q_r(\mathbf{u})$. A martingale d *succeeds* at a point $\mathbf{x} \in [0,1)^n$ if

$$\limsup_{r \to \infty} d^{(r)}(\mathbf{x}) = \infty.$$

Let

$$J = \{(r, \mathbf{u}) \in \mathbb{N} \times \mathbf{Z}^n \mid \mathbf{u} \in \{0, \ldots, 2^r - 1\}^n\}.$$

We say that a martingale $d : \mathcal{Q} \to [0, \infty)$ is *computable* if there is a computable function $\hat{d} : \mathbb{N} \times J \to \mathbb{Q} \cap [0, \infty)$ such that for all $(s, r, \mathbf{u}) \in \mathbb{N} \times J$,

$$|\hat{d}(s, r, \mathbf{u}) - d(Q_r(\mathbf{u}))| \leq 2^{-s}. \tag{5.3}$$

A martingale $d : \mathcal{Q} \to [0, \infty)$ is *p-computable* (resp. *pspace-computable*) if there is a function $\hat{d} : \mathbb{N} \times J \to \mathbb{Q} \cap [0, \infty)$ that satisfies (5.3) and is computable in $(s+r)^{O(1)}$ time (resp. space). A point $x \in \mathbb{R}^n$ is *p-random* (resp. *pspace-random*) if no p-computable (resp. pspace-computable) martingale succeeds at x.

5.1.3 Resource Bounded Computation in Euclidean Space

In this section, we review Ko's framework for complexity theory in $\mathbb{R}^n$ [21]. For the remainder of the chapter, we include the write tape when considering polynomial space bounds of Turing machines.

An infinite sequence $\{S_m\}_{m \in \mathbb{N}}$ of finite unions of open boxes is *polynomial space computable* if there exists a polynomial space TM M such that for all $m > 0$, and all $d \in \mathbf{D}^n$,

$$M(0^m, d) = \begin{cases} 1 & \text{if } d \in S_m \\ -1 & \text{if } d \text{ is a boundary point of } S_m \\ 0 & \text{otherwise} \end{cases}$$

A set $S \subseteq [0,1]^n$ is *polynomial space approximable* if S is measurable and there exists a polynomial space computable sequence of sets $\{S_m\}_{m\in\mathbb{N}}$ such that, for every $m > 0$,

1. there is a polynomial p such that all endpoints of S_m are in $\mathbf{D}^n_{p(m)}$ and
2. $\mu(S \Delta S_m) \leq 2^{-m}$.

Note that we may assume that the polynomial p is increasing; that is $p(i) \leq p(i+1)$, for all $i \in \mathbb{N}$.

5.2 Uniformly Approximable Sequences

We now generalize Ko's definition of approximable sets to approximable *arrays* of sets. We follow Ko in first defining computability, then leveraging this to define approximability.

An infinite array $\{S^k_m\}_{k,m\in\mathbb{N}}$ of finite unions of open boxes is *uniformly polynomial space computable* if there exists a polynomial space TM M such that for all $k, m > 0$, and all $d \in \mathbf{D}^n$,

$$M(0^m, 0^k, d) = \begin{cases} 1 & \text{if } d \in S^k_m \\ -1 & \text{if } d \text{ is an boundary point of } S^k_m \\ 0 & \text{otherwise} \end{cases}$$

If $\{S^k_m\}$ is uniformly polynomial space computable and M is a TM satisfying the definition, we say M *computes* $\{S^k_m\}$.

A sequence of sets $\{U_m\}_{m\in\mathbb{N}}$ is *uniformly polynomial space approximable* if there exists a uniformly polynomial space computable array of sets $\{S^k_m\}$ and a polynomial p such that

1. all endpoints of S^k_m are in $\mathbf{D}^n_{p(m+k)}$ and

2. $\mu(U_m \Delta S_m^k) \leq 2^{-k}$.

If a polynomial p and a uniformly polynomial space computable sequence $\{S_m^k\}$ satisfies (1) and (2), we say that $\{S_m^k\}_{k,m\in\mathbb{N}}$ *approximates* $\{U_m\}$ *at precision* p. Note that we may assume that the polynomial p is increasing.

We now show that we can construct *uniformly* pspace computable sequences from pspace computable sequences. This lemma will be useful, as polynomial space computability is an easier property to verify than its uniform counterpart.

Lemma 5.4. *Let $\{T_i\}_{i\in\mathbb{N}}$ be a pspace computable sequence, and q_1, q_2 be polynomials. For every k, $m > 0$, define the set S_m^k by*

$$S_m^k = \bigcup_{i=q_1(m)}^{q_2(k)} T_i.$$

Then the array $\{S_m^k\}$ is uniformly polynomial space computable.

Proof. It is clear that S_m^k is a finite union of open boxes for each k and $m > 0$. Let M' be the polynomial space TM computing $\{T_i\}$. For every k, $m > 0$, and $d \in \mathbf{D}^n$, define the TM M by

$$M(0^m, 0^k, d) = \begin{cases} 1 & \text{if } M'(0^i, d) = 1 \text{ for any } q_1(m) \leq i \leq q_2(k) \\ -1 & \text{else, if } M'(0^i, 0^{2k+2}, d) = -1 \text{ for any } q_1(m) \leq i \leq q_2(k) \cdot \\ 0 & \text{otherwise} \end{cases}$$

Clearly, M is computable in polynomial space. Hence, $\{S_m^k\}_{k,m\in\mathbb{N}}$ is uniformly polynomial space computable. □

Similarly, we are able to construct uniformly pspace approximable sequences from other uniformly approximable sequences.

Lemma 5.5. *Let q be a polynomial, $j \in \mathbb{N}$, and $\{V_i\}_{i\in\mathbb{N}}$ be a uniformly pspace approximable sequence, such that $\mu(V_i) \leq 2^{-i+j}$. Define the sequence $\{U_m\}_{m\in\mathbb{N}}$ by*

$$U_m = \bigcup_{i=q(m)}^{\infty} V_i.$$

Then $\{U_m\}_{m\in\mathbb{N}}$ is a uniformly pspace approximable sequence.

Proof. Let $\{V_i\}$ be a uniformly approximable sequence, approximated by the uniformly pspace computable array $\{T_i^s\}$ at precision p. For each k, $m > 0$, define the set

$$S_m^k = \bigcup_{i=q(m)}^{k+j+1} T_i^{2k+2}.$$

It is clear that $\{S_m^k\}_{k,m\in\mathbb{N}}$ is a array of finite unions of open boxes. Let M' be the polynomial space TM computing $\{T_i^s\}$. For every k, $m > 0$ and $d \in \mathbf{D}^n$, define the TM M by

$$M(0^m, 0^k, d) = \begin{cases} 1 & \text{if } M'(0^i, 0^{2k+2}, d) = 1 \text{ for any } q(m) \leq i \leq k+j+1 \\ -1 & \text{else, if } M'(0^i, 0^{2k+2}, d) = -1 \text{ for any } q(m) \leq i \leq k+j+1 \cdot \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that M is a polynomial space TM. Hence, $\{S_m^k\}_{k,m\in\mathbb{N}}$ is a uniformly pspace computable sequence. Recall that we are able to assume that the polynomial p is increasing. Therefore, all endpoints of S_m^k are in $\mathbf{D}^n_{p(3k+3)}$. Finally, we have

$$\begin{aligned}
\mu(U_m \Delta S_m^k) &= \mu\Big(\bigcup_{i=q(m)}^{\infty} V_i \,\Delta \bigcup_{i=q(m)}^{k+j+1} T_i^{2k+2}\Big) \\
&\leq \mu\Big(\bigcup_{i=q(m)}^{k+j+1} V_i \,\Delta \bigcup_{i=q(m)}^{k+j+1} T_i^{2k+2}\Big) + \mu\Big(\bigcup_{i=k+j+2}^{\infty} V_i\Big) \\
&\leq \sum_{i=q(m)}^{k+j+1} \mu(V_i \Delta T_i^{2k+2}) + \sum_{i=k+j+2}^{\infty} \mu(V_i) \\
&\leq \sum_{i=q(m)}^{k+1} 2^{-2k-2} + \sum_{i=k+j+2}^{\infty} 2^{-i+j} \\
&\leq 2^{-k}.
\end{aligned}$$

So then $\{S_m^k\}_{k,m\in\mathbb{N}}$ approximates $\{U_m\}_{m\in\mathbb{N}}$ at precision p, and therefore $\{U_m\}_{m\in\mathbb{N}}$ is a uniformly polynomial space approximable sequence. □

5.3 Weak Polynomial Space Randomness in Euclidean Space

Using uniformly polynomial space approximable sequences, we give an open-cover definition of polynomial space randomness.

Let $a, b \in \mathbb{Z}$. An infinite sequence of open sets $\{U_m\}_{m\in\mathbb{N}} \subseteq [a,b]^n$ is a *polynomial space $\mathcal{W}$-test (pspace $\mathcal{W}$-test)* if the following hold.

1. For every m, $\mu(U_m) \leq 2^{-m}$.

2. There is a uniformly pspace computable array $\{S_m^k\}$ approximating $\{U_m\}$ such that, for all m,

$$U_m \subseteq \liminf_{k\to\infty} S_m^k,$$

A point x *passes* a polynomial space $\mathcal{W}$-test $\{U_m\}_{m\in\mathbb{N}}$ if $x \notin \bigcap_{m=1}^{\infty} U_m$. We say that x is *weakly pspace random* if x passes every polynomial space $\mathcal{W}$-test.

The approximability of pspace $\mathcal{W}$-tests allows us to estimate the measure of the open covers in polynomial space.

Lemma 5.6. *If $\{U_m\}_{m\in\mathbb{N}}$ is a pspace $\mathcal{W}$-test, then there exists a polynomial space TM M such that for every s, r, $m \in \mathbb{N}$ and $\mathbf{u} \in \{0,\dots,2^r-1\}^n$*

$$|M(0^s, 0^r, \mathbf{u}, 0^m) - \mu(U_m \cap Q_r(\mathbf{u}))| \leq 2^{-s}.$$

Proof. Let p be a polynomial, and $\{U_m\}_{m\in\mathbb{N}}$ be a pspace $\mathcal{W}$-test, approximated by the uniformly pspace computable array $\{S_m^k\}$ at precision p. Let M' be the polynomial space TM computing $\{S_m^k\}_{k,m\in\mathbb{N}}$. For every s, r, $m \in \mathbb{N}$ and $\mathbf{u} \in \{0,\dots,2^r-1\}^n$, define the TM M by,

$$M(0^s, 0^r, \mathbf{u}, 0^m) = \mu(S_m^s \cap Q_r(\mathbf{u})).$$

Then,

$$\begin{aligned}
|M(0^s, 0^r, \mathbf{u}, 0^m) - \mu(U_m \cap Q_r(\mathbf{u}))| &= |\mu(S_m^s \cap Q_r(\mathbf{u})) - \mu(U_m \cap Q_r(\mathbf{u}))| \\
&\leq \mu((S_m^s \Delta U_m) \cap Q_r(\mathbf{u})) \\
&\leq 2^{-s}.
\end{aligned}$$

It remains to be shown that M is a polynomial space machine. To compute $\mu(S_m^s \cap Q_r(\mathbf{u}))$, M enumerates over all dyadic cubes Q of precision $p(s+m)$. For each Q, M computes the center of Q, the dyadic rational d_Q of precision $p(s+m)+1$. If $M'(0^m, 0^s, d_Q) = 1$, then M adds $\mu(Q \cap Q_r(\mathbf{u}))$ to the current measure. After enumerating over all $Q \in \mathbf{B}_{p(s+m)}$, M outputs the total measure. Hence, M is a polynomial space machine, and the proof is complete. □

We are now able to relate weakly polynomial space randomness with Lutz's pspace randomness. The following lemma shows that pspace randomness implies weakly pspace randomness.

Theorem 5.7. *Let $\{U_m\}_{m\in\mathbb{N}}$ be a polynomial space $\mathcal{W}$-test. Then there exists a pspace martingale d succeeding on all points $x \in \bigcap_{m=1}^{\infty} U_m \bigcap [0,1]^n$.*

Proof. Let $\{U_m\}_{m\in\mathbb{N}}$ be a polynomial space $\mathcal{W}$-test. For each $m > 0$, define the function $d_m : \mathcal{Q} \to [0,\infty)$ by

$$d_m(Q_r(\mathbf{u})) = \frac{1}{\mu(Q_r(\mathbf{u}))}\,\mu(U_m \cap Q_r(\mathbf{u})).$$

We then have

$$\begin{aligned}
2^{-n} \sum_{\mathbf{a}\in\{0,1\}^n} d_m(Q_{r+1}(2\mathbf{u}+\mathbf{a})) &= 2^{-n} \sum_{\mathbf{a}\in\{0,1\}^n} \frac{1}{\mu(Q_{r+1}(2\mathbf{u}+\mathbf{a}))}\,\mu(U_m \cap Q_{r+1}(2\mathbf{u}+\mathbf{a})) \\
&= 2^{rn} \sum_{\mathbf{a}\in\{0,1\}^n} \mu(U_m \cap Q_{r+1}(2\mathbf{u}+\mathbf{a})) \\
&= 2^{rn}\mu(U_m \bigcap (\bigcup_{\mathbf{a}\in\{0,1\}^n} Q_{r+1}(2\mathbf{u}+\mathbf{a}))) \\
&= 2^{rn}\mu(U_m \cap Q_r(\mathbf{u})) \\
&= \frac{1}{\mu(Q_r(\mathbf{u}))}\,\mu(U_m \cap Q_r(\mathbf{u})) \\
&= d_m(Q_r(\mathbf{u})),
\end{aligned}$$

and so d_m is a martingale. Define the function $d : \mathcal{Q} \to [0,\infty)$ by

$$d(Q_r(\mathbf{u})) = \sum_{m=1}^{\infty} d_m(Q_r(\mathbf{u})).$$

Then,

$$\begin{aligned} d(Q_0(\mathbf{0})) &= \sum_{m=1}^{\infty} d_m(Q_0(\mathbf{0})) \\ &\leq \sum_{m=1}^{\infty} 2^{-m} \\ &\leq 1, \end{aligned}$$

and since each d_m is a martingale, d is a martingale. We now show that d is a pspace martingale by constructing a polynomial space TM M computing $\hat{d}$. By Lemma 5.6, there exists a polynomial space TM M' such that

$$|M'(0^s, 0^r, \mathbf{u}, 0^m) - \mu(U_m \cap Q_r(\mathbf{u}))| \leq 2^{-s}.$$

For every $s \in \mathbb{N}$ and $(r, \mathbf{u}) \in J$, define the TM M by

$$\begin{aligned} M(0^s, 0^r, \mathbf{u}) &= \sum_{m=1}^{s+nr+1} \frac{1}{\mu(Q_r(\mathbf{u}))} M'(0^{s+nr+2}, 0^r, \mathbf{u}, 0^m) \\ &= \sum_{m=1}^{s+nr+1} 2^{nr} M'(0^{s+nr+2}, 0^r, \mathbf{u}, 0^m) \end{aligned}$$

Clearly, M runs in polynomial space. Moreover,

$$\begin{aligned} |M(0^s, 0^r, \mathbf{u}) - d(Q_r(\mathbf{u}))| &= |M(0^s, 0^r, \mathbf{u}) - \sum_{m=1}^{\infty} d_m(Q_r(\mathbf{u}))| \\ &\leq |M(0^s, 0^r, \mathbf{u}) - \sum_{m=1}^{s+nr+1} d_m(Q_r(\mathbf{u}))| + \sum_{m=s+nr+2}^{\infty} d_m(Q_r(\mathbf{u})). \end{aligned}$$

By the definition of M,

$$\begin{aligned} |M(0^s, 0^r, \mathbf{u}) - \sum_{m=1}^{s+nr+1} d_m(Q_r(\mathbf{u}))| &= 2^{nr} |\sum_{m=1}^{s+nr+1} M'(0^{s+nr+2}, 0^r, \mathbf{u}, 0^m) - \mu(U_m \cap Q_r(\mathbf{u}))| \\ &\leq 2^{nr} \sum_{m=1}^{s+nr+1} 2^{-s-nr-2} \\ &\leq \sum_{m=1}^{s+nr+1} 2^{-s-2} \\ &\leq 2^{-s-1}. \end{aligned}$$

Combining the two inequalities, we have

$$\begin{aligned}
|M(0^s, 0^r, \mathbf{u}) - d(Q_r(\mathbf{u}))| &\leq 2^{-s-1} + \sum_{m=s+nr+2}^{\infty} d_m(Q_r(\mathbf{u})) \\
&\leq 2^{-s-1} + \sum_{m=s+nr+2}^{\infty} 2^{nr}\, 2^{-m} \\
&\leq 2^{-s-1} + 2^{nr} 2^{-s-nr-1} \\
&\leq 2^{-s}.
\end{aligned}$$

Therefore, d is a pspace martingale.

Assume $x \in \bigcap_{m=1}^{\infty} U_m \bigcap [0,1]^n$. Let $i > 0$. Then, since U_i is an open set, there exists an N such that for all $r \geq N$, $Q_r(\mathbf{u}) \subseteq U_i$, where $Q_r(\mathbf{u})$ is the unique dyadic cube containing x. Hence, for all $r \geq N$, $d_i(Q_r(\mathbf{u})) = 1$. Therefore,

$$\lim_{r \to \infty} d^{(r)}(x) = \infty,$$

and so d succeeds on x. □

5.4 Weak Resource Bounded Randomness of Sequences

In this section we give the equivalent definition of weak randomness for infinite sequences, rather than Euclidean points. In so doing, we generalize weak randomness to arbitrary time and space bounded computation.

Definition 5.8. *A sequence of open sets $\{U_n\}$ is polynomial time (space) approximable if there is an array $\{S_n^k\}_{k,n \in \mathbb{N}}$, $S_n^k \subseteq \Sigma^*$, such that the following hold.*

1. *There is a polynomial time (space) computable function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that, for every k and n,*

$$\max\{|w| \,:\, w \in S_n^k\} \leq f(k, n).$$

2. *The language* $L = \{\langle w, 0^k, 0^n\rangle \mid w \in S_n^k\}$ *is decidable in polynomial time (space).*

3. *For every* n, $U_n = \bigcup_{k\geq 1} [S_n^k]$.

4. *For every* n *and* k, $\mu(U_n - \bigcup_{k\geq 1} [S_n^k]) \leq 2^{-k}$.

Definition 5.9. *A weak polynomial time (space) test is a polynomial time (space) approximable sequence of descending open sets* $\{U_n\}$ *such that* $\mu(U_n) \leq 2^{-n}$. *An infinite sequence* A *passes a weak polynomial time (space) test if* $A \notin \cap_n U_n$. *An infinite sequence* A *is weakly polynomial time (space) random if* A *passes every weak polynomial time (space) test.*

In the polynomial space setting, we are able to compute, using property (1) of Definition 5.8, the measures of U_n uniformly. The proof is nearly identical to that in the previous section.

Proposition 5.10. *Let* $\{U_n\}$ *be a weak polynomial space test. Then the function* $f : \Sigma^* \times \mathbb{N} \to [0, 1]$ *defined by* $f(w, n) = \mu(C_w \cap U_n)$ *is polynomial space computable.*

This proposition allows us to show that weak polynomial space randomness is no stronger than polynomial space randomness.

Lemma 5.11. $\mathsf{RAND}_{\mathsf{pspace}}$ *is a subset of* $\mathsf{RAND}_{\mathsf{W-pspace}}$.

We now show that weak polynomial space randomness is, in fact, strictly weaker than polynomial space randomness. We also show that there is a sequence which is weakly polynomial time random, yet *not* polynomial time random.

Lemma 5.12. *There is a sequence* A *such that* A *is weakly polynomial time random and there is a* $O(n)$*-time martingale* d *succeeding on* A.

Proof. Let B be Martin-Löf random, and define the sequence A by

$$A[n] = \begin{cases} B[n] & \text{if } n = 2^{2^m} \text{ for some } m \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

It is easy to construct an $O(n)$-time martingale d succeeding on A. We now show that if A is not weakly polynomial time random, then B is not ML random, contradicting our assumption.

Assume that A fails a weak polynomial time test $\{U_n\}$. Let $\{S_n^k\}$ be the computable array approximating $\{U_n\}$ and let n^r be its associated polynomial. For every string w, define the set

$$N(w) = \{x \in \Sigma^* \mid x[n] = w[n] \text{ for every } n \neq 2^{2^m}\}.$$

Note that, for every $w \in \Sigma^*$, $|N(w)| \leq \log |w|$. For every $n \in \mathbb{N}$ inductively define the sets T_n^k by

$$T_n^1 = \{w \mid (\exists j \leq 2n)\ w \in S_n^j\}$$
$$T_n^{k+1} = \{w \mid (\exists j \leq 2(k+1)n)\ w \in S_n^j\} - \cup_{i \leq k} T_n^k.$$

Note that, for every $k > 1$,

$$\sum_{w \in T_n^k} 2^{-|w|} \leq 2^{-2(k-1)n}.$$

For each n, let

$$U_n = \{x \in \Sigma^* \mid (\exists k)\ x \in N(w) \text{ and } \in T_n^k\}.$$

It is clear that U_n is c.e. and that $B \in \cap_n U_n$. Furthermore, for every n,

$$\begin{aligned}
\mu(U_n) &= \sum_k \sum_{w \in T_n^k} |N(w)|\, 2^{-|w|} \\
&= \sum_k \sum_{w \in T_n^k} \log |w|\, 2^{-|w|} \\
&\leq \sum_k \log(2kn)^r \sum_{w \in T_n^k} 2^{-|w|} \\
&\leq \sum_k \log(2kn)^r 2^{-2(k-1)n}.
\end{aligned}$$

By the ratio test this sum converges. Therefore $\{U_n\}$ is a Martin-Löf test covering B, contradicting our hypothesis. Hence A is weakly polynomial time random. $\square$

Combining Lemmas 5.11 and 5.12 shows that weak polynomial space randomness is, in fact, weaker than polynomial space randomness.

Corollary 5.13. *The set* $\mathsf{RAND}_{\mathsf{W-pspace}}$ *is a (strict) subset of* $\mathsf{RAND}_{\mathsf{pspace}}$.

Unfortunately, we do not know if Proposition 5.10 holds in the polynomial time setting, so the method of Lemma 5.11 cannot be used. Lemma 5.12 shows that either weak polynomial time - and polynomial time -randomness are independent, or weak polynomial time randomness is strictly weaker. We conjecture that the latter holds.

Conjecture 5.14. *The set* $\mathsf{RAND}_{\mathsf{W-p}}$ *is a (strict) subset of* $\mathsf{RAND}_{\mathsf{p}}$.

CHAPTER 6. RESOURCE BOUNDED RANDOMNESS AND THE LEBESGUE DIFFERENTIATION THEOREM

In this chapter, we use the Lebesgue differentiation theorem, a fundamental theorem of analysis, to characterize the notion of weak randomness given in the previous chapter. This chapter is joint work with Xiang Huang and some portions of it have appeared in [20].

Recently, research in algorithmic randomness has used computable analysis to study the connection between randomness and classical analysis [3, 15, 16, 41, 42, 47]. With the rise of measure theory, many fundamental theorems of analysis have been "almost everywhere" results. Theorems of this type state that a certain property holds for almost every point; i.e., the set of points that does not satisfy the property is of measure zero. However, almost everywhere theorems typically give no information about which points satisfy the stated property. By adding computability restrictions, tools from algorithmic randomness are able to strengthen a theorem from a property simply holding almost everywhere, to one that holds for all random points. For example, an important classical result of analysis is Lebesgue's theorem on nondecreasing functions. Lebesgue showed that every nondecreasing continuous function $f : [0, 1] \to \mathbb{R}$ is differentiable almost everywhere. Brattka, Miller and Nies characterized computable randomness using Lebesgue's theorem by proving the following result [5].

Theorem 6.1. *Let $z \in [0,1]$. Then z is computably random if and only if $f'(z)$ exists for every nondecreasing computable function $f : [0,1] \to \mathbb{R}$.*

This paper concerns a related theorem, also due to Lebesgue [23].

Theorem 6.2. *For each $f \in L_1([0,1]^n)$,*

$$f(x) = \lim_{Q \to x} \frac{\int_Q f d\mu}{\mu(Q)}$$

for almost every $x \in [0,1]^n$. The limit is taken over all open cubes Q containing x as the diameter of Q tends to 0.

Pathak first studied the Lebesgue differentiation theorem in the context of Martin-Löf randomness [45]. Under the assumption that the function is L_1-computable, Pathak showed that the Lebesgue differentiation theorem holds for every Martin-Löf random point. Subsequently, Pathak, Rojas and Simpson improved this theorem [46]. They showed that the Lebesgue differentiation theorem holds at a point z for every L_1 computable function if and only if z is Schnorr random [46]. Independently, and using very different techniques, Rute also showed that the Lebesgue differentiation theorem holds for Schnorr random points [47].

This chapter concerns the connection between *resource-bounded* randomness and analysis. While there has been work on this interaction [6, 28, 44], resource-bounded randomness in analysis is still poorly understood. Recently, Nies extended the result of Brattka, Miller and Nies to the polynomial time domain [44]. Specifically, Nies characterized polynomial time randomness using the differentiability of nondecreasing polynomial time computable functions. We extend this research of the Lebesgue differentiation theorem to the context of resource-bounded randomness. We show that the Lebesgue differentiation theorem characterizes weak polynomial space randomness. That is, we prove that a point x is weakly polynomial space random if and only if the Lebesgue differentiation theorem holds at x for every polynomial space L_1-computable function.

6.1 Preliminaries

Throughout this chapter, μ will always denote the Lebesgue measure on $\mathbb{R}^n$. We denote the set of all Lebesgue integrable functions $f : [0,1]^n \to \mathbb{R}$ by $L_1([0,1]^n)$. A *dyadic rational number* d is a rational number that has a finite binary expansion; that is $d = \frac{m}{2^r}$ for some integers m, r with $r \geq 0$. We denote the set of all dyadic rational numbers by $\mathbf{D}$. We denote the set of all dyadic rationals d of precision r by $\mathbf{D}_r$. Formally,

$$\mathbf{D}_r = \{\frac{m}{2^r} \,|\, m \in \mathbb{Z}\}.$$

We denote the set of dyadic rationals in the interval $[0,1]$ by $\mathbf{D}[0,1]$. We denote the set of dyadic rationals of precision r in the interval $[0,1]$ by $\mathbf{D}_r[0,1]$. An *open dyadic cube of precision* r is a subset $Q \subseteq \mathbb{R}^n$ such that

$$Q = (\frac{a_1}{2^r}, \frac{a_1+1}{2^r}) \times \ldots \times (\frac{a_n}{2^r}, \frac{a_n+1}{2^r}),$$

where $a_i \in \mathbb{Z}$, and $r \in \mathbb{N}$. We say that the points $\{\frac{a_1}{2^r}, \frac{a_1+1}{2^r}, \ldots \frac{a_n}{2^r}, \frac{a_n+1}{2^r}\}$ are the *endpoints of* Q. In the same manner, we define closed dyadic cubes, and half-open dyadic cubes. We denote the set of all open dyadic cubes of precision r by

$$\mathbf{B}_r = \{Q \,|\, Q \text{ is an open dyadic cube of precision } r\}.$$

For an open set $Q \subseteq \mathbb{R}^n$ and $t \in \mathbb{R}^n$, define the translation of Q by t to be the set

$$t + Q = \{t + x \,|\, x \in Q\}.$$

6.1.1 Polynomial Space Computability in Euclidean Space

In this section, we review Ko's framework for complexity theory in $\mathbb{R}^n$ [21]. For the remainder of the chapter, we include the write tape when considering polynomial space bounds of Turing machines.

We first introduce the polynomial space L_1-computable functions, the class of functions we will be using in the proof of the Lebesgue differentiation theorem. This definition is equivalent to Ko's notion of *pspace approximable functions*. It is a direct analog of the L_1-computable functions used in computable analysis.

A function $f : [0, 1]^n \to \mathbb{R}$ is a *simple step function* if f is a step function such that

1. $f(x) \in \mathbf{D}$ for all $x \in [0, 1]^n$ and
2. there exists a finite number of (disjoint) dyadic boxes $Q_1, \ldots, Q_k$ and dyadic rationals $d_1, \ldots, d_k$ such that $f(x) = \sum_{i=1}^{k} d_i \chi_{Q_i}(x)$, where χ_Q is the characteristic function of a set Q.

A function $f \in L_1([0, 1]^n)$ is *polynomial space L_1-computable* if there exists a sequence of simple step functions, $\{f_m\}_{m \in \mathbb{N}}$, and a polynomial p such that for all $d \in \mathbf{D}^n$,

1. $f_m(x) = \sum_{i=1}^{k} d_i \chi_{Q_i}(x)$, such that the endpoints of each Q_i are in $\mathbf{D}^n_{p(m)}$,
2. there is a polynomial space TM M computing f_m in the sense that
$$M(0^m, d) = \begin{cases} f_m(d) & \text{if } d \text{ is not a breakpoint of } f_m \\ \# & \text{otherwise} \end{cases}$$
3. $\|f - f_m\|_1 \leq 2^{-n}$.

Note that we may assume that the polynomial p is increasing. We will frequently use the following nice property of polynomial space L_1-computable functions. If $f \in L_1([0,1]^n)$ is approximated by sequence of simple step function $\{f_m\}$ at precision p, then for every $i > 0$, f_i is a constant function on every $Q \in \mathbf{B}_{p(i)}$.

6.2 Weak Randomness and the Lebesgue Differentiation Theorem

In this section we prove our main theorem, that the Lebesgue differentiation theorem characterizes weakly pspace-randomness. Recall the statement of Lebesgue's theorem.

Theorem 6.3. *For each $f \in L_1([0,1]^n)$,*

$$f(x) = \lim_{Q \to x} \frac{\int_Q f d\mu}{\mu(Q)}$$

for almost every $x \in [0,1]^n$. The limit is taken over all open cubes Q containing x as the diameter of Q tends to 0.

A point x that satisfies the Lebesgue differentiation theorem is called a *Lebesgue point*. We will prove the following theorem,

Theorem 6.4. *A point x is weakly pspace-random if and only if for every polynomial space L_1-computable $f \in L_1([0,1]^n)$, and every polynomial space computable sequence of simple functions $\{f_m\}_{m\in\mathbb{N}}$ approximating f,*

$$\lim_{m\to\infty} f_m(x) = \lim_{Q \to x} \frac{\int_Q f d\mu}{\mu(Q)} \tag{6.1}$$

where the limit is taken over all cubes Q containing x as the diameter of Q tends to 0.

We first make several remarks regarding the form of our main theorem. The use of polynomial space L_1-computability is not simply for the sake of generality. It is well-known that if a function is continuous, the Lebesgue differentiation theorem holds for *every* point. Thus, to get a non-trivial randomness result, we must allow the function to be discontinuous. Our second remark concerns the limit of the approximating functions. In the statement of the classical theorem, the integral limit is equal to $f(x)$; whereas in our main theorem, it is equal to $\lim_{m\to\infty} f_m(x)$. This concession is necessary. For any point x, it is trivial to construct a polynomial space L_1-computable function f such that

$$f(x) \neq \lim_{Q\to x} \frac{\int_Q f d\mu}{\mu(Q)}.$$

Consider the function f which is 0 for all points, except at the given point x, $f(x) = 1$. Clearly, f is polynomial space L_1-computable, but x does not satisfy the Lebesgue differentiation theorem.

6.2.1 Random points satisfy the Lebesgue differentiation theorem

The outline of our proof roughly follows that of the classical proof of the Lebesgue differentiation theorem [46]. However, the restriction to polynomial space computation significantly changes the internal methods. We first show that if a point $x \in [0,1]^n$ is weakly pspace-random, then it must be contained in an open dyadic cube. This is a useful property of weakly pspace-random points that we take advantage of in later theorems.

Lemma 6.5. *Let $x = (x_1, \ldots, x_n) \in [0,1]^n$ be weakly pspace-random. Then, for every i, x_i is not a dyadic rational.*

Proof. Let $x = (x_1, \ldots, x_n) \in [0,1]^n$ be weakly pspace-random. We show that x_1 cannot be a dyadic rational, the proof for the other components is similar. For every

$i > 0$, define the set

$$S_i = \bigcup_{d \in \mathbf{D}_i[0,1]} (d - 2^{-2i-2}, d + 2^{-2i-2}) \times (0,1) \times \ldots \times (0,1).$$

For every $m > 0$, define the set

$$U_m = \bigcup_{i=m}^{\infty} S_i.$$

We now prove that the sequence $\{U_m\}_{m\in\mathbb{N}}$ is a pspace $\mathcal{W}$-test. It is clear that for every $m > 0$, U_m is an open set. Let $m > 0$, then,

$$\begin{aligned}
\mu(U_m) &= \mu(\bigcup_{i=m}^{\infty} S_i) \\
&\leq \sum_{i=m}^{\infty} \mu(S_i) \\
&\leq \sum_{i=m}^{\infty} 2^i 2^{-2i-1} \\
&\leq 2^{-m}.
\end{aligned}$$

It remains to be shown that $\{U_m\}_{m\in\mathbb{N}}$ is uniformly pspace approximable. For every k, $m > 0$, define the set

$$T_m^k = \bigcup_{i=m}^{k-1} S_i.$$

It is easy to verify that $\{S_i\}$ is a polynomial space computable sequence. Hence, by Lemma 5.4, $\{T_m^k\}$ is a uniformly polynomial space computable sequence. Finally, for every k, $m > 0$,

$$\begin{aligned}
\mu(U_m \Delta T_m^k) &= \mu(\bigcup_{i=k}^{\infty} S_i) \\
&\leq \sum_{i=k}^{\infty} \mu(S_i) \\
&\leq 2^{-k},
\end{aligned}$$

and so the sequence $\{U_m\}$ is uniformly polynomial space approximable. It is clear that for every m, and all $x \in U_m$, $x \in \liminf_k T_m^k$. Therefore, $\{U_m\}_{m\in\mathbb{N}}$ is a polynomial space $\mathcal{W}$-test. By assumption $x \notin \cap U_m$, therefore x_1 is not a dyadic rational.

Using a similar argument we see that, for all $1 \leq i \leq n$, x_i is not a dyadic rational. □

Let f be a polynomial space L_1-computable function, approximated by the pspace computable sequence of simple step functions $\{f_m\}_{m\in\mathbb{N}}$. We now show that for every weakly pspace-random point x, the limit $\lim\limits_{m\to\infty} f_m(x)$ exists. We will need the following inequality due to Chebyshev. For every $f \in L_1([0,1]^n)$ and $\epsilon > 0$, define the set

$$S(f,\epsilon) = \{x \mid |f(x)| > \epsilon\}.$$

Lemma 6.6. *Let $f \in L_1([0,1]^n)$ and $\epsilon > 0$. Then $\mu(S(f,\epsilon)) \leq \frac{\|f\|_1}{\epsilon}$.*

Lemma 6.7. *Let $f \in L_1([0,1]^n)$ be polynomial space L_1 computable, approximated by the polynomial space computable sequence of simple step functions $\{f_m\}_{m\in\mathbb{N}}$. If x is weakly pspace-random, the limit $\lim\limits_{m\to\infty} f_m(x)$ exists.*

Proof. Let p be a polynomial and $f \in L_1([0,1]^n)$ be polynomial space L_1 computable, approximated by the polynomial space computable sequence of simple step functions $\{f_m\}_{m\in\mathbb{N}}$ at precision p. Recall that we may assume that p is increasing. For each $i \geq 1$, define the set

$$S_i = (S(f_{2i-1} - f_{2i}, 2^{-i}) \cup S(f_{2i} - f_{2i+1}, 2^{-i})) \cap (\bigcup_{Q\in\mathbf{B}_{p(2i+1)}} Q).$$

We intersect with the open dyadic cubes of precision $p(2i+1)$ to ensure that S_i is an open set. For each $m \geq 1$ define the set

$$U_m = \bigcup_{i=m+4}^{\infty} S_i.$$

We now prove that the sequence $\{U_m\}_{m\in\mathbb{N}}$ is a pspace $\mathcal{W}$-test. Using the properties of simple step functions, it is routine to verify that, for every $i > 0$, S_i is the union of all open dyadic cubes $Q \in \mathbf{B}_{p(2i+1)}$, such that either

$$\begin{aligned} &|f_{2i-1}(Q) - f_{2i}(Q)| > 2^{-i}, \text{or} \\ &|f_{2i}(Q) - f_{2i+1}(Q)| > 2^{-i}. \end{aligned}$$

Therefore, for every $m > 0$, U_m is an open set. By Chebyshev's inequality,

$$\begin{aligned} \mu(S_i) &\leq 2^i(\|f_{2i-1} - f_{2i}\| + \|f_{2i} - f_{2i+1}\|) \\ &\leq 2^i(2^{-2i+2} + 2^{-2i+1}) \\ &\leq 2^{-i+3}. \end{aligned}$$

Using this upper bound on the measure of S_i we obtain

$$\begin{aligned} \mu(U_m) &\leq \sum_{i=m+4}^{\infty} \mu(S_i) \\ &\leq \sum_{i=m+4}^{\infty} 2^{-i+3} \\ &\leq 2^{-m}. \end{aligned}$$

It remains to be shown that the sequence $\{U_m\}_{m\in\mathbb{N}}$ is uniformly polynomial space approximable. For every k, $m > 0$, define the set

$$T_m^k = \bigcup_{i=m+4}^{k+3} S_i.$$

It is clear that $\{S_i\}$ is a polynomial space computable sequence. Hence, by Lemma 5.4, $\{T_m^k\}$ is a uniformly pspace computable array. Finally, we have

$$\begin{aligned}\mu(U_m \Delta T_m^k) &= \mu(U_m \Delta(\bigcup_{i=m+4}^{k+3} S_i)) \\ &\leq \mu((\bigcup_{i=k+4}^{\infty} S_i)) \\ &\leq \sum_{i=k+4}^{\infty} \mu(S_i) \\ &\leq \sum_{i=k+4}^{\infty} 2^{-i+3} \\ &\leq 2^{-k}.\end{aligned}$$

Finally, it is clear that, for every $m \in \mathbb{N}$ and all $x \in U_m$, $x \in \liminf_k T_m^k$. Hence, $\{U_m\}_{m\in\mathbb{N}}$ is a pspace $\mathcal{W}$-test.

Assume x is weakly pspace-random. Then there exists an N such that for all $m > N$, $x \notin U_m$, and therefore $x \notin S_i$, for all $i > N + 4$. By Lemma 6.5, x cannot have any dyadic rational components; i.e., $x \in Q$, for some $Q \in \mathbf{B}_{2i+1}$. Hence, $|f_{2i-1}(x) - f_{2i}(x)| \leq 2^{-i}$ and $|f_{2i}(x) - f_{2i+1}(x)| \leq 2^{-i}$. Let $j > 2N + 8$, then $|f_j(x) - f_{j+1}(x)| \leq 2^{-\frac{j}{2}}$. Therefore, the limit $\lim\limits_{m\to\infty} f_m(x)$ exists. □

We now focus on the limit

$$\lim_{Q\to x} \frac{\int_Q f d\mu}{\mu(Q)}$$

on the right hand side of our main theorem (equation (6.1)). The restriction to polynomial space computation creates difficulties in considering arbitrary open cubes. Intuitively, we overcome this obstacle through the use of translations of dyadic cubes, which are more amenable to polynomial space computation. Formally, for $t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n$, define the set

$$\mathbf{B}_r^t = \{I_r^t \,|\, I_r^t = t + Q, \text{ where } Q \in \mathbf{B}_r\}.$$

That is, $\mathbf{B}_r^t$ is the set of all translations of dyadic cubes of precision r by points $t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n$. For every $x \in [0,1]^n$, let $I_r^t(x)$ denote the (unique) element of $\mathbf{B}_r^t$ containing x. The following theorem of Rute [47], using results due to Morayne and Solecki [43], shows that it suffices to prove that the right hand limit of equation (6.1) exists for these translations.

Theorem 6.8 ([47]). *Let $f \in L_1([0,1]^n)$, and $x \in [0,1]^n$. Then the following are equivalent,*

1. *the limit $\lim\limits_{Q \to x} \frac{\int_Q f d\mu}{\mu(Q)}$ exists, where the limit is taken over all cubes containing x, as the diameter goes to* 0
2. *the limit $\lim\limits_{k \to \infty} \frac{\int_{I_k^t(x)} f d\mu}{\mu(I_k^t(x))}$ exists, for all $t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n$.*

We now show that the limit

$$\lim_{m \to \infty} \frac{\int_{I_r^t(x)} |f - f_m| d\mu}{\mu(I_r^t(x))}$$

exists, for every $t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n$ and $r > 0$. We will need the following inequality due to Hardy and Littlewood. For every $f \in L_1([0,1]^n)$ and $\epsilon > 0$, define the set

$$T(f, \epsilon) = \{x \mid \sup_{r,t} \frac{\int_{I_r^t(x)} f d\mu}{\mu(I_r^t)} > \epsilon\},$$

where the supremum is taken over all $r > 0$ and $t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n$.

Theorem 6.9. *There is a constant c such that, for every $f \in L_1([0,1]^n)$ and $\epsilon > 0$, $\mu(T(f, \epsilon)) \leq \frac{c\|f\|_1}{\epsilon}$.*

Lemma 6.10. *Let $f \in L_1([0,1]^n)$ be polynomial space L_1 computable, approximated by the polynomial space computable sequence of step functions $\{f_m\}_{m \in \mathbb{N}}$. If x is weakly pspace-random, then*

$$\lim_{m \to \infty} \frac{\int_{I_r^t(x)} |f - f_m| d\mu}{\mu(I_r^t(x))} = 0,$$

for every $t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n$ and $r > 0$.

Proof. Let p be a polynomial, and $f \in L_1([0,1]^n)$ be polynomial space L_1 computable, approximated by the polynomial space computable sequence of simple step functions $\{f_m\}_{m\in\mathbb{N}}$ at precision p. For every $i > 0$, define the set

$$T_i = T(f_{2i-1} - f_{2i}, 2^{-i}) \cup T(f_{2i} - f_{2i+1}, 2^{-i}).$$

For every $m \geq 1$ define the set

$$U_m = \bigcup_{i=m+4+c}^{\infty} T_i.$$

We now prove that the sequence $\{U_m\}_{m\in\mathbb{N}}$ is a pspace $\mathcal{W}$-test. Clearly, for every $m > 0$, U_m is an open set. By the Hardy/Littlewood inequality,

$$\begin{aligned}
\mu(T_i) &\leq 2^i\, c\,(\|f_{2i-1} - f_{2i}\| + \|f_{2i} - f_{2i+1}\|)\\
&\leq 2^i\, c\,(2^{-2i+2} + 2^{-2i+1})\\
&\leq c\,2^{-i+3}.
\end{aligned}$$

Using this upper bound on the measure of T_i we obtain

$$\begin{aligned}
\mu(U_m) &\leq \sum_{i=m+4+c}^{\infty} \mu(T_i)\\
&\leq \sum_{i=m+4+c}^{\infty} c\,2^{-i+3}\\
&< 2^{-m}.
\end{aligned}$$

It remains to be shown that the sequence $\{U_m\}_{m\in\mathbb{N}}$ is uniformly polynomial space approximable. By Lemma 5.5, it suffices to prove that the sequence (T_i) is uniformly polynomial space approximable. For every k, i, define the sets

$$V_i^k = \{I_r^t \,|\, r \leq p(2i+1) + k + 2,\; t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n, \text{ and } \frac{\int_{I_r^t(x)} |f_{2i-1} - f_{2i}| d\mu}{\mu(I_r^t(x))} > 2^{-i}\},$$

$$W_i^k = \{I_r^t \,|\, r \leq p(2i+1) + k + 2,\; t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n, \text{ and } \frac{\int_{I_r^t(x)} |f_{2i} - f_{2i+1}| d\mu}{\mu(I_r^t(x))} > 2^{-i}\},$$

and

$$A_i^k = W_i^k \bigcup V_i^k.$$

We now show that $\mu(T_i \Delta A_i^k) \leq 2^{-k}$. Intuitively, we bound the measure using the property that simple step functions are constant on dyadic cubes. Let $I_r^t \subseteq Q$, for some $Q \in \mathbf{B}_{p(2i+1)}$; i.e., I_r^t is fully contained in an open dyadic cube of precision $p(2i+1)$. Assume

$$\frac{\int_{I_r^t} |f_{2i-1} - f_{2i}| d\mu}{\mu(I_r^t)} > 2^{-i}.$$

Since $|f_{2i-1} - f_{2i}|$ is a simple step function whose break points are in $\mathbf{D}^n_{p(2i+1)}$, $|f_{2i-1} - f_{2i}|$ must be a constant function on Q. Thus, $|f_{2i-1}(Q) - f_{2i}(Q)| > 2^{-i}$, and so $I_r^t \subseteq Q \subseteq A_i^1$. Similarly, if

$$\frac{\int_{I_r^t} |f_{2i} - f_{2i+1}| d\mu}{\mu(I_r^t)} > 2^{-i},$$

then $I_r^t \subseteq Q \subseteq A_i^1$. So then, the set of points in $T_i - A_i^k$ must be contained in some translate I_r^t that is not contained in a dyadic cube of precision $p(2i+1)$; that is,

$$T_i - A_i^k \subseteq \bigcup_{r=p(2i+1)+k+3}^{\infty} N_r. \tag{6.2}$$

We now bound the measure of these points. For $r \in \mathbb{N}$ define the set

$$N_r = \{I_r^t \,|\, t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n, \text{ and } I_r^t \nsubseteq Q \text{ for any box } Q \text{ of precision } p(2i+1)\}.$$

If I_r^t is not contained in a dyadic cube of precision $p(2i+1)$, then I_r^t must contain at least one dyadic rational of precision $p(2i+1)$. Hence,

$$|N_r| \leq 3^n 2^{np(2i+1)} \tag{6.3}$$

and so,

$$\mu(N_r) \leq 3^n 2^{np(2i+1)}\, 2^{-rn}. \tag{6.4}$$

By equation (6.2) and inequality (6.4), we obtain

$$\begin{aligned}
\mu(T_i - A_i^k) &\le \mu\Big(\bigcup_{r=p(2i+1)+k+3}^{\infty} N_r\Big) \\
&\le \sum_{r=p(2i+1)+k+3}^{\infty} \mu(N_r) \\
&\le \sum_{r=p(2i+1)+k+3}^{\infty} 3^n\, 2^{np(2i+1)}\, 2^{-rn} \\
&\le 3^n\, 2^{np(2i+1)} \sum_{r=p(2i+1)+k+3}^{\infty} 2^{-rn} \\
&\le 2^{-k-1}.
\end{aligned}$$

We would like $\{A_i^k\}$ to be a uniformly polynomial space computable sequence. However, there is a minor technical detail which complicates the argument. The definition of uniformly pspace computable sequences requires the endpoints to be dyadic rationals. Unfortunately, translating the dyadic cubes by $t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n$ violates this requirement. In order to overcome this, we will approximate $\{A_i^k\}$ by boxes with dyadic endpoints. For any open cube Q, define $D_i^k(Q)$ to be the open dyadic box containing Q such that

$$\mu(D_i^k(Q) - Q) < 2^{-n(p(2i+1)+2k+3)}.$$

Formally, if $Q = (a_1, b_1) \times \ldots \times (a_n, b_n)$, let

$$D_i^k(Q) = (d_1, d_1') \times \ldots \times (d_n, d_n')$$

where d_i, d_i' are dyadic rationals at precision $p(2i+1)+2k+n+3$, and $d_i \le a_i < b_i \le d_i'$.

Define the set

$$S_i^k = \bigcup_{Q \in A_i^k} D_i^k(Q).$$

It is easy to verify that $\{S_i^k\}$ is a uniformly pspace computable array such that the endpoints of S_i^k are in $\mathbf{D}^n_{p(2i+1)+2k+n+3}$, and $\mu(T_i \Delta S_i^k) \le 2^{-k}$ for every i, $k > 0$. It is clear that, for every i and all $x \in T_i$, $x \in \liminf_k S_i^k$. Hence, $\{T_i\}$ is a uniformly polynomial space approximable sequence, and $\{U_m\}_{m\in\mathbb{N}}$ is a pspace $\mathcal{W}$-test.

Assume x is weakly pspace-random. Then there exists an N such that for all $m > N$, $x \notin U_m$. Let $i > 2N + 8 + 2c$, $t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n$ and $r > 0$. Choose $j > rn + i$. Then,

$$
\begin{aligned}
\frac{\int_{I_r^t(x)} |f - f_i| d\mu}{\mu(I_r^t(x))} &\leq \frac{\int_{I_r^t(x)} |f - f_j| d\mu}{\mu(I_r^t(x))} + \frac{\int_{I_r^t(x)} |f_j - f_i| d\mu}{\mu(I_r^t(x))} \\
&\leq 2^{rn}\, 2^{-j} + \frac{\int_{I_r^t(x)} |f_j - f_i| d\mu}{\mu(I_r^t(x))} \\
&\leq 2^{-i} + \sum_{m=i}^{j-1} \frac{\int_{I_r^t(x)} |f_m - f_{m+1}| d\mu}{\mu(I_r^t(x))} \\
&\leq 2^{-i} + \sum_{m=i}^{j-1} 2^{-\frac{m}{2}} \\
&\leq 2^{-i} + 2^{-\frac{i}{2}+2} \\
&< 2^{-\frac{i}{2}+3}.
\end{aligned}
$$

Since $t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n$ and $r > 0$ were arbitrary,

$$
\lim_{m\to\infty} \frac{\int_{I_r^t(x)} |f - f_m| d\mu}{\mu(I_r^t(x))} = 0,
$$

for every $t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n$ and $r > 0$. □

We are now able to prove that weakly pspace random points satisfy the Lebesgue differentiation theorem.

Theorem 6.11. *If x is weakly pspace-random, then for every polynomial space L_1-computable $f \in L_1([0,1]^n)$, and every polynomial space computable sequence of simple functions $\{f_m\}_{m\in\mathbb{N}}$ approximating f,*

$$
\lim_{m\to\infty} f_m(x) = \lim_{Q\to x} \frac{\int_Q f d\mu}{\mu(Q)}
$$

where the limit is taken over all cubes Q containing x as the diameter of Q tends to 0.

Proof. Let x be weakly pspace-random. By Theorem 6.8, it suffices to show that

$$\lim_{m\to\infty} f_m(x) = \lim_{k\to\infty} \frac{\int_{I_k^t(x)} f d\mu}{\mu(I_k^t(x))}$$

for all $t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n$.

Let $\epsilon > 0$. By Lemmas 6.7 and 6.10, there exists an N such that for all $i > N$,

$$|f_i(x) - \lim_{m\to\infty} f_m(x)| < \frac{\epsilon}{2}, \tag{6.5}$$

and

$$\frac{\int_{I_k^t(x)} |f - f_i| d\mu}{\mu(I_k^t(x))} < \frac{\epsilon}{2}, \tag{6.6}$$

for every $t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n$ and $k > 0$. Let $i > N$. Then, using (6.5) we obtain

$$|\lim_{m\to\infty} f_m(x) - \lim_{k\to\infty} \frac{\int_{I_k^t(x)} f d\mu}{\mu(I_k^t(x))}| < \frac{\epsilon}{2} + |f_i(x) - \lim_{k\to\infty} \frac{\int_{I_k^t(x)} f d\mu}{\mu(I_k^t(x))}|. \tag{6.7}$$

By Lemma 6.5, for every $r > 0$, $x \in Q$ for some $Q \in \mathbf{B}_r$. Since f_i is a simple step function, f_i is constant on every $Q \in \mathbf{B}_{p(i)}$. So there exists an N' so that for all $r > N'$,

$$f_i(x) = \frac{\int_{I_r^t(x)} f_i d\mu}{\mu(I_r^t(x))},$$

for every $t \in \{-\frac{1}{3}, 0, \frac{1}{3}\}^n$. Therefore, by inequality (6.6), for every $r > N'$,

$$|f_i(x) - \frac{\int_{I_r^t(x)} f d\mu}{\mu(I_r^t(x))}| = |\frac{\int_{I_r^t(x)} f_i d\mu}{\mu(I_r^t(x))} - \frac{\int_{I_r^t(x)} f d\mu}{\mu(I_r^t(x))}| \tag{6.8}$$

$$\leq \frac{\int_{I_r^t(x)} |f - f_i| d\mu}{\mu(I_r^t(x))} \tag{6.9}$$

$$< \frac{\epsilon}{2}. \tag{6.10}$$

Combining inequalities (6.7) and (6.10) we have

$$|\lim_{m\to\infty} f_m(x) - \lim_{k\to\infty} \frac{\int_{I_k^t(x)} f d\mu}{\mu(I_k^t(x))}| < \epsilon.$$

Since ϵ was arbitrary, the proof is complete. $\square$

6.2.2 Non-random points are not Lebesgue points

We now show that converse of our main theorem holds. That is, we show that if a point x is not weakly pspace random, the limit $\lim\limits_{Q \to x} \frac{1}{\mu(Q)} \int_Q f d\mu$ does not exist. Our approach is largely similar from the construction of Pathak, et al [46]. However, due to the restriction of polynomial space computation, the implementation is significantly different. To adapt the construction of Pathak et al, we first introduce a notion that will partition a pspace $\mathcal{W}$-test $\{U_m\}$ into a tree of dyadic cubes.

Recall that the *level* of a node in a rooted tree is the length of the (unique) path from the root to the node. We denote the set of all nodes of a tree $\mathbf{T}$ at level i by $Level_i(\mathbf{T})$.

A *dyadic tree decomposition of* $[0,1]^n$ is a tree $\mathbf{T}$ of dyadic cubes rooted at $[0,1]^n$ such that the following hold:

1. For every cube $Q \in \mathbf{T}$, the children of Q, are subsets of Q.

2. For any two cubes $Q_1, Q_2 \in \mathbf{T}$, either Q_1 and Q_2 are disjoint, or one contains the other.

3. For any cube $Q \in \mathbf{T}$,
$$\mu\Big(\bigcup_{B \in Child(Q)} B\Big) \leq \frac{\mu(Q)}{4}.$$

A dyadic tree decomposition $\mathbf{T}$ is *polynomial space approximable* if there exists a polynomial p and uniformly pspace computable array $\{T_m^k\}_{k,m\in\mathbb{N}}$ such that the following hold.

1. For every $k, m \in \mathbb{N}$, T_m^k is a finite union of disjoint dyadic cubes.

2. For every $\mu(Level_m(\mathbf{T})\Delta T_m^k) \leq 2^{-(k+m)}$.

Intuitively, for every k and m, T^k_m is a good approximation of the mth level of the tree $\mathbf{T}$. The following technical lemma will be used to show that every pspace $\mathcal{W}$-test admits a pspace approximable dyadic tree decomposition.

Lemma 6.12. *For every uniformly pspace computable array $\{R^k_m\}_{k,m\in\mathbb{N}}$, there exists a uniformly pspace computable array $\{S^k_m\}_{k,m\in\mathbb{N}}$ such that*

1. *For every m, k, $\mu(\cup_{i\leq k} R^i_m \Delta \cup_{i\leq k} S^i_m) = 0$, and*

2. *For every m, $\cup_k S^k_m$ is a set of disjoint open dyadic cubes.*

Proof. We can, and do assume that, for every k, m, R^k_m is a finite union of disjoint open dyadic cubes, whose endpoints are dyadic rationals at precision $p(k+m)$. For every m, define $S^1_m = R^1_m$. Let $m \in \mathbb{N}$ and $k > 1$. Define the set

$$A^k_m = \{Q \in \mathbf{B}_{p(k+m)} \,|\, (\exists i < k)\, Q \subseteq B \text{ where } B \in R^i_m\}.$$

That is, A^k_m is the set of all cubes in $\cup_{i<k} R^i_m$ broken into dyadic cubes of precision $p(k+m)$. Define $S^k_m = R^k_m - A^k_m$.

It is clear that $\{S^k_m\}_{k,m\in\mathbb{N}}$ satisfies both properties of the lemma. Note that $\{A^k_m\}_{m\in\mathbb{N},k>1}$ is a uniformly pspace computable array. It therefore follows that $\{S^k_m\}_{k,m\in\mathbb{N}}$ is pspace computable. □

We now show that every pspace $\mathcal{W}$-test admits a pspace approximable dyadic tree decomposition. We build the tree inductively, using the uniformly pspace computable sequence of the previous lemma.

Lemma 6.13. *Let $\{U_m\}_{m\in\mathbb{N}}$ be a pspace $\mathcal{W}$-test. Then there exists a pspace approximable dyadic tree decomposition $\mathbf{T}$ such that, for every non-dyadic $x \in \bigcap U_m$, x is contained in an infinite path in $\mathbf{T}$.*

Proof. Let $\{U_m\}_{m\in\mathbb{N}}$ be a pspace $\mathcal{W}$-test. Let $\{R_m^k\}_{k,m\in\mathbb{N}}$ be a uniformly pspace computable array approximating $\{U_m\}_{m\in\mathbb{N}}$. We can and do assume that for all $k,m\in\mathbb{N}$, $\mu(U_m \Delta R_m^k) < 2^{-(k+m)}$. Let $\{S_m^k\}_{k,m\in\mathbb{N}}$ be the uniformly pspace computable array of obtained from $\{R_m^k\}$ satisfying the properties of Lemma 6.12. For every m, define the set

$$S_m = \{Q \,|\, Q \in S_m^k \text{ for some } k \geq 1\}.$$

We define the dyadic tree decomposition $\mathbf{T}$ inductively. Define the first level of $\mathbf{T}$ to be

$$Level_1(\mathbf{T}) = S_1.$$

For $i > 1$, define level i as follows. For every $Q \in Level_{i-1}(\mathbf{T})$, let $m \in \mathbb{N}$ be the smallest integer such that $2^{-m} < \frac{\mu(Q)}{8}$. Define the set

$$Child(Q) = \{B \,|\, B \in S_m \text{ and } B \subseteq Q\}.$$

Finally, define the ith level to be

$$Level_i(\mathbf{T}) = \bigcup_{Q\in Level_{i-1}(\mathbf{T})} Child(Q).$$

We now prove that $\mathbf{T}$ is a dyadic tree decomposition of $[0,1]^n$. By our construction of $\mathbf{T}$, it is clear that for any $Q \in \mathbf{T}$, the children of Q are subsets of Q. We prove item (2) of the definition dyadic tree decompositions by induction on the level of the tree. For the root $[0,1]^n$, the claim is immediate. Let $i > 0$. Let Q_1, Q_2 be dyadic cubes at level i. If Q_1 and Q_2 have different parents, the claim holds by our inductive hypothesis. Assume that Q_1 and Q_2 have the same parent. Then $Q_1, Q_2 \in \cup_k S_m^k$ for some $m \in \mathbb{N}$, and therefore Q_1 and Q_2 are disjoint. Let $Q \in \mathbf{T}$ and m be the smallest integer such that $2^{-m} < \frac{\mu(Q)}{8}$. By the construction of $\mathbf{T}$,

$$\begin{aligned}
\mu(\bigcup_{B \in Child(Q)} B) &\leq \mu(\cup_{k \geq 1} S_m^k) \\
&\leq \sum_{k=1}^{\infty} \mu(R_m^k) \\
&\leq \mu(U_m) + \mu(\bigcup_{k \geq 1} U_m \Delta R_m^k) \\
&\leq 2^{-m} + \sum_{k=1}^{\infty} 2^{-(k+m)} \\
&= 2^{-m+1} \\
&\leq \frac{\mu(Q)}{4}
\end{aligned}$$

We now show that $\mathbf{T}$ is pspace approximable. We define the array $\{T_m^k\}_{k,m \in \mathbb{N}}$ inductively on m. For $m = 1$, set

$$T_1^k = \bigcup_{i=1}^{k} S_1^i.$$

Let $m > 1$ and $k \in \mathbb{N}$. For every $Q \in T_{m-1}^k$, let $j \in \mathbb{N}$ be the smallest integer such that $2^{-j} < \frac{\mu(Q)}{8}$. Define the set

$$C_Q^k = \{B \in Child(Q) \,|\, B \in S_j^i \text{ for some } i \leq k+2\}.$$

Since,

$$\begin{aligned}
\mu(\bigcup_{i=k+3}^{\infty} S_j^i) &\leq \sum_{i=k+2}^{\infty} \mu(S_j^i) \\
&\leq \sum_{i=k+2}^{\infty} \mu(R_j^i - R_j^{i-1}) \\
&\leq \sum_{i=k+2}^{\infty} (\mu(U_j \Delta R_j^i) + \mu(U_j \Delta R_j^{i-1})) \\
&\leq \sum_{i=k+2}^{\infty} (2^{-(j+i)} + 2^{-(j+i-1)}) \\
&\leq 2^{-(j+k}
\end{aligned}$$

we have

$$\begin{aligned}\mu(Child(Q) - C_Q^k) &\leq \mu(\bigcup_{i=k+3}^{\infty} S_j^i) \\ &\leq 2^{-(j+k)} \\ &\leq \frac{\mu(Q)}{8} 2^{-(k)}.\end{aligned}$$

Finally, define

$$T_m^k = \bigcup_{Q \in T_{m-1}^k} C_Q^k.$$

We now show that $\{T_m^k\}_{k,m\in\mathbb{N}}$ approximates $\mathbf{T}$ by induction on the level m. It is clear that for all k, $\mu(Level_1(\mathbf{T})\Delta T_1^k) < 2^{-k}$. Let $k, m \in \mathbb{N}$. Define the set

$$N = \{Q \,|\, Q \in Level_{m-1}(\mathbf{T}) - T_{m-1}^k\}.$$

Then,

$$\begin{aligned}\mu(Level_m(\mathbf{T})\Delta T_m^k) &= \mu(\bigcup_{Q\in T_{m-1}^k} Child(Q)\Delta T_m^k) + \mu(\bigcup_{Q\in N} Child(Q)) \\ &\leq \sum_{Q\in T_{m-1}^k} \mu(Child(Q) - C_Q^k) + \sum_{Q\in N} \mu(Child(Q)) \\ &\leq \sum_{Q\in T_{m-1}^k} (\frac{\mu(Q)}{8} 2^{-k}) + 2^{-(k+3)} \\ &\leq 2^{-k}.\end{aligned}$$

Since $\{S_m^k\}_{k,m\in\mathbb{N}}$ is pspace computable, $\{T_m^k\}_{k,m\in\mathbb{N}}$ is a uniformly pspace computable array. Hence $\mathbf{T}$ is a pspace approximable dyadic tree decomposition.

Let $x = (x_1, \ldots, x_n) \in \cap_{m\geq 1} U_m$ be a point so that x_i is not a dyadic rational. We prove that there is an infinite path in $\mathbf{T}$ containing x by induction on the level of $\mathbf{T}$. By the definition of pspace $\mathcal{W}$-tests, it is clear that there exists a dyadic cube Q in S_1 such that $x \in Q$. Hence $Q \in Level_1(\mathbf{T})$. Let $i > 1$. By our inductive hypothesis, there exists a dyadic rational cube $Q \in Level_{i-1}(\mathbf{T})$ containing x. Let m be the smallest integer such that $2^{-m} < \frac{\mu(Q)}{8}$. Since there exists a dyadic cube $Q \in S_m$ containing x, the conclusion follows. $\square$

We are now able to prove the converse of Theorem 6.11, thereby completing the proof of our main theorem. The proof of this theorem involves constructing a function that takes advantage of the dyadic tree decomposition of a pspace W-test succeeding on x. We construct the function so that it assigns different values to alternating levels of the tree. As we are guaranteed that x is in an infinite path of the tree, the function oscillates around x.

Theorem 6.14. *If $x \in [0,1]^n$ is not weakly pspace random, then there exists a pspace L_1 computable function f such that the limit $\lim\limits_{Q \to x} \frac{1}{\mu(Q)} \int_Q f d\mu$ does not exist.*

Proof. We first assume that $x = (x_1, \ldots, x_n)$ so that some component x_i of x is a dyadic rational. Without loss of generality assume that $x_1 = d \in \mathbf{D}$. Define the function $f : [0,1]^n \to \mathbb{R}$ to be

$$f(y) = \begin{cases} 1 & \text{if } y \in [0,d] \times [0,1] \times \ldots \times [0,1] \\ 0 & \text{otherwise} \end{cases}$$

It is clear that f is pspace L_1-computable, and that the limit $\lim\limits_{Q \to x} \frac{1}{\mu(Q)} \int_Q f d\mu$ does not exist.

Assume that $x = (x_1, \ldots, x_n)$ so that x_i is not a dyadic rational for all $i \leq n$. Let $\{U_m\}_{m \in \mathbb{N}}$ be a pspace $\mathcal{W}$-test succeeding on x. Let $\mathbf{T}$ be a pspace computable dyadic tree partition of $\{U_m\}_{m \in \mathbb{N}}$ given by Lemma 6.13. Define $f : [0,1]^n \to \mathbb{R}$ as follows. For every $Q \in \mathbf{T}$,

$$f(Q - \bigcup_{B \in Child(Q)} B) = \begin{cases} 1 & \text{if the level of } Q \text{ in } \mathbf{T} \text{ is even} \\ 0 & \text{if the level of } Q \text{ in } \mathbf{T} \text{ is odd} \end{cases}$$

It is clear that f is integrable and well defined for all points that are not in the intersection $\bigcap U_m$. We now show that f is pspace L_1-computable. Let $\{T_m^k\}_{k,m\in\mathbb{N}}$ be the uniformly pspace computable array approximating $\mathbf{T}$. For every $m \in \mathbb{N}$, define

$$\mathbf{T}_m = \bigcup_{i=1}^{m} T_i^{m+2}.$$

We can consider $\mathbf{T}_m$ as a finite subtree of $\mathbf{T}$ which well approximates $\mathbf{T}$. For every $m \in \mathbf{N}$ and every $Q \in \mathbf{T}_m$, define the set of children of Q in the approximation $\mathbf{T}_m$ by

$$C_m(Q) = Child(Q) \cap \mathbf{T}_m.$$

For every $m \in \mathbb{N}$, define $f_m : [0,1]^n \to \mathbb{R}$ as follows.

$$f_m(Q - \bigcup_{B\in C_m(Q)} B) = \begin{cases} 1 & \text{if the level of } Q \text{ in } \mathbf{T}_m \text{ is even} \\ \\ 0 & \text{if the level of } Q \text{ in } \mathbf{T}_m \text{ is odd} \end{cases}$$

It is clear that f_m is a simple step function. Since the array $\{T_m^k\}$ approximating $\mathbf{T}$ is uniformly pspace computable, on input $(0^m, d)$ we are able to compute the level of the largest dyadic cube in $\mathbf{T}$ containing d in polynomial space. Therefore the sequence of functions $\{f_m\}$ is pspace computable.

We now prove that $\{f_m\}_{m\in\mathbb{N}}$ approximates f. For $m \in \mathbb{N}$, define the set $A = \mathbf{T} - \mathbf{T}_m$, the set of all cubes in $\mathbf{T}$ that are not in the approximation $\mathbf{T}_m$. We now bound the error of our approximation $\mathbf{T}_m$. From the definition of tree decompositions, we have

$$\begin{aligned}
\mu(A) &= \mu(\mathbf{T} - \mathbf{T}_m) \\
&= \mu(\bigcup_{i=1}^{m} Level_i(\mathbf{T}) - T_i^{m+2}) + \mu(\bigcup_{i=m+1}^{\infty} Level_i(\mathbf{T})) \\
&\leq \sum_{i=1}^{m} \mu(Level_i(\mathbf{T}) - T_i^{m+2}) + \sum_{i=m+1}^{\infty} \mu(Level_i(\mathbf{T})) \\
&\leq \sum_{i=1}^{m} 2^{-(i+m+2)} + \sum_{i=m+1}^{\infty} 2^{-2i} \\
&\leq 2^{-m}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|f - f_m\|_1 &= \int_0^1 |f - f_m| \\
&= \int_A |f - f_m| \\
&\leq \mu(A) \\
&\leq 2^{-m}.
\end{aligned}$$

Hence, f is a pspace L_1 computable function.

Finally, we show that the limit $\lim\limits_{Q \to x} \frac{1}{\mu(Q)} \int_Q f d\mu$ does not exist. We first show that

$$\limsup_{Q \to x} \tfrac{1}{\mu(Q)} \int_Q f d\mu \geq \tfrac{3}{4}.$$

Let $N \in \mathbb{N}$. By Lemma 6.13, x is contained in an infinite path of $\mathbf{T}$. Choose a dyadic cube $Q \in \mathbf{T}$ containing x so that $\mu(Q) < 2^{-N}$ and the level of Q in $\mathbf{T}$ is even. Then, by our construction of f,

$$\begin{aligned}
\frac{1}{\mu(Q)} \int_Q f d\mu &\geq \frac{1}{\mu(Q)} \int_{Q - Child(Q)} 1 d\mu \\
&= \frac{1}{\mu(Q)} \mu(Q - Child(Q)) \\
&\geq \frac{3}{4}.
\end{aligned} \tag{6.11}$$

Similarly, we show that $\liminf\limits_{Q\to x} \frac{1}{\mu(Q)} \int_Q f d\mu \leq \frac{1}{4}$. Let $N \in \mathbb{N}$. Choose a dyadic cube $Q \in \mathbf{T}$ containing x so that $\mu(Q) < 2^{-N}$ and the level of Q in $\mathbf{T}$ is odd. Then, by our construction of f,

$$\begin{aligned} \frac{1}{\mu(Q)} \int_Q f d\mu &\leq \frac{1}{\mu(Q)} \int_{Child(Q)} 1 d\mu \\ &= \frac{1}{\mu(Q)} \mu(Child(Q)) \\ &\leq \frac{1}{4}. \end{aligned} \tag{6.12}$$

Combining the equalities (6.11) and (6.12), we see that the limit $\lim\limits_{Q\to x} \frac{1}{\mu(Q)} \int_Q f d\mu$ does not exist. □

Finally, by Theorems 6.11 and 6.14, the Lebesgue differentiation theorem characterizes weakly pspace randomness.

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APPENDIX . ADDITIONAL PROOFS

Kolmogorov Complexity Using Initial Segments

In this section we formalize the relationship between $K_r(x)$ and the *initial segment complexity* $K(x{\upharpoonright}r)$. The three lemmas in this section are proved by standard techniques. We use these results elsewhere in the technical appendix, but not in the body of the thesis.

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $r \in \mathbb{N}$, let $x{\upharpoonright}r = (x_1{\upharpoonright}r, \ldots, x_n{\upharpoonright}r)$, where each $x_i{\upharpoonright}r = 2^{-r}\lfloor 2^r x_i \rfloor$, the truncation of x_i to r bits to the right of the binary point. For $r \in (0, \infty)$, let $x{\upharpoonright}r = x{\upharpoonright}\lceil r \rceil$.

Lemma A.1. *For every $m, n \in \mathbb{N}$, there is a constant c such that for all $x \in \mathbb{R}^m$, $p \in \mathbb{Q}^n$, and $r \in \mathbb{N}$,*

$$|\hat{K}_r(x|p) - K(x{\upharpoonright}r \,|\, p)| \leq K(r) + c\,.$$

Proof. Let $m, n, r \in \mathbb{N}$, $x \in \mathbb{R}^m$, and $p \in \mathbb{Q}^n$. Observe that $x{\upharpoonright}r \in B_{2^{-r}\sqrt{m}}(x)$, and therefore $K(x{\upharpoonright}r \,|\, p) \geq \hat{K}_{r-\log(m)/2}(x|p)$. Thus, by Lemma 2.3, there exists $c_1 \in \mathbb{N}$ depending only on m such that

$$\hat{K}_r(x|p) \leq K(x{\upharpoonright}r \,|\, p) + K(r) + c_1\,.$$

For the other direction, observe that for every $q \in \mathbb{Q}^n \cap B_{2^{-r}}(x)$, we have $x{\upharpoonright}r \in B_{2^{-r}(1+\sqrt{m})}(q)$, and that $B_{2^{-r}(1+\sqrt{m})}(q)$ contains at most $(2(1+\sqrt{m}))^m$ r-dyadic points, i.e., points in the set

$$\mathcal{Q}_t^m = \{2^{-r}z : z \in \mathbb{Z}^m\}\,.$$

Let M be a Turing machine that, on input $(\pi, p') \in \{0,1\}^* \times \mathbb{Q}^n$, does the following. If $\pi = \pi_1\pi_2\pi_3$, with $U(\pi_1, p') = q \in \mathbb{Q}^m$, $U(\pi_2) = t \in \mathbb{N}$, and $U(\pi_3) = k \in \mathbb{N}$, then M outputs the (lexicographically) k^{th} point in $\mathcal{Q}_r^m \cap B_{2^{-t}(1+\sqrt{m})}(q)$.

Now let π_q testify to $\hat{K}_r(x|p)$, let π_r testify to $K(r)$, and let $q = U(\pi_q, p)$. There is some $k \leq (2(1+\sqrt{m}))^m$ such that $x{\restriction}r$ is the k^{th} point in $\mathcal{Q}_r^m \cap B_{2^{-r}(1+\sqrt{m})}(q)$; let π_k testify to $K(k)$. Then $M(\pi_q\pi_r\pi_k, p) = x{\restriction}r$, so there is some machine constant c_M for M such that

$$\begin{aligned} K(x{\restriction}r\,|\,p) &\leq \ell(\pi_q) + \ell(\pi_r) + \ell(\pi_k) + c_M \\ &= \hat{K}_r(x|p) + K(x) + K(k) + c_M \end{aligned}$$

It is well known (see, e.g., [13]) that there is some constant c_2 such that

$$\begin{aligned} K(k) &\leq \log k + 2\log\log k + c_2 \\ &\leq m\log(2(1+\sqrt{m})) + 2\log(m\log(2(1+\sqrt{m}))) + c_2\,. \end{aligned}$$

The above value depends only on m, as does c_M; let c_3 be their sum. Then

$$K(x{\restriction}r\,|\,p) \leq \hat{K}_r(x|p) + K(r) + c_3\,,$$

so $c = \max\{c_1, c_3\}$ affirms the lemma. □

Observing that there exists a constant c_0 such that, for all $m \in \mathbb{N}$ and $q^m \in \mathbb{Q}$, $|K(q) - K(q|0)| \leq c_0$, we also have the following.

Corollary A.2. *For every $m \in \mathbb{N}$, there is a constant c such that for every $x \in \mathbb{R}^m$ and $r \in \mathbb{N}$,*

$$|K_r(x) - K(x{\restriction}r)| \leq K(r) + c\,.$$

Corollary A.3. *For every $m, n \in \mathbb{N}$, there is a constant c such that for all $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $r, s \in \mathbb{N}$,*

$$|K_{r,s}(x|y) - K(x{\upharpoonright}r \,|\, y{\upharpoonright}s)| \leq K(r) + K(s) + c\,.$$

Proof. Let $m, n, r, s \in \mathbb{N}$, $x \in \mathbb{R}^m$, and $y \in \mathbb{R}^n$. Let $p \in \mathbb{Q}^2 \cap B_{2^{-s}}(y)$ be such that $K_{r,s}(x|y) = \hat{K}_r(x|p)$. Since $y{\upharpoonright}s \in B_{2^{-s}\sqrt{n}}(y)$, we have $\hat{K}_r(x \,|\, y{\upharpoonright}s) \geq K_{r,s-\log(n)/2}(x|y)$. Thus, by Lemma 2.3there is a constant c_1 (depending on n) such that $\hat{K}_r(x \,|\, y{\upharpoonright}s) \geq K_{r,s}(x|y) - K(s) - c_1$. Lemma A.1 tells us that there is a constant c_2 (depending on m) such that $K(x{\upharpoonright}r \,|\, y{\upharpoonright}s) \geq \hat{K}_r(x \,|\, y{\upharpoonright}s) - K(r) - c_2$, so we have

$$K_{r,s}(x|y) \leq K(x{\upharpoonright}r \,|\, y{\upharpoonright}s) + K(r) + K(s) + c_1 + c_2\,.$$

For the other direction, we use essentially the same technique as in the proof of Lemma A.1, and we describe a Turing machine M' that is very similar to the machine M used above. On every input $(\pi, p') \in \{0,1\}^* \times \mathbb{Q}^n$ such that $\pi = \pi_1\pi_2\pi_3$, $U(\pi_1, p') = q \in \mathbb{Q}$, $U(\pi_2) = t \in \mathbb{N}$, and $U(\pi_3) = k \in \mathbb{N}$, M' outputs $U(\pi_1, q')$, where q' is the k^{th} point in $\mathcal{Q}^n_t \cap B_{2^{-t}(1+\sqrt{n})}(p')$.

Much as before, let π_x testify to $K(x{\upharpoonright}r \,|\, y{\upharpoonright}s)$, let π_s testify to $K(s)$, and let π_k testify to $K(k)$, where $y{\upharpoonright}s$ is the k^{th} point in $\mathcal{Q}^n_s \cap B_{2^{-t}(1+\sqrt{n})}(p)$. Then

$$M'(\pi_x, \pi_s, \pi_k) = U(\pi_x, y{\upharpoonright}s) = x{\upharpoonright}r\,,$$

As $k \leq |\mathcal{Q}^n_s \cap B_{2^{-t}(1+\sqrt{n})}(p)| \leq (2(1+\sqrt{n}))^n$, there exist constants $c_{M'}$ and c_k (depending on n) such that

$$\begin{aligned}
K(x{\upharpoonright}r \,|\, p) &\leq \ell(\pi_x) + \ell(\pi_s) + \ell(\pi_k) + c_{M'} \\
&= K(x{\upharpoonright}r \,|\, y{\upharpoonright}s) + K(s) + K(k) + c_{M'} \\
&= K(x{\upharpoonright}r \,|\, y{\upharpoonright}s) + K(s) + c_k + c_{M'}\,,
\end{aligned}$$

Applying Lemma A.1 again, there is a constant c_3 (depending on m) such that $K(x{\upharpoonright}r\,|p) \leq \hat{K}_r(x|p) + K(r) + c_3$. We conclude that

$$K(x{\upharpoonright}r\,|\,y{\upharpoonright}s) \leq K(r) + K(s) + c_k + c_{M'} + c_3\,,$$

therefore $c = \max\{c_1 + c_2, c_k + c_{M'} + c_3\}$ affirms the lemma. □

Proofs from Chapter 2

Lemma 2.3. *Let $m, n \in \mathbb{N}$, $x \in \mathbb{R}^m$, $z \in \mathbb{R}^n$, $\varepsilon > 0$ and $r \in \mathbb{N}$. If $K^x_r(z) \geq K_r(z) - \varepsilon r$, then the following hold for all $s \leq r$.*

(i) $|K^x_s(z) - K_s(z)| \leq \varepsilon r - O(\log r)\,.$

(ii) $|K_{s,r}(x\,|\,z) - K_s(x)| \leq \varepsilon r - O(\log r)\,.$

Proof. We first prove item (i). By Lemma 2.2(ii),

$$\begin{aligned}
\varepsilon r &\geq K_r(z) - K^x_r(z)\\
&\geq K_s(z) + K_{r,s}(z\,|\,z) - (K^x_s(z) + K^x_{r,s}(z\,|\,z)) - O(\log r)\\
&\geq K_s(z) - K^x_s(z) + K_{r,s}(z\,|\,z) - K^x_{r,s}(z\,|\,z) - O(\log r).
\end{aligned}$$

Rearranging, this implies that

$$\begin{aligned}
K_s(z) - K^x_s(z) &\leq \varepsilon r + K^x_{r,s}(z\,|\,z) - K_{r,s}(z\,|\,z) + O(\log r)\\
&\leq \varepsilon r + O(\log r),
\end{aligned}$$

and the proof of item (i) is complete.

To prove item (ii), by Lemma 2.2(i) we have

$$\begin{aligned}
\varepsilon r &\geq K_r(z) - K_r(z \mid x) \\
&\geq K_r(z) - (K_r(z,x) - K_r(x)) - O(\log r) \\
&\geq K_r(z) - (K_r(z) + K_r(x \mid z) - K_r(x)) - O(\log r) \\
&= K_r(x) - K_r(x \mid z) - O(\log r).
\end{aligned}$$

Therefore, by Lemma 2.2(ii),

$$\begin{aligned}
K_s(x) - K_{s,r}(x \mid z) &= K_r(x) - K_{r,s}(x \mid x) - (K_r(x \mid z) - K_{r,s,r}(x \mid x, z)) \\
&\leq \varepsilon r + O(\log r) + K_{r,s,r}(x \mid x, z) - K_{r,s}(x \mid x) \\
&\leq \varepsilon r + O(\log r),
\end{aligned}$$

and the proof is complete. □

Proofs from Chapter 3

Here we construct the oracles used in the proof of Theorems 3.7 and Lemma 4.6. Our proof of this lemma uses the fact that conditional Kolmogorov complexity is essentially equivalent to Kolmogorov complexity relative to a finite oracle set.[1]

Observation A.4. *For every $k \in \mathbb{N}$ and $\tau = (\tau_1, \dots, \tau_n) \in \{0,1\}^k$, define the oracle set*

$$C(\tau) = \left\{ j \leq 2k : \tau_{\lfloor j/2 \rfloor} = 1 \right\} \cup \{2k+1\} \subseteq \mathbb{N}\,.$$

Then there is a constant c such that for every $\sigma, \tau \in \{0,1\}^*$,

$$\left| K(\sigma|\tau) - K^{C(\tau)}(\sigma) \right| \leq c\,.$$

[1] In fact, [13] defines conditional Kolmogorov complexity in terms of a finite oracle, using a construction similar to the one described here.

Proof. Let $\pi \in \{0,1\}^*$ be such that $U(\pi,\tau) = \sigma$. Then given the oracle $C(\tau)$ and input π, a machine can discern τ from $2\ell(\tau)+2$ queries to $C(\tau)$ and use it to simulate $U(\pi,\tau)$. Let $\pi \in \{0,1\}^*$ such that $U^{C(\tau)}(\pi) = \sigma$. Likewise, given input (π',τ), a machine can compute any bit $C(\tau)$ queried in a simulation of $U^{C(\tau)}(\pi)$. □

Lemma 3.6. *Let $n, r \in \mathbb{N}$, $z \in \mathbb{R}^n$, and $\eta \in \mathbb{Q} \cap [0, \dim(z)]$. Then there is an oracle $A = A(n,r,z,\eta)$ with the following properties.*

(i) *For every $t \leq r$, $K_t^A(z) = \min\{\eta r, K_t(z)\} + O(\log r)$.*

(ii) *For every $m, t \in \mathbb{N}$ and $y \in \mathbb{R}^m$, $K_{t,r}^A(y|z) = K_{t,r}(y|z) + O(\log r)$ and $K_t^{z,A}(y) = K_t^z(y) + O(\log r)$.*

Proof. Let $s = \max\{t \leq r : K_{t-1}(z) < \eta r\}$. Observe that

$$\eta r \leq K_s(z) \leq \eta r + K(s) + c\,.$$

Let σ be the lexicographically first time-minimizing witness to $K(z{\upharpoonright}r \,|\, z{\upharpoonright}s)$, and let $A = C(\sigma)$, as defined in Observation A.4.

Suppose $s \leq t \leq r$. Then applying a relativized version of Corollary A.2 and Observation A.4,

$$\begin{aligned} K_t^A(z) &\leq K_r^A(z) \\ &\leq K^A(z{\upharpoonright}r) + K(r) + O(1) \\ &\leq K(z{\upharpoonright}r \,|\, \sigma) + K(r) + O(1)\,. \end{aligned}$$

There exists a Turing machine M_1 that, on input (π,σ), for $\pi \in \{0,1\}^*$, simulates $U(\sigma, U(\pi,\sigma))$. If π is a witness to $K(z{\upharpoonright}s \,|\, \sigma)$, then

$$M(\pi,\sigma) = U(\sigma, U(\pi,\sigma)) = U(\sigma, z{\upharpoonright}s) = z{\upharpoonright}r\,.$$

Thus, $K(z{\upharpoonright}r \,|\, \sigma) \leq K(z{\upharpoonright}s \,|\, \sigma) + c_{M_1}$, where c_{M_1} is a constant for the description length of M_1. We now have

$$\begin{aligned}
K_t^A(z) &\leq K(z{\upharpoonright}s \,|\, \sigma) + K(r) + O(1)\\
&\leq K(z{\upharpoonright}s) + K(r)\\
&\leq K_s(z) + 2K(r) + O(1)\\
&\leq \eta r + 2K(r) + K(s) + O(1)\,.
\end{aligned}$$

For the other direction, since $K_t^A(z) \geq K_s^A(z)$ whenever $t \geq s$, it is sufficient to show that $K_s^A(z) \geq \eta r$. We use Corollary A.2, Observation A.4, and the symmetry of information:

$$\begin{aligned}
K_s^A(z) &\geq K^A(z{\upharpoonright}s) - K(s) - O(1)\\
&\geq K(z{\upharpoonright}s \,|\, \sigma) - K(s) - O(1)\\
&\geq K(z{\upharpoonright}r \,|\, \sigma) - K(s) - O(1)\\
&\geq K(z{\upharpoonright}r) - K(\sigma) - K(s) - O(1)\\
&= K(z{\upharpoonright}r) - K(z{\upharpoonright}r \,|\, z{\upharpoonright}s) - K(s) - O(1)\\
&\geq K(z{\upharpoonright}r, z{\upharpoonright}s) - K(z{\upharpoonright}r \,|\, z{\upharpoonright}s, K(z{\upharpoonright}s)) - K(K(z{\upharpoonright}s)) - 2K(s) - O(1)\\
&= K(z{\upharpoonright}s) - K(K(z{\upharpoonright}s)) - 2K(s) - O(1)\\
&\geq K_z(s) - K(K(z{\upharpoonright}s)) - 3K(s) - O(1)\\
&= K_z(s) - O(\log r)\,.
\end{aligned}$$

Since $K_s(z) \geq \eta r$, property (i) holds in this case.

Now suppose instead that $t \leq s \leq r$. We again use Corollary A.2, Observation A.4, and the symmetry of information.

$$\begin{aligned}
K^A_t(z) =& K^A(z{\upharpoonright}t) - K(t) - O(1)\\
=& K(z{\upharpoonright}t \,|\, \sigma) - K(t) - O(1)\\
\geq& K(z{\upharpoonright}t \,|\, \sigma, K(\sigma)) - K(t) - O(1)\\
=& K(\sigma \,|\, z{\upharpoonright}t, K(z{\upharpoonright}t)) + K(z{\upharpoonright}t) - K(\sigma) - K(t) - O(1)\\
\geq& K(\sigma \,|\, z{\upharpoonright}t) - K(K(z{\upharpoonright}t)) + K(z{\upharpoonright}t) - K(\sigma) - K(t) - O(1)\\
\geq& K(\sigma \,|\, z{\upharpoonright}s, t) - K(K(z{\upharpoonright}t)) + K(z{\upharpoonright}t) - K(\sigma) - K(t) - O(1)\\
\geq& K(z{\upharpoonright}t) + K(\sigma \,|\, z{\upharpoonright}s, K(z{\upharpoonright}s)) - K(\sigma) - K(K(z{\upharpoonright}t)) - 2K(t) - O(1)\\
=& K(z{\upharpoonright}t) + K(z{\upharpoonright}s \,|\, \sigma, K(\sigma)) - K(z{\upharpoonright}s) - K(K(z{\upharpoonright}t)) - 2K(t) - O(1)\\
\geq& K(z{\upharpoonright}t) + K(z{\upharpoonright}s \,|\, \sigma) - K(z{\upharpoonright}s) - K(K(\sigma)) - K(K(z{\upharpoonright}t))\\
&\quad - 2K(t) - O(1)\\
\geq& K_t(z) + K^A_s(z) - K_s(z) - K(K(\sigma)) - K(K(z{\upharpoonright}t))\\
&\quad - 3K(t) - 2K(s) - O(1)\\
=& K_t(z) + K^A_s(z) - K_s(z) - O(\log r)\,.
\end{aligned}$$

As we have already shown that $K^A_s(z) - K_s(z) = O(\log r)$, we conclude that property (i) holds in this case as well.

For property (ii), we again apply Corollary A.2, relativized to (z, A), and Observation A.4, relativized to z, to see that

$$\begin{aligned} K_t^{z,A}(y) &\geq K^{z,A}(y{\upharpoonright}t) - K(t) - O(1) \\ &= K^z(y{\upharpoonright}t \mid \sigma) - K(t) - O(1) \\ &\geq K^z(y{\upharpoonright}t) - K^z(\sigma) - K(t) - O(1) \\ &\geq K_t^z(y) - K^z(\sigma) - 2K(t) - O(1) \\ &\geq K_t^z(y) - K(\sigma \mid z{\upharpoonright}r) - 2K(t) - O(1)\,, \end{aligned}$$

where the last inequality is due to Lemma 2.3. We argue that $K(\sigma \mid z{\upharpoonright}r)$ is at most logarithmic in r.

$$\begin{aligned} K(\sigma \mid z{\upharpoonright}r) &\leq K(\sigma, s, \ell(\sigma) \mid z{\upharpoonright}r) + O(1) \\ &\leq K(\sigma \mid s, \ell(\sigma), z{\upharpoonright}r) + K(s) + K(\ell(\sigma)) + O(1) \\ &\leq K(\sigma \mid s, \ell(\sigma), z{\upharpoonright}r) + O(\log r)\,. \end{aligned}$$

To see that the first term is constant, define a Turing machine M_2 that does the following. Given input (j, k, x), M_2 simulates, for every $\pi \in \{0,1\}^k$ in parallel, $U(\pi, x{\upharpoonright}j)$. It outputs the first such π whose simulation halts with output x. We defined σ in such a way that $M_2^z(s, \ell(\sigma), z{\upharpoonright}r) = \sigma$, so

$$K(\sigma \mid s, \ell(\sigma), z{\upharpoonright}r) \leq c_{M_2}\,,$$

where c_{M_2} is a constant for the length of M_2's description. We conclude that $K(\sigma \mid z{\upharpoonright}r) = O(\log r)$, so $K_t^{z,A}(y) \geq K_t^z(y) - O(\log r)$.

The argument for conditional complexity is essentially identical. By a relativized version of Corollary A.3 and Observation A.4,

$$\begin{aligned}
K^A_{t,r}(y|z) &\geq K^{z,A}(y{\upharpoonright}t \,|\, z{\upharpoonright}r) - K(t) - O(1)\\
&= K(y{\upharpoonright}t \,|\, z{\upharpoonright}r, \sigma) - K(t) - O(1)\\
&\geq K(y{\upharpoonright}t \,|\, z{\upharpoonright}r) - K(\sigma \,|\, z{\upharpoonright}r) - K(t) - O(1)\\
&\geq K_{t,r}(y|z) - K(\sigma \,|\, z{\upharpoonright}r) - 2K(t) - O(1)\\
&\geq K_{t,r}(y|z) - K(\sigma \,|\, z{\upharpoonright}r) - O(\log r)\,,
\end{aligned}$$

and we have already shown that $K(\sigma \,|\, z{\upharpoonright}r) = O(\log r)$. □

Lemma A.5. *Let $z, w \in \mathbb{R}^n$, $e \in S^{n-1}$, and $r \in \mathbb{N}$ such that $P_e(z) = P_e(w)$. Let $q = (q_1, \dots, q_n) \in \mathbb{Q}^n$ and $p = (p_1, \dots, p_n) \in \mathbb{Q}^n$ be r-approximations of z and w, respectively. Then*

$$\left|\tfrac{-b+\sqrt{b^2-4ac}}{2a} - \tfrac{-b'+\sqrt{b'^{,2}-4a'c'}}{2a'}\right| \leq 2^{-r+t+\alpha},$$

where a, b, c, a', b' and c' are as defined in Lemma 3.5, $t = -\log\|z - w\|$ and α is a constant depending only on e.

Proof. We begin by recalling that $|z_1 - w_1|$ is maximal and

$$\begin{aligned}
&sgn((z_i - w_i)e_i) \neq sgn((z_j - w_j)e_j), \text{ and}\\
&|z_j - w_j| > 0,
\end{aligned}$$

where sgn denotes the sign.

We now bound a, b and c. By our assumption of e_1, e_2, we have

$$\begin{aligned}
|a| &= (z_1 - w_1)^2 + (z_2 - w_2)^2\\
&\geq (z_1 - w_1)^2\\
&\geq 2^{-2t+\alpha}.
\end{aligned}$$

Similarly, we have $|a| \leq 2^{-2t+\alpha}$, resulting in

$$|a| = 2^{-2t+\alpha} \tag{A.1}$$

It is routine, although tedious, to calculate the following bounds:

$$|b| \leq 2^{-2t+\alpha} \tag{A.2}$$

$$|c| \leq 2^{-2t+\alpha} \tag{A.3}$$

By our assumption and repeated use of the triangle inequality, we deduce the following.

$$\begin{aligned}
|a - a'| \leq & |(z_1 - w_1)^2 - (q_1 - p_1)^2| + |(z_2 - w_2)^2 - (q_2 - p_2)^2| && \text{(A.4)} \\
= & |(z_1 - w_1) + (q_1 - p_1)|\,|(z_1 - w_1) - (q_1 - p_1)| && \text{(A.5)} \\
& + |(z_2 - w_2) + (q_2 - p_2)|\,|(z_2 - w_2) - (q_2 - p_2)| && \text{(A.6)} \\
\leq & \, 2|(z_1 - w_1)|\,|z_1 - q_1| + |p_1 - w_1| && \text{(A.7)} \\
& + \; 2|(z_2 - w_2)|\,|z_2 - q_2| + |p_2 - w_2| && \text{(A.8)} \\
\leq & 2^{-t+1}(2^{-r} + 2^{-r}) + 2^{-t+1}(2^{-r} + 2^{-r}) && \text{(A.9)} \\
= & 2^{-r-t+3}. && \text{(A.10)}
\end{aligned}$$

In a similar manner we can prove the following inequalities.

$$|b - b'| \leq 2^{-r-t-\alpha} \tag{A.11}$$

$$|c - c'| \leq 2^{-r-t-\alpha} \tag{A.12}$$

We now show that $c < 0$. This follows from

$$\begin{aligned}
c &= (\sum_{i=3}^{n}(w_i - z_i)e_i)^2 + (z_1 - w_1)^2 \sum_{i=3}^{n} e_i^2 - 1 \\
&= ((z_1 - w_1)e_1 + (z_2 - w_2)e_2)^2 + (z_1 - w_1)^2(-e_1^2 - e_2^2) \\
&= (z_1 - w_1)(z_2 - w_2)e_1e_2 + e_2^2((z_2 - w_2)^2 - (z_1 - w_1)^2)
\end{aligned}$$

Since $e_1, e_2 > 0$, $|z_1 - w_1|$ is maximal and $sgn((z_1 - w_1)e_1) \neq sgn((z_2 - w_2)e_2)$, we see that $c < 0$. Let e_2 and e_2' be the two solutions to our quadratic formula. Then

$$\begin{aligned} e_2 e_2' &= \frac{c}{a} \\ |e_2 - e_2'| &= \frac{\sqrt{b^2 - 4ac}}{|a|}. \end{aligned}$$

The first equality implies that $e_2' < 0$. The second, in conjunction with equation (A.1), implies that

$$\sqrt{b^2 - 4ac} = 2^{-2t+\alpha} |e_2 - e_2'|$$

Since e_2 is positive and e_2' is negative,

$$2^{-2t+\alpha}|e_2| \leq \sqrt{b^2 - 4ac} \leq 2^{-2t+\alpha}|e_2 + 1|. \tag{A.13}$$

Let $\alpha, \beta > 0$. Then it can easily be seen that

$$|\sqrt{\alpha} - \sqrt{\beta}| = \frac{\alpha - \beta}{\sqrt{\alpha} + \sqrt{\beta}}.$$

Using this fact, and the bounds (A.10), (A.11), (A.12) and (A.13) we have

$$\begin{aligned} |\sqrt{b^2 - 4ac} - \sqrt{b'^{,2} - 4a'c'}| &= \frac{|b^2 - 4ac - (b')^2 + 4a'c'|}{\sqrt{b^2 - 4ac} + \sqrt{b'^{,2} - 4a'c'}} \\ &\leq \frac{2^{-r-3t}}{\sqrt{b^2 - 4ac} + \sqrt{b'^{,2} - 4a'c'}} \\ &\leq \frac{2^{-r-3t+\alpha}}{\sqrt{b^2 - 4ac}} \\ &\leq \frac{2^{-r-3t+\alpha}}{2^{-2t-\alpha}} \\ &\leq 2^{-r-t+2\alpha}. \end{aligned}$$

Putting everything together, we therefore have

$$|e_2 - -\frac{-b' + \sqrt{b'^{,2} - 4a'c'}}{2a'}| \leq 2^{-r+t+\alpha} \tag{A.14}$$

and the proof is complete ☐

Observation A.6. *Let $z \in \mathbb{R}^n$, $p \in \mathbb{Q}^n$, $e \in S^{n-1}$, and $r \in \mathbb{N}$ such that $|P_e(z) - P_e(p)| \leq 2^{-r}$. Then there is a $w \in \mathbb{R}^n$ such that $\|p - w\| \leq 2^{\gamma - r}$ and $P_e(z) = P_e(w)$, for some constant γ depending only on z and e.*

Proofs from Chapter 4

The following two lemmas from our previous work (stated in slightly different forms here) are precursors to Lemmas 4.4 and 4.6. The proof of Lemma 4.4 is similar to that of Lemma A.7, and the proof of Lemma 4.6 is an induction on Lemma 3.6.

Lemma A.7 (N. Lutz and Stull [33])**.** *Suppose that $a, b, x \in \mathbb{R}$, $r \in \mathbb{N}$, $\delta \in \mathbb{R}_+$, and $\varepsilon, \eta \in \mathbb{Q}_+$ satisfy the following conditions.*

1. $r \geq \log(2|a| + |x| + 5) + 1$.
2. $K_r(a, b) \leq (\eta + \varepsilon)\, r$.
3. *For every $(u, v) \in \mathbb{R}^2$ such that $t = -\log \|(a, b) - (u, v)\| \in (0, r]$ and $ux + v = ax + b$, $K_r(u, v) \geq (\eta - \varepsilon)\, r + \delta \cdot (r - t)$.*

Then for every oracle set $A \subseteq \mathbb{N}$,

$$K_r^A(x, ax + b) \geq K_r^A(a, b, x) - \frac{4\varepsilon}{\delta} r - K(\varepsilon) - K(\eta) - O(\log r)\,.$$

For our purposes, we will need the following corollary to Lemma A.7. Informally, that lemma gives conditions under which precision-r estimates for $(x, ax + b)$ and (a, b, x) contain similar amounts of information. This corollary shows that, under the same conditions, those two approximations are furthermore nearly "interchangeable," in the sense that there is a short program which, given a precision-r estimate for $(x, ax + b)$ as input, will output a precision-r estimate for (a, b, x), and, as we argue in the proof, vice versa.

Corollary A.8. *If the conditions of Lemma A.7 are satisfied, then*

$$K_r^A(a,b,x|x,ax+b) \leq \frac{4\varepsilon}{\delta}r + K(\varepsilon) + K(\eta) + O(\log r)\,.$$

Proof. It is easy to see that $K_r(x,ax+b|a,b,x) = O(\log r)$: consider a constant-length program that, given $(u,v,y) \in \mathbb{Q}^3$, outputs $(y, uy+v)$. If $(u,v,y) \in B_{2^{-r}}(a,b,x)$, then $(y,uy+v) \in B_{2^{c-r}}(x,ax+b)$, where c is constant in r, so $K_{r-c,r}(ax+b|a,b,x) = O(1)$. Thus, by Lemma 2.1, $K_r(ax+b|a,b,x) = O(\log r)$.

Now suppose that the conditions of Lemma 6 are satisfied. Then by symmetry of information and Lemma A.7,

$$\begin{aligned} K_r^A(a,b,x|x,ax+b) &= K_r^A(a,b,x) - K_r^A(x,ax+b) + K_r^A(x,ax+b|a,b,x) \\ &= K_r^A(a,b,x) - K_r^A(x,ax+b) + O(\log r) \\ &\leq \frac{4\varepsilon}{\delta}r + K(\varepsilon) + K(\eta) + O(\log r)\,. \end{aligned}$$

□

We will also need the following pair of geometric facts.

Observation A.9 (N. Lutz and Stull [33])**.** *Let $a,x,b \in \mathbb{R}$ and $r \in \mathbb{N}$. Let $(q_1,q_2) \in B_{2^{-r}}(x,ax+b)$.*

1. *If $(p_1,p_2) \in B_{2^{-r}}(a,b)$, then $|p_1q_1 + p_2 - q_2| < 2^{-r}(|p_1| + |q_1| + 3)$.*
2. *If $|p_1q_1 + p_2 - q_2| \leq 2^{-r}(|p_1| + |q_1| + 3)$, then there is some pair $(u,v) \in B_{2^{-r}(2|a|+|x|+5)}(p_1,p_2)$ such that $ax+b = ux+v$.*