



Determinantal representations of singular hypersurfaces in \mathbb{P}^n

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Abstract

A (global) determinantal representation of projective hypersurface $X \subset \mathbb{P}^n$ is a matrix whose entries are linear forms in homogeneous coordinates and whose determinant defines the hypersurface.

We study the properties of such representations for singular (possibly reducible or non-reduced) hypersurfaces. In particular, we obtain the decomposability criteria for determinantal representations of globally reducible hypersurfaces.

Further, we classify the determinantal representations in terms of the corresponding kernel sheaves on X . Finally, we extend the results to the case of symmetric/self-adjoint representations, with implications to hyperbolic polynomials and the generalized Lax conjecture.

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1. Introduction

1.1. Setup

Let \mathbb{k} be an algebraically closed, normed, complete field of zero characteristic, e.g. the complex numbers, \mathbb{C} . Let \mathbb{k}^n be the corresponding affine space, let $(\mathbb{k}^n, 0)$ be the germ at the origin, i.e. a small neighborhood. Let $\mathcal{O}_{(\mathbb{k}^n, 0)}$ denote the corresponding local ring of regular functions, i.e.

- rational functions that are regular at the origin, $\mathbb{k}[x_1, \dots, x_n]_{(m)}$, or
- locally converging series, $\mathbb{k}\{x_1, \dots, x_n\}$, or

– formal series $\mathbb{k}[[x_1, \dots, x_n]]$.

We denote the identity matrix by $\mathbb{1}$ and the zero matrix by $\mathbb{0}$. Let \mathcal{M} be a $d \times d$ matrix with the entries in either of:

- (local case) $\mathcal{O}_{(\mathbb{k}^n, 0)}$
- (global case) linear forms in x_0, \dots, x_n (the later being the homogeneous coordinates of \mathbb{P}^n), i.e. the global sections of the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$.

We always assume $f := \det(\mathcal{M}) \neq 0$ and $d > 1$. Such a matrix defines:

- (local case) the germ of hypersurface near the origin, $(X, 0) := \{\det(\mathcal{M}) = 0\} \subset (\mathbb{k}^n, 0)$,
- (global case) the projective hypersurface $X := \{\det(\mathcal{M}) = 0\} \subset \mathbb{P}^n$.

This hypersurface is called determinantal and the matrix \mathcal{M} is its determinantal representation. The determinant, $f = \det \mathcal{M}$, can be reducible or non-reduced (i.e. not square-free). Let $f = \prod f_\alpha^{p_\alpha}$ be the (local/global) decomposition, i.e. $\{f_\alpha\}$ are reduced, irreducible and mutually prime. Correspondingly the hypersurface is (locally/globally) decomposable: $(X, 0) = \cup (p_\alpha X_\alpha, 0) \subset (\mathbb{k}^n, 0)$ or $X = \cup p_\alpha X_\alpha \subset \mathbb{P}^n$. Sometimes we consider the *reduced locus*: $X_{red} = \cup X_\alpha = \{\prod f_\alpha = 0\}$.

The local/global determinantal representations are considered up to the *local/global equivalence*: $\mathcal{M} \sim A\mathcal{M}B$, where $A, B \in GL(d, \mathcal{O}_{(\mathbb{k}^n, 0)})$ or $A, B \in GL(d, \mathbb{k})$. Both equivalences preserve the hypersurface pointwise.

Such “matrices of functions” appear constantly in various fields. Hence the interest in the determinantal representations of *arbitrary* hypersurfaces (not only smooth or irreducible). In this work we study the global determinantal representations of singular (possibly reducible, non-reduced) hypersurfaces/plane curves. The symmetric and self-adjoint determinantal representations are treated separately at the end of the paper.

1.2. A brief history

A good summary of 19th century’s works on determinantal representations is in [55]. A modern introduction is in [17].

- The question “For which pairs (n, d) is the generic hypersurface of degree d in \mathbb{P}^n determinantal?” has been studied classically. Already [15] has shown that this happens only for $(2, d)$ and $(3, d \leq 3)$. For a recent description see [9], [17, Section 4] and [35].
- Any (projective) curve in \mathbb{P}^2 admits a *symmetric* determinantal representation. For smooth curves this was constructed (using ineffective theta characteristics) in [16]. For singular curves this was proved in [6, Sections 2 and 7] and in [13, Proposition 2.28]. For some related works, see [46,2].

For *smooth* (irreducible, reduced) plane curves the ordinary/symmetric/self-adjoint determinantal representations have been classified in [53,54]; see also [9, Proposition 1.11 and Corollary 1.12] and [17, Section 4]. In [5, Theorem 3.2] the classification of ordinary determinantal representations was extended to the case of multiple nodal curves i.e. curves of the form $\{f^p = 0\} \subset \mathbb{P}^2$ for $p \in \mathbb{N}$ and $\{f = 0\}$ irreducible, reduced, nodal curve.

- A cubic surface in \mathbb{P}^3 is determinantal iff it contains at least two lines [11, Proposition 4.3]. In particular, the only cubic surfaces not admitting determinantal representations are those with a singularity of E_6 type, e.g. $\{x_0x_1^2 + x_1x_2^2 + x_3^3 = 0\} \subset \mathbb{P}^3$. For the classification of determinantal representations of smooth cubics cf. [12], in [17, Section 9.3] the classification was extended to all cubic surfaces.

- Determinantal quartic surfaces in \mathbb{P}^3 form a subvariety of codimension one in the family of all the quartics (i.e. the complete linear family $|\mathcal{O}_{\mathbb{P}^3}(4)|$). Such a surface may have on it any number of lines up to 64, [47]. In [26] one studies determinantal representations of quartics in \mathbb{P}^3 possessing two lines L_1, L_2 of multiplicities $mult_1 + mult_2 = 4$.
- In higher dimensions the determinantal hypersurfaces are necessarily singular and the singular locus is of dimension at least $(n - 4)$. (For symmetric determinantal representations the dimension is at least $(n - 3)$, in fact the subset of X over which the corank of \mathcal{M} is at least two is of dimension at least $(n - 3)$.) The singularities occurring at the points of $corank \mathcal{M} \geq 2$ are called *essential*, all the others: *accidental*. The general linear symmetric determinantal hypersurface of degree d has only essential singularities [49, p. 495]. For their properties and the classification of singularities of determinantal cubic/quartic surfaces, i.e. $n = 3$, see [43]. Nodal quartics in \mathbb{P}^4 were studied in [42].
- The symmetric determinantal representations can be considered as n -dimensional linear families of quadrics in \mathbb{P}^{d-1} . For various applications to Hilbert schemes of complete intersections, see [51]. In [55] the determinantal representations of plane quartics (corresponding to nets of quadrics in \mathbb{P}^3) are studied in detail.
- The natural objects associated to a determinantal representation are the kernel and cokernel of \mathcal{M} . At each point of X these are just vector spaces; as the point travels along the hypersurface these spaces glue into torsion-free sheaves supported on X . The determinantal representation is determined (up to the local/global equivalence) by its kernel/cokernel, e.g. [14, Theorem 1.1]. For the precise definition, see Section 2.4.
- As was proved in [31], any affine hypersurface in \mathbb{A}^n admits a symmetric determinantal representation, i.e. any polynomial $f(x_1, \dots, x_n)$ can be presented as the determinant of a symmetric matrix of the type $A_0 + \sum x_i A_i$.
- There are several reasons to consider non-reduced hypersurfaces, i.e. the cases when $\det \mathcal{M}$ is not square-free. For example, consider *matrix factorizations*, [20]: $AB = f\mathbb{1}$ with $\det(A) = (\text{a power of } f)$. So matrix factorizations correspond to some determinantal representations of hypersurfaces with multiple components. While the general hypersurface in \mathbb{P}^n does not admit a determinantal representation, unless $(d, n) = (3, 3)$ or $n = 2$, its higher multiples, $\{f^p = 0\} \subset \mathbb{P}^n$, do. For example, by [3,33], any homogeneous polynomial f admits a matrix factorization in linear matrices: $f\mathbb{1} = \mathcal{M}_1 \dots \mathcal{M}_d$, i.e. all the entries of $\{\mathcal{M}_\alpha\}$ are linear.
- The problem can be reformulated as the study of $(n + 1)$ -tuples of matrices up to the two-sided equivalence, $(\mathcal{M}_0, \dots, \mathcal{M}_n) \sim A(\mathcal{M}_0, \dots, \mathcal{M}_n)B$. Hence the applications in linear algebra, operator theory and dynamical systems (see e.g. [5,4,50] or [39]). In particular, these applications ask for the properties of determinantal representations of an *arbitrary* hypersurface, i.e. with arbitrary singularities, possibly reducible and non-reduced.
- In applications one meets determinantal representations with specific properties, e.g. symmetric or self-adjoint (in the real case). The self-adjoint determinantal representations are important in relation to the Lax conjecture as they produce hyperbolic polynomials; see [36,27,8,38,48,41,10].
- Finally, we mention the fast developing field of semi-definite-programming and matrix inequalities, i.e. presentability of the boundary of a convex set in \mathbb{R}^n by the determinant of a self-adjoint, positive definite matrix. For the introduction, cf. [32,39].

1.3. Results and contents of the paper

We tried to make the paper readable by non-specialists in commutative algebra/algebraic geometry. Thus in Section 2 and further in the paper we recall some notions and results.

In particular in Section 2.1 we recall sheaves on singular (possibly reducible and non-reduced) hypersurfaces and the Hirzebruch–Riemann–Roch theorem for locally free sheaves.

In this paper we study the *global* determinantal representations. But the *local* version of the problem appears constantly, due to the presence of singular points of curves/hypersurfaces and points where the kernel sheaves are not locally free. The relevant results on the local version of the problem are obtained in [34] and are restated in Section 2.3. Every global determinantal representation can be localized and every local *algebraic* determinantal representation comes from a global one. The localization process preserves the equivalence in a strong sense, etc.

In Section 2.4 we introduce the sheaves of kernels (or kernel modules in the local case) and prove some of their properties. The kernel sheaves can be also defined in a completely geometric way as follows. Taking the kernel of a matrix provides a natural map $X \ni pt \rightarrow Ker(\mathcal{M}|_{pt}) \subset \mathbb{k}^d$. The image of X in \mathbb{P}^{d-1} under this map determines the kernel sheaf. This map is studied in Section 2.4.2, the equivalence of the two definitions is proven in Proposition 2.12. Then we study particular types of determinantal representations/kernel sheaves: *maximally generated* determinantal representations (in Section 2.4.4) and *X'/X -saturated* (in Section 2.4.5). They possess especially nice properties and tend to be *decomposable*.

1.3.1. Decomposability

Suppose the determinant is reducible, $\det \mathcal{M} = f_1 f_2$, so the corresponding hypersurface is globally reducible: $X = X_1 \cup X_2$. Is \mathcal{M} globally decomposable? Namely, is it globally equivalent to a block-diagonal matrix with blocks defining the components of the hypersurface: $\mathcal{M} \sim \mathcal{M}_1 \oplus \mathcal{M}_2$. We study this question in Section 3. The global decomposability obviously implies the local one. A probably unexpected feature is the converse implication.

Theorem 3.1. *Let $X = X_1 \cup X_2 \subset \mathbb{P}^n$ be a global decomposition of the hypersurface. Here X_1, X_2 can be further reducible, non-reduced, but without common components, i.e their defining polynomials are relatively prime. \mathcal{M} is globally decomposable, i.e. $\mathcal{M} \stackrel{\text{globally}}{\sim} \mathcal{M}_1 \oplus \mathcal{M}_2$, iff it is locally decomposable at each point $pt \in X_1 \cap X_2$, i.e. $\mathcal{M} \stackrel{\text{locally}}{\sim} \mathbb{1} \oplus \mathcal{M}_1|_{(\mathbb{P}^n, pt)} \oplus \mathcal{M}_2|_{(\mathbb{P}^n, pt)}$. Here $\mathcal{M}_\alpha|_{(\mathbb{P}^n, pt)}$ is the local determinantal representation of (X_α, pt) .*

The proof of this property is heavily based on Noether’s $AF + BG$ theorem [1, p. 139], in fact one might consider the statement as Noether’s-type theorem for matrices.

Similarly, suppose at each point of the intersection $X_1 \cap X_2$ the determinantal representation is locally equivalent to an upper-block-triangular, $\mathcal{M} \sim \begin{pmatrix} \mathcal{M}_1 & * \\ \mathbb{0} & \mathcal{M}_2 \end{pmatrix}$, where $\det(\mathcal{M}_\alpha)$ defines X_α . Then the global determinantal representation is globally equivalent to an upper-block-triangular, Proposition 3.3.

Both statements are non-trivial from linear algebra point of view, but almost tautological when considered as statements on kernel sheaves. In the first case the sheaf is the direct sum, $E \approx E_1 \oplus E_2$, in the second case it is an extension: $0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$.

These results completely reduce the global decomposability problem for determinantal representations to the local problem. In Sections 2.4.4 and 2.4.5 we state various necessary and sufficient criteria for local decomposability of determinantal representations; they are formulated and proved in [34].

1.3.2. Properties and classification of kernel sheaves

In Section 4 we study the kernel sheaves on the hypersurfaces in \mathbb{P}^n . First we summarize their properties Theorem 4.1. It is possible to classify those sheaves arising as kernels of determinantal

representations. For the smooth case this was done in [53]; see also [17, Section 4]. The classification for an arbitrary hypersurface is done in [9, Theorem A]. We give a direct proof of this result.

(Theorems 4.3 and 4.1) Consider a hypersurface $X = \cup p_\alpha X_\alpha \subset \mathbb{P}^n$. The torsion-free sheaf E_X of multi-rank (p_1, \dots, p_k) is the kernel of a determinantal representation of X , in the sense of Eq. (14), iff $h^0(E_X(-1)) = 0, h^i(E_X(j)) = 0$ for $0 < i < n - 1, j \in \mathbb{Z}$ and $h^{n-1}(E_X(1 - n)) = 0$.

(Note that over a non-reduced hypersurface the torsion-free sheaf can be nowhere locally free. For the definition of multi-rank, see Section 2.1.1.)

We prove this theorem by an explicit construction, generalizing [16,53], where it is done for smooth plane curves. One advantage of this proof is that it is easily adjustable to symmetric/self-adjoint cases (see below).

An immediate corollary, in the case when E_X is locally free, is the information about the Chern class of the kernel, Theorem 4.1.

1.3.3. Relation to matrix factorizations and descent to the reduced locus.

Suppose $AB = \mathbb{1} \prod f_\alpha$, where A has homogeneous linear entries, the factors $\{f_\alpha\}$ are irreducible and A is non-invertible at any point of the hypersurface $\{f = 0\}$. Then $\det(A) = \prod f_\alpha^{p_\alpha}$, for some multiplicities $\{p_\alpha\}$. So, A is a determinantal representation of the non-reduced hypersurface. A natural question: Which determinantal representations arise from matrix factorizations of reduced hypersurfaces? An immediate consequence of our results is the following.

Corollary 1.1. Let \mathcal{M} be a determinantal representation of $\prod f_\alpha^{p_\alpha}$. There exists a matrix \mathcal{N} satisfying $\mathcal{M}\mathcal{N} = \prod f_\alpha \mathbb{1}$ iff \mathcal{M} is maximally generated at generic smooth points of the reduced locus $\{\prod f_\alpha = 0\}$.

(For the definition of maximally generated see Section 2.4.4.) In such a case the kernel E of \mathcal{M} , a sheaf over the non-reduced hypersurface X , has natural descent to the reduced locus X_{red} : $E \rightsquigarrow E_X^{red}$. In Section 4.2 we classify the sheaves obtained in this way.

Proposition 4.7. A torsion free sheaf E_X^{red} of multirank (p_1, \dots, p_k) on the reduced locus X_{red} arises by descent from X iff $h^0(E_X^{red}(-1)) = 0, h^i(E_X^{red}(j)) = 0$ for $0 < i < n - 1, j \in \mathbb{Z}$ and $h^{n-1}(E_X^{red}(1 - n)) = 0$.

1.3.4. Ascent to the modification, for curves

Let $C = \cup p_\alpha C_\alpha$ be the global decomposition of a plane curve. One often considers normalization: $\tilde{C} := \coprod (p_\alpha \tilde{C}_\alpha) \xrightarrow{\nu} \cup p_\alpha C_\alpha$, here each $\tilde{C}_\alpha \rightarrow C_\alpha$ is the normalization of an irreducible curve. Correspondingly the kernel sheaf is pulled back: $\nu^*(E)/Torsion$. For the normalization $\tilde{C} \xrightarrow{\nu} C$ the pullback $\nu^*(E)/Torsion$ is locally quasi-free (or just locally free in the reduced case).

Sometimes the pullback is locally quasi-free already for some intermediate modification: $\tilde{C} \rightarrow C' \xrightarrow{\nu} C$. It is important to classify those sheaves on C' whose pushforward to C produces kernel sheaves.

Corollary 4.8. Given a modification $C' \rightarrow \cup p_\alpha C_\alpha$, the torsion free sheaf $E_{C'}$ descends to the kernel of a determinantal representation of C iff $h^i(E_{C'}^{red}(-1)) = 0$ for $i \geq 0$.

A more complicated question is: Which sheaves on C' are pullbacks (modulo torsion) of kernel sheaves on C ? (Note that in general $E \subsetneq \nu_*\nu^*(E)/\text{Torsion}$.) We give a criterion in Proposition 4.11.

Once the general properties of kernel sheaves are established one can study the determinantal representations for particular hypersurfaces. In Section 4.4 we give some simplest examples of kernel sheaves on curves/surfaces.

1.3.5. Symmetric and self-adjoint determinantal representations

In Sections 5 and 6 we work with $\mathbb{k} = \mathbb{R} \subset \mathbb{C}$. If \mathcal{M} is symmetric or self-adjoint then it is natural to consider symmetric or self-adjoint equivalence ($\mathcal{M} \stackrel{s}{\sim} A\mathcal{M}A^T$ or $\mathcal{M} \stackrel{\tau}{\sim} A\mathcal{M}A^\tau$). Many of the previous results are extended to this setup.

Being symmetric or self-adjoint can be formulated in terms of the kernel sheaves (Proposition 5.2 and Proposition 6.3). Two symmetric representations are equivalent (in the ordinary sense) iff they are symmetrically equivalent (Proposition 5.3). For self-adjoint representations this is true up to a diagonal matrix, the precise statement is Proposition 6.3.

The symmetric determinantal representations of singular hypersurfaces are studied in Section 5. In particular, we characterize the kernel sheaves of symmetric determinantal representations of hypersurfaces. In Section 6 we characterize the self-adjoint determinantal representations of hypersurfaces.

1.3.6. Applications to hyperbolic polynomials

Recall that if \mathcal{M} is self-adjoint then $\det \mathcal{M}$ is a hyperbolic polynomial; see Section 6.3. Then the real locus of X can have at most one singular point with a non-smooth locally irreducible component; see Theorem 6.7. In the later case the region of hyperbolicity degenerates to this singular point. Thus, if the hypersurface is defined by a self-adjoint positive-definite determinantal representation, then all the locally irreducible components of its reduced locus are smooth. In Theorem 6.7 we prove that any self-adjoint positive-definite determinantal representation of a real hypersurface is \tilde{X}/X saturated (at real points), i.e. its kernel arises as the push-forward of a locally free sheaf from the normalization $\tilde{X} \xrightarrow{\nu} X$.

2. Preliminaries and notations

For local considerations we always assume the (singular) point to be at the origin and mostly use the ring of locally convergent power series $\mathbb{k}\{x_1, \dots, x_n\} = \mathcal{O}_{(\mathbb{k}^n, 0)}$. Let $\mathfrak{m} = \langle x_1, \dots, x_n \rangle \subset \mathcal{O}_{(\mathbb{k}^n, 0)}$ be the maximal ideal.

The *tangent cone* $T_{(X,0)} \subset \mathbb{k}^n$ is formed as the limit of all the tangent hyperplanes at smooth points. For the hypersurface $(X, 0) = \{f = 0\}$, with the Taylor expansion $f = f_p + f_{p+1} + \dots$, the tangent cone is $\{f_p = 0\} \subset (\mathbb{k}^n, 0)$. For curves the tangent cone is the collection of tangent lines, each with the corresponding multiplicity.

The tangent cone is in general reducible. Associated to it is the *tangential decomposition*: $(X, 0) = \cup_{\alpha \in T_{(X,0)}} (X_\alpha, 0)$. Here α runs over all the (set-theoretical) components of the tangent cone, each $(X_\alpha, 0)$ can be further reducible, non-reduced.

Example 2.1. Consider the curve singularity $(X, 0) = \{(y^2 - x^4)(x^2 - y^4) = 0\} \subset (\mathbb{k}^2, 0)$. Here the tangent cone is $T_{(X,0)} = \{y^2x^2 = 0\} \subset \mathbb{k}^2$. Accordingly, the tangential decomposition is: $\{y^2 = x^4\} \cup \{x^2 = y^4\} \subset (\mathbb{k}^2, 0)$.

The basic invariant of the hypersurface singularity $\{f_p + f_{p+1} + \dots = 0\}$ is the *multiplicity* $\text{mult}(X, 0) = p$. For the tangential components denote $p_\alpha = \text{mult}(X_\alpha, 0)$.

2.1. Sheaves on singular hypersurfaces

The theory of coherent sheaves on multiple smooth curves, i.e. pC_{red} , for C_{red} irreducible and smooth, is developed in [18].

A coherent sheaf on a pure dimensional scheme X is called torsion-free if it has no subsheaf whose support is of strictly smaller dimension than $dim(X)$.

2.1.1. Multi-rank of pure sheaves on reducible, non-reduced hypersurfaces

Let F_X be a torsion-free sheaf; its singular locus $Sing(F_X) \subset X$ is the set of (closed) points where F_X is not locally free. If X is reduced then F is generically locally free and to the decomposition $X = \cup X_\alpha$ is associated the multi-rank $(r_1, \dots, r_k): r_i = rank(F|_{X_i})$.

In the non-reduced case a torsion free sheaf can be *nowhere locally free*. To define its multi-rank we need a preliminary construction. Consider a multiple hypersurface, $X = pX_{red} = \{f^p = 0\}$, where X_{red} is irreducible. We define the rank of F_X . Let $I_{X_{red}} \subset \mathcal{O}_X$ be the ideal of the reduced locus. As X is a hypersurface, this ideal is principal, $I_{X_{red}} = \langle f \rangle$. Consider the multiplication by f on F . Its successive kernels define a useful filtration on F :

$$0 \subseteq Ker(f) \subseteq Ker(f^2) \subseteq \dots \subseteq Ker(f^{p-2}) \subseteq Ker(f^{p-1}) \subseteq Ker(f^p) = F. \tag{1}$$

Associated to this filtration is the graded sheaf:

$$Gr_f(F) := \bigoplus_{j=0}^{p-1} Ker(f^{p-j})/Ker(f^{p-j-1}) = \bigoplus_{j=0}^{p-1} Gr_j. \tag{2}$$

By definition $fGr_j = 0$; hence Gr_j is naturally a module over $\mathcal{O}_{X_{red}}$.

Example 2.2. • Though F is torsion-free, its reduction, $F \otimes_{\mathcal{O}_X} \mathcal{O}_{X_{red}} = F/IF$, in general has torsion. For example, let $X = \{x_1^2 = 0\} \subset \mathbb{P}^n$. Let F be an \mathcal{O}_X module, generated by $s_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ and $s_2 = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$. So $F = \mathcal{O}_X \langle s_1, s_2 \rangle / (x_1s_2 - x_2s_1, x_1s_1)$. Then $F/IF = \mathcal{O}_{X_{red}} \langle s_1, s_2 \rangle / (x_2s_1)$, i.e. the element s_1 is annihilated by a non-zero divisor x_2 .

• For simplicity consider the local case of curves. Let $\mathcal{O}_C = \mathbb{k}[x, \epsilon]/\epsilon^p$ and $F = \bigoplus_0^{p-1} \epsilon^j (\mathbb{k}[x])^{\oplus l_j}$, for $l_0 \leq l_1 \leq \dots \leq l_{p-1}$. Then the filtration is

$$0 \subset \epsilon^{p-1} (\mathbb{k}[x])^{\oplus l_{p-1}} \subset \dots \subset \bigoplus_1^{p-1} \epsilon^j (\mathbb{k}[x])^{\oplus l_j} \subset \bigoplus_0^{p-1} \epsilon^j (\mathbb{k}[x])^{\oplus l_j}. \tag{3}$$

A sheaf is called *locally quasi-free* (at a point) if its graded version is locally free (at this point). Every torsion free sheaf on a hypersurface is generically locally quasi-free.

Definition 2.3. 1. Let $X = pX_{red}$, where X_{red} is irreducible and reduced. The rank of F on X is the rank of $Gr_f(F)$ as a module over X_{red} .

2. For $X = \cup p_\alpha X_\alpha$ the multi-rank of F is the collection of ranks $\{rank F|_{p_\alpha X_\alpha} / Torsion\}$.

Proposition 2.4. Let $L \subset \mathbb{P}^n$ be the generic line, and let $pt \in X \cap L$. Then $length(F|_{pt}) = rank(F)$.

2.1.2. Hirzebruch–Riemann–Roch theorem

In this paper we use the Hirzebruch–Riemann–Roch theorem for locally free sheaves on (singular, reducible, possibly non-reduced) hypersurfaces, [22, p. 354]:

$$\chi(F_X) = (ch(F_X)Td(T_X))_{top.\,dimensional}. \tag{4}$$

Here we have the following.

- ★ The Euler characteristic of the sheaf is $\chi(F_X) = \sum_{i=0}^{n-1} (-1)^i h^i(F_X)$.
- ★ The Chern character of the sheaf is $ch(F_X) = \prod \exp(\alpha_i)$ where $\{\alpha_i\}_i$ are the Chern roots of F_X , i.e. $c(F_X) = \prod (1 + \alpha_i)$.
- ★ The Todd class of the hypersurface is $Td(T_X)$. For a smooth variety it equals $\prod_i \frac{\alpha_i}{1 - \exp(-\alpha_i)}$, where $\{\alpha_i\}_i$ are the Chern roots of the tangent bundle, i.e. $c(T_X) = \prod (1 + \alpha_i)$. Suppose the scheme X is singular but embeddable as a locally complete intersection into a smooth variety, $X \subset Y$. For example, this is the case for hypersurfaces in \mathbb{P}^n . Then $Td(T_X)$ is the Todd class of the virtual tangent bundle, $T_X := T_Y|_X - N_{X/Y}$.
- ★ Both the Chern and the Todd classes are graded, from their product one extracts the top dimensional part.

Example 2.5. Let $X_d \subset \mathbb{P}^n$ be an arbitrary hypersurface of degree d . Let \mathcal{L} be a line bundle on X . The total Chern class is $c(\mathcal{L}) = 1 + c_1(\mathcal{L})$. Then $ch(\mathcal{L}) = \sum \frac{c_i(\mathcal{L})}{i!}$. The virtual tangent bundle of X is defined by

$$0 \rightarrow T_X \rightarrow T_{\mathbb{P}^n}|_X \rightarrow N_{X/\mathbb{P}^n} \rightarrow 0, \quad c(T_{\mathbb{P}^n}) = (1 + L)^{n+1},$$

$$c(N_{X/\mathbb{P}^n}) = 1 + dL. \tag{5}$$

Here L is the class of a hyperplane in \mathbb{P}^n . Hence

$$c(T_X) = \frac{(1 + L)^{n+1}}{1 + dL} = 1 + (n + 1 - d)L + \left(\binom{n + 1}{2} - d(n + 1) + d^2 \right) L^2 + \dots \tag{6}$$

The Hirzebruch–Riemann–Roch theorem in this case reads:

$$\chi(\mathcal{L}) = \left[\left(\sum \frac{c_i(\mathcal{L})}{i!} \right) \left(1 + \frac{c_1(T_X)}{2} + \frac{c_1^2(T_X) + c_2(T_X)}{12} + \frac{c_1(T_X)c_2(T_X)}{24} + \dots \right) \right]_{top.dim.} \tag{7}$$

For example, in the case of plane curves, see [29, Section IV.I Exercise 1.9]:

$$h^0(F) - h^1(F) = deg(F) + (1 - p_a)rank(F). \tag{8}$$

Here $p_a = \binom{d-1}{2}$ is the arithmetic genus of the plane curve, it does not depend on the singularities.

The same formula holds sometimes for sheaves that are torsion free, but not locally free. For example, for torsion-free sheaves on an integral curve this was proved in [30, Theorem 1.3]. See also [23]. The theorem was also proved for sheaves on multiple smooth curves in [18].

2.1.3. The dualizing sheaf and Serre duality

For torsion free sheaves on varieties with (at most) Gorenstein singularities, e.g. on any hypersurface in \mathbb{P}^n , the dualizing sheaf w_C is invertible. By the adjunction formula for a hypersurface in \mathbb{P}^n of degree d , with arbitrary singularities: $w_X = \mathcal{O}_X(d - n - 1)$. Then the usual Serre duality holds: $H^i(F_X) = H^{\dim(X)-i}(F_X^* \otimes w_X)^* = H^{\dim(X)-i}(F_X^*(d - n - 1))^*$.

2.2. The matrix and its adjoint

We work with (square) matrices, their sub-blocks and particular entries. Sometimes to avoid confusion we emphasize the dimensionality, e.g. $\mathcal{M}_{d \times d}$. Then $\mathcal{M}_{i \times i}$ denotes an $i \times i$ block in $\mathcal{M}_{d \times d}$ and $\det(\mathcal{M}_{i \times i})$ the corresponding minor. On the other hand by \mathcal{M}_{ij} we mean a particular entry.

Let \mathcal{M} be a determinantal representation of $X \subset \mathbb{P}^n$ or $X \subset (\mathbb{k}^n, 0)$. Let \mathcal{M}^\vee be the adjoint matrix of \mathcal{M} , so $\mathcal{M}\mathcal{M}^\vee = \det(\mathcal{M})\mathbb{1}_{d \times d}$. Then \mathcal{M} is non-degenerate outside the hypersurface X and its corank over the hypersurface satisfies:

$$1 \leq \text{corank}(\mathcal{M}|_{pt \in X}) \leq \text{mult}(X, pt) \tag{9}$$

(as is checked e.g. by taking derivatives of the determinant). The adjoint matrix \mathcal{M}^\vee is not zero at smooth points of X . As $\mathcal{M}^\vee|_X \times \mathcal{M}|_X = \mathbb{O}$ the rank of \mathcal{M}^\vee at any smooth point of X is 1 (for the reduced hypersurface). Note that $(\mathcal{M}^\vee)^\vee = f^{d-2}\mathcal{M}$ and $\det \mathcal{M}^\vee = f^{d-1}$.

2.2.1. The case $\det(\mathcal{M}) \equiv 0$

A natural question in this case is whether \mathcal{M} is equivalent to a matrix with a zero row/column. In general this does not hold, e.g. for $\mathcal{M} = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ x & z & 0 \end{pmatrix}$. Indeed, if \mathcal{M} was equivalent to a matrix with zero row/column then \mathcal{M}^\vee would be equivalent to a matrix with at most 3 non-zero entries. But the ideal of \mathcal{M}^\vee is (x^2, xy, xz, yz) , i.e. is generated by 4 elements. And this ideal is invariant under equivalence.

2.3. Local determinantal representations

Here we review some aspects of local determinantal representations and quote the necessary results, all the proofs are in [34]. Essentially this is the part of commutative algebra, the theory of Cohen–Macaulay modules; see [56,37].

2.3.1. The global-to-local reduction

This is the way to pass from global to local determinantal representations. Replace the homogeneous coordinates of \mathbb{P}^n by the local coordinates: $(x_0, \dots, x_n) \rightarrow (x_1, \dots, x_n)$ with $x_0 = 1$.

Proposition 2.6. *Suppose the multiplicity of $(X, 0)$ is $m \geq 1$ and $\mathcal{M}_{d \times d}$ is a corresponding (local or global) determinantal representation.*

1. *Locally $\mathcal{M}_{d \times d}$ is equivalent to $\begin{pmatrix} \mathbb{1}_{(d-p) \times (d-p)} & \mathbb{O} \\ \mathbb{O} & \mathcal{M}_{p \times p} \end{pmatrix}$ with $\mathcal{M}_{p \times p}|_{(0,0)} = \mathbb{O}$ and $1 \leq p \leq m$.*
2. *The stable local equivalence (i.e. $\mathbb{1} \oplus \mathcal{M}_1 \sim \mathbb{1} \oplus \mathcal{M}_2$) implies ordinary local equivalence ($\mathcal{M}_1 \sim \mathcal{M}_2$). So, the global-to-local reduction is unique up to the local equivalence.*

From the algebraic point of view the first statement is the reduction to the minimal free resolution of the kernel module [21, Section 20]. The first claim is proved in symmetric case e.g. in [43, Lemma 1.7]. Both bounds are sharp, regardless of the singularity of hypersurface. For the second statement see [34].

Definition 2.7. In the notations as above, $\mathcal{M}_{p \times p}$ is the reduction of $\mathcal{M}_{d \times d}$ or the local representation.

Any matrix whose entries are *rational* functions, regular at the origin, is the reduction of some global determinantal representation:

- Lemma 2.8.** 1. For any $\mathcal{M}_{local} \in Mat(p \times p, \mathbb{k}[x_1, \dots, x_n]_{(m)})$, there exists a matrix of homogeneous linear forms, $\mathcal{M}_{global} \in Mat(d \times d, H^0(\mathcal{O}_{\mathbb{P}^n}(1)))$, whose reduction is \mathcal{M}_{local} .
2. In particular, if $\mathcal{M}_1, \mathcal{M}_2$ are locally equivalent and \mathcal{M}_1 is the reduction of some \mathcal{M}_{global} then \mathcal{M}_2 is also the reduction of \mathcal{M}_{global} .

Note that in the lemma \mathcal{M}_{global} or $\det(\mathcal{M}_{global})$ are not unique in any sense, even the dimension of \mathcal{M}_{global} is not fixed.

Proof. We can assume that \mathcal{M} is a matrix with polynomial entries. Indeed, all the denominators of entries of \mathcal{M} do not vanish at the origin; hence one can multiply \mathcal{M} by them.

Let $x_1^{a_1} \dots x_n^{a_n}$ be a monomial in \mathcal{M}_{local} with the highest total degree $\sum a_i$. By permutation assume that it belongs to the entry \mathcal{M}_{11} . Consider the augmented matrix:

$$\begin{pmatrix} 1 & 0 & & & \\ 0 & x_1^{a_1} \dots x_n^{a_n} + \dots & \mathcal{M}_{12} & \dots & \\ 0 & \mathcal{M}_{21} & \mathcal{M}_{22} & \dots & \\ 0 & \dots & \dots & \dots & \end{pmatrix}. \tag{10}$$

It is locally equivalent to

$$\begin{pmatrix} 1 & x_1 & 0 & & \\ -x_1^{a_1-1} \dots x_n^{a_n} & 0 + \dots & \mathcal{M}_{12} & \dots & \\ 0 & \mathcal{M}_{21} & \mathcal{M}_{22} & \dots & \\ 0 & \dots & \dots & \dots & \end{pmatrix}. \tag{11}$$

For the new matrix the number of monomials with highest total degree is less by one. Continue in the same way till all the monomials of the highest total degree ($\sum a_i$) are removed. Now the highest order degree is less than $\sum a_i$. Continue by induction till one gets a matrix with entries of degree at most 1. \square

The last lemma is formulated as a purely linear-algebraic statement. A reformulation in terms of sheaves (using Proposition 2.14):

For any kernel sheaf the stalk (at any point) is a kernel module. The isomorphism class of the stalk is well defined. Every kernel module is the stalk of some kernel sheaf.

Example 2.9. Let \mathcal{M} be a determinantal representation of the plane curve $\{y^2 = x^{k+1}\} \subset (\mathbb{k}^2, 0)$, this is the A_k singularity. Suppose \mathcal{M} is local, i.e. $\mathcal{M}|_{(0,0)} = \mathbb{O}$. As the multiplicity of this curve singularity is 2, the dimensionality of \mathcal{M} is either 1 or 2. The first case is trivial, in the second case one can show that \mathcal{M} is (locally) equivalent to $\begin{pmatrix} y & x^l \\ x^{k+1-l} & y \end{pmatrix}$.

Finally we prove that for global determinantal representations the local equivalence is not weaker than the global one.

Lemma 2.10. *Suppose two global determinantal representations are locally equivalent, i.e. $\mathcal{M}_1 = A\mathcal{M}_2B$ for $A, B \in GL(d, \mathbb{k}[[x_0, \dots, x_n]])$. Then $\mathcal{M}_1, \mathcal{M}_2$ are globally equivalent too.*

Proof. Expand $A = jet_0(A) + A_{\geq 1}$, where $A_{\geq 1}|_0 = \mathbb{O}$, similarly $B = jet_0(B) + B_{\geq 1}$. Note that $jet_0(A), jet_0(B) \in GL(d, \mathbb{k})$. Therefore

$$(jet_0(A))^{-1} \mathcal{M}_1 (jet_0(B))^{-1} = (\mathbb{1} + A'_{\geq 1})\mathcal{M}_2(\mathbb{1} + B'_{\geq 1}). \tag{12}$$

Hence, by comparing the degrees (in x_i) we get: $(jet_0(A))^{-1} \mathcal{M}_1 (jet_0(B))^{-1} = \mathcal{M}_2$. \square

2.4. Kernels and cokernels of determinantal representations

Let $\mathcal{M}_{d \times d}$ be a determinantal representation of the hypersurface $X \subset \mathbb{P}^n$. At each point of X the matrix has some (co-)kernel. These vector spaces glue to sheaves on X , or to vector bundles in nice situations. The sheaf structure can be defined in two equivalent ways.

2.4.1. Algebraic definition of the kernel

The cokernel sheaf is defined by the sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus d}(-1) \xrightarrow{\mathcal{M}} \mathcal{O}_{\mathbb{P}^n}^{\oplus d} \rightarrow Coker \rightarrow 0. \tag{13}$$

As \mathcal{M} is invertible at the points of $\mathbb{P}^n \setminus X$ the cokernel is supported on the hypersurface. Restrict the sequence to the hypersurface (and twist), then the kernel appears:

$$0 \rightarrow E_X \rightarrow \mathcal{O}_X^{\oplus d}(d-1) \xrightarrow{\mathcal{M}} \mathcal{O}_X^{\oplus d}(d) \rightarrow Coker(\mathcal{M})_X \rightarrow 0. \tag{14}$$

Sometimes we consider also the “left” kernel, E_X^l , the kernel of \mathcal{M}^T (called the Auslander transpose):

$$0 \rightarrow E_X^l \rightarrow \mathcal{O}_X^{\oplus d}(d-1) \xrightarrow{\mathcal{M}^T} \mathcal{O}_X^{\oplus d}(d) \rightarrow Coker(\mathcal{M}^T)_X \rightarrow 0. \tag{15}$$

From now on all the sheaves are considered on curves/hypersurfaces.

Example 2.11. Consider a smooth quadric surface $X = \{x_0x_1 = x_2x_3\} \subset \mathbb{P}^3$. By direct check, it has two (non-equivalent) determinantal representations: $\begin{pmatrix} x_0 & x_2 \\ x_3 & x_1 \end{pmatrix}$ and $\begin{pmatrix} x_0 & x_3 \\ x_2 & x_1 \end{pmatrix}$. Consider the first case, the kernel E_X is the line bundle spanned by two sections: $\begin{pmatrix} -x_3 \\ x_0 \end{pmatrix}$ and $\begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}$. To identify this line bundle recall that $X \approx \mathbb{P}_{left}^1 \times \mathbb{P}_{right}^1$ and the isomorphism can be written explicitly: $(x_0, x_1, x_2, x_3) \rightarrow ((x_0, x_2), (x_0, x_3))$. Note that both maps are well defined, using $(x_0, x_2) = (x_3, x_1)$ and $(x_0, x_3) = (x_2, x_1)$. The sections of E_X vanish at $x_0 = 0 = x_3$ and $x_2 = 0 = x_1$. Note that both cases define the divisors $pt \times \mathbb{P}_{right}^1 \subset X$. Thus $E_X \approx \mathcal{O}_{\mathbb{P}_{left}^1}^1(1) \boxtimes \mathcal{O}_{\mathbb{P}_{right}^1}^1$.

To work with singular points we consider the stalks of the kernel sheaves, i.e. kernel modules over the local ring $\mathcal{O}_{(X,0)}$. For them one has the corresponding exact sequence of modules. Many properties of kernels hold both in local and in global situation, usually we formulate and prove them together.

Both in local and global cases the kernel module/sheaf is spanned by the columns of the adjoint matrix \mathcal{M}^\vee ; see Theorem 4.1.

2.4.2. Geometric definition of the kernel

The kernel sheaves E_X, E_X^l can be defined also in a more geometric way [53, Section 3]. Suppose X is reduced, so for generic point $pt \in X$ the kernel $Ker(\mathcal{M}|_{pt})$ is a one-dimensional vector subspace of \mathbb{k}^d . Consider the rational map $\phi : X \dashrightarrow \mathbb{P}^{d-1}$ defined on the smooth points of X by

$$X \ni pt \xrightarrow{\phi} \{Ker(\mathcal{M}|_{pt}) \subset \mathbb{k}^d\} \rightarrow \mathbb{P}^{d-1}. \tag{16}$$

It extends to a morphism of algebraic varieties iff $corank(Ker \mathcal{M}|_X) \equiv 1$, i.e. $E|_X$ is a locally free sheaf. In general, consider a birational morphism resolving the singularities of the map ϕ ; see the diagram. Hence $v^*(E)/Torsion$ is a locally free sheaf and the pull-back $v^*(\phi)$ extends to a morphism. Recall that $\mathcal{O}_{\mathbb{P}^{d-1}}(-1)$ is the tautological bundle, its fiber over (x_0, \dots, x_{d-1}) is spanned by the vector (x_0, \dots, x_{d-1}) .

$$\begin{array}{ccc} \tilde{X} & & v^*(E)/Torsion \\ v \downarrow \searrow & & \\ X & \xrightarrow{\phi} & \mathbb{P}^{d-1} \quad \mathcal{O}_{\mathbb{P}^{d-1}}(-1) \end{array}$$

Proposition 2.12. *Let $X \subset \mathbb{P}^n$ be a reduced hypersurface.*

1. *If E_X is locally free and $\phi : X \dashrightarrow \mathbb{P}^{d-1}$ is the corresponding morphism, then $\phi^*(\mathcal{O}_{\mathbb{P}^{d-1}}(-1)) = E_X(1-d) \subset \mathcal{O}_X^{\oplus d}$.*
2. *If $v^*(E_X)/Torsion$ is locally free and $\tilde{X} \xrightarrow{\phi \circ v} \mathbb{P}^{d-1}$ is the corresponding morphism then $(\phi \circ v)^*(\mathcal{O}_{\mathbb{P}^{d-1}}(-1)) = v^*(E_X)(1-d)/Torsion$.*
3. *The kernel sheaf E_X is determined uniquely by $\phi^*(\mathcal{O}_{\mathbb{P}^{d-1}}(-1))$.*

Proof. Use the definition, i.e. Eq. (14), to get the following.

1. In the locally free case $E_X(1-d)$ and $\phi^*(\mathcal{O}_{\mathbb{P}^{d-1}}(-1))$ are two line subbundles of $\mathcal{O}_X^{\oplus d}$, whose fibers coincide at each point. So the bundles coincide tautologically.
2. Similarly, $v^*(E_X(1-d))/Torsion$ and $(\phi \circ v)^*(\mathcal{O}_{\mathbb{P}^{d-1}}(-1))$ are line subbundles on $\mathcal{O}_{\tilde{X}}^{\oplus d}$, with coinciding fibers.
3. Suppose there are two kernel sheaves corresponding to $\phi^*(\mathcal{O}_{\mathbb{P}^{d-1}}(-1))$, their restrictions onto the smooth locus of X coincide, as sub-sheaves of $\mathcal{O}^{\oplus d}(d-1)|_{X \setminus Sing(X)}$. Then, by Proposition 2.14, part 1, we get two determinantal representations, \mathcal{M}_1 and \mathcal{M}_2 , satisfying locally $\mathcal{M}_1 = A\mathcal{M}_2$. Here the entries of A are in $\mathcal{O}_{(\mathbb{k}^n, 0) \setminus Sing(X)}$ and A is locally invertible at each point of $(\mathbb{k}^n, 0) \setminus Sing(X)$. So, each entry of A is regular in codimension one, i.e. its possible locus of irregularity is of codimension at least two. But then this entry, being a rational function, is regular on $(\mathbb{k}^n, 0)$. Similarly, $\det(A) \neq 0$ except possibly for a subset of codimension two. Thus $\det(A) \neq 0$ on the whole $(\mathbb{k}^n, 0)$. Therefore the stalks of two kernel sheaves coincides everywhere on X . \square

For some determinantal representations of *non-reduced* hypersurfaces the kernel can be defined geometrically too; see Section 4.2.

Example 2.13. 1. Let $X \subset \mathbb{P}^2$ be a line arrangement, consider the simplest determinantal representation: the diagonal matrix $\mathcal{M} = (l_1, \dots, l_d)$. Here $\{l_i\}_i$ are linear forms defining the lines. The map $X \xrightarrow{\phi} \mathbb{P}^{d-1}$ is defined on the smooth locus of X , it sends each line to a point in \mathbb{P}^{d-1} .

2. In general, for a determinantal hypersurface $X = \cup_{\alpha} X_{\alpha} \subset \mathbb{P}^n$, let $X^0 \subset X$ be an open dense subset on which ϕ is defined. Then \mathcal{M} is decomposable, $\mathcal{M} \sim \oplus \mathcal{M}_{\alpha}$, iff $\phi(X^0) = \bigsqcup \text{Span}(X_{\alpha})$, where the spans ($\text{Span}(X_{\alpha}) \subset \mathbb{P}^{d-1}$ is the minimal linear subspace that contains $\phi(X_{\alpha})$) are mutually generic, i.e. $\forall \alpha: \text{Span}(X_{\alpha}) \cap \text{Span}(\prod'_{\beta \neq \alpha} X_{\beta}) = \emptyset$.

2.4.3. *Kernels vs determinantal representations*

Finally we formulate the relation between the embedded kernels and the determinantal representations. As always, in the local case, we assume $\mathcal{M}|_0 = \mathbb{O}$.

Proposition 2.14. *0. The kernel module of $\mathcal{M}_{(X,0)}$ is an $\mathcal{O}_{(X,0)}$ module minimally generated by the columns of the adjoint matrix $\mathcal{M}^{\vee}_{(X,0)}$. Similarly, for the kernel sheaf of \mathcal{M}_X , the columns of the adjoint matrix give the natural basis of the space $H^0(E_X)$. In particular $h^0(E_X) = d$.*

1. Let $\mathcal{M}_1, \mathcal{M}_2 \in \text{Mat}(d \times d, \mathcal{O}_{(\mathbb{k}^n,0)})$ be two local determinantal representations of the same hypersurface germ and $E_1, E_2 \subset \mathcal{O}_{(\mathbb{k}^n,0)}^{\oplus d}$ the corresponding embedded kernel modules. Then

$$\mathcal{M}_1 = \mathcal{M}_2 \quad \text{or} \quad \mathcal{M}_1 = A\mathcal{M}_2 \quad \text{or} \quad \mathcal{M}_1 = A\mathcal{M}_2B, \tag{17}$$

for A, B locally invertible on $(\mathbb{k}^n, 0)$

iff

$$\begin{aligned} (E_1, \{s^1_1 \cdots s^1_d\}) &= (E_2, \{s^2_1 \cdots s^2_d\}) \subset \mathcal{O}_{(\mathbb{k}^n,0)}^{\oplus d} \quad \text{or} \\ E_1 = E_2 \subset \mathcal{O}_{(\mathbb{k}^n,0)}^{\oplus d} \quad \text{or} \quad E_1 &\approx E_2 \end{aligned} \tag{18}$$

- 2. In particular, if two kernel modules of the same hypersurface are abstractly isomorphic then their isomorphism is induced by a unique ambient automorphism of $\mathcal{O}_{(\mathbb{k}^n,0)}^{\oplus d}$.
- 3. \mathcal{M} is decomposable (or equivalent to an upper-block-triangular form) iff E is a direct sum (or an extension).
- 4. Let $\mathcal{M}_1, \mathcal{M}_2 \in \text{Mat}(d \times d, |\mathcal{O}_{\mathbb{P}^n}(1)|)$ be two global determinantal representations of the same hypersurface, for $n > 1$. Let E_1, E_2 be the corresponding kernel sheaves. Then the global versions of all the statements above hold.

Remark 2.15. • In Eq. (18), in the first case the coincidence of the natural bases, in the second case the coincidence of the embedded modules, and in the third case the abstract isomorphism of modules are meant.

- Part 2 of the last proposition does not hold for arbitrary modules (not kernels). For example the ideals $\langle x^l \rangle \subset \mathbb{k}[x]$ for $l \geq 0$, are all abstractly isomorphic as (non-embedded) modules but certainly not as ideals, i.e. embedded modules.
- Note that the coincidence/isomorphism of kernel sheaves is a much stronger property than the pointwise coincidence of kernels as embedded vector spaces. For example, let \mathcal{M} be a local determinantal representation of the plane curve $C = \{f(x, y) = 0\}$. Let v_1, \dots, v_p be the columns of \mathcal{M}^{\vee} , i.e. the generators of the kernel E_C . Let $\{g_i = 0\}_{i=1}^p$ be some local curves intersecting C at the origin only. Then $\text{Span}(g_1v_1, \dots, g_pv_p)$ coincide pointwise with $\text{Span}(v_1, \dots, v_p)$ as a collection of embedded vector spaces on C . Though the two modules correspond to determinantal representations of distinct curves.

Proof of Proposition 2.14. 0. For modules. As $(\mathcal{M}\mathcal{M}^{\vee})|_{(X,0)} = \mathbb{O}$, the columns of $\mathcal{M}^{\vee}|_{(X,0)}$ generate a submodule of $E_{(X,0)}$. For any element $s \in E$, one has $\mathcal{M}s = \det(\mathcal{M})v$, where v is

some d -tuple. Then $\mathcal{M}(s - \mathcal{M}^\vee v) = 0$ on $(\mathbb{k}^n, 0)$. And \mathcal{M} is non-degenerate on $(\mathbb{k}^n, 0)$, so $s = \mathcal{M}^\vee v$.

For sheaves: the columns of \mathcal{M}^\vee generate a subsheaf of E . If $s \in H^0(E_X)$ then s is the column whose entries are sections of $\mathcal{O}_X(d - 1)$. By the surjection $H^0(\mathcal{O}_{\mathbb{P}^n}) \rightarrow H^0(\mathcal{O}_X) \rightarrow 0$ the entries of s are restrictions of some sections of $\mathcal{O}_{\mathbb{P}^n}$. Hence s is the restriction of some globally defined section S , for which $\mathcal{M}S = \det(\mathcal{M})(\cdot)$. But then $S = \mathcal{M}^\vee(\cdot)$. Hence s belongs to the span of the columns of \mathcal{M}^\vee , thus $h^0(E_X) = d$, i.e. $H^0(E_X)$ is generated by the columns of \mathcal{M}^\vee .

1, 2. As the kernel is spanned by the columns of \mathcal{M}^\vee the statement is straightforward, except possibly for the last part: if $E_1 \approx E_2$ then $\mathcal{M}_1 = A\mathcal{M}_2B$.

Let $\phi : E_1 \xrightarrow{\sim} E_2$ be an abstract isomorphism, i.e. an $\mathcal{O}_{(X,0)}$ -linear map. This provides an additional minimal free resolution of E_1 :

$$\begin{array}{ccccccc}
 0 & \rightarrow & E_1 & \rightarrow & \mathcal{O}_{(X,0)}^{\oplus d} & \xrightarrow{\mathcal{M}_1} & \mathcal{O}_{(X,0)}^{\oplus d} \cdots \\
 & & \phi \downarrow & & \psi \downarrow & & \\
 0 & \rightarrow & E_2 & \rightarrow & \mathcal{O}_{(X,0)}^{\oplus d} & \xrightarrow{\mathcal{M}_2} & \mathcal{O}_{(X,0)}^{\oplus d} \cdots
 \end{array} \tag{19}$$

By the uniqueness of minimal free resolution, [21, Section 20], we get that ψ is an isomorphism.

3. Suppose $E = E_1 \oplus E_2$, let $F_2 \xrightarrow{\mathcal{M}} F_1 \rightarrow E \rightarrow 0$ be the minimal resolution. Let $F_2^{(\alpha)} \xrightarrow{\mathcal{M}_\alpha} F_1^{(\alpha)} \rightarrow E_\alpha \rightarrow 0$ be the minimal resolutions of E_1, E_2 . Consider their direct sum:

$$F_2^{(1)} \oplus F_2^{(2)} \xrightarrow{\mathcal{M}_1 \oplus \mathcal{M}_2} F_1^{(1)} \oplus F_1^{(2)} \rightarrow E_1 \oplus E_2 = E \rightarrow 0. \tag{20}$$

This resolution of E is minimal. Indeed, by the decomposability assumption the number of generators of E is the sum of those of E_1, E_2 ; hence $rank(F_1) = rank(F_1^{(2)}) + rank(F_1^{(1)})$. Similarly, any linear relation between the generators of E (i.e. a syzygy) is the sum of relations for E_1 and E_2 . Hence $rank(F_2) = rank(F_2^{(2)}) + rank(F_2^{(1)})$.

Finally, by the uniqueness of the minimal resolution we get that the two proposed resolutions of E are isomorphic, hence the statement.

4. The statement, $E_1 \approx E_2$ implies $\mathcal{M}_1 = A\mathcal{M}_2B$, is proved for sheaves in [14, Theorem 1.1]. Note that in general it fails for $n = 1$; see [14, p. 425].

From this part the rest of the statements follows. \square

2.4.4. Maximally generated determinantal representations

Note that at each point $corank \mathcal{M}|_{pt} \leq mult(X, pt)$ (see Proposition 2.6). This motivates the following.

Definition 2.16. • A determinantal representation of a hypersurface is called *maximally generated* at the point $pt \in X$ (or Ulrich-maximal [52]) if $corank \mathcal{M}|_{pt} = mult(X, pt)$.

- A determinantal representation of a hypersurface is called *maximally generated near* the point if it is maximally generated at each point of some neighborhood of $pt \in X$.
- A determinantal representation of a hypersurface is called *generically maximally generated* if it is maximally generated at the generic smooth point of X_{red} .

Example 2.17. 1. The determinantal representations in Example 2.9 and in the first part of Example 2.13 are maximally generated. In fact, as follows from Property 2.18 below, for the ordinary multiple point (i.e. curve singularity with several smooth pairwise non-tangent branches) the diagonal matrix is the *unique* local maximally generated representation.

2. Any determinantal representation of a smooth hypersurface is maximally generated and any determinantal representation of a reduced hypersurface is generically maximally generated. If the (reduced) hypersurface germ has an isolated singularity then any determinantal representation is maximally generated on the punctured neighborhood of the singular point.
3. If \mathcal{M} is maximally generated at the generic smooth point of $X_{(red)}$ then it is maximally generated at any smooth of $X_{(red)}$, as the corank of the matrix does not increase under deformations.

Maximally generated determinantal representations are studied in [34]. They possess various excellent properties, in particular tend to be decomposable.

Property 2.18. 1. Let \mathcal{M} be a determinantal representation maximally generated at the generic smooth points of the (non-reduced) hypersurface $\{\prod f_\alpha^{p_\alpha} = 0\} \subset (\mathbb{k}^n, 0)$. Then any entry of \mathcal{M}^\vee is divisible by $\prod f_\alpha^{p_\alpha-1}$.

2. For the case $n = 2$, plane curves. Let $(X, 0) = (X_1, 0) \cup (X_2, 0) \subset (\mathbb{k}^2, 0)$; here $\{(X_\alpha, 0)\}_\alpha$ can be further reducible, non-reduced but without common components. Let \mathcal{M} be a maximally generated local determinantal representation of $(X, 0)$. Then \mathcal{M} is locally equivalent to $\begin{pmatrix} \mathcal{M}_1 & * \\ 0 & \mathcal{M}_2 \end{pmatrix}$, where \mathcal{M}_α are maximally generated determinantal representations of $(X_\alpha, 0)$.

If in addition the curve germs $(X_i, 0)$ have no common tangents then any maximally generated determinantal representation is decomposable, $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$.

3. For the case $n > 2$. Let $(X, 0) = (X_1, 0) \cup (X_2, 0) \subset (\mathbb{k}^n, 0)$, let $\mathcal{M}_{d \times d}$ be a determinantal representation of $(X, 0)$ and E its kernel module. Let $E|_{(X_i, 0)}/Torsion$ be the restriction to a component. Suppose that it is minimally generated by d_i elements. Similarly, suppose $E^l|_{(X_i, 0)}/Torsion$ is minimally generated by d_i^l elements.

The determinantal representation is equivalent to an upper block-triangular iff $d_1 + d_2^l = d$ or $d_1^l + d_2 = d$.

2.4.5. X'/X -saturated determinantal representations

Let $X' \xrightarrow{\nu} X$ be a finite modification, i.e. X' is a pure dimensional scheme, ν is a finite surjective proper morphism that is an isomorphism over $X \setminus Sing(X_{red})$. If X is reduced then the “maximal” modification is the normalization $\tilde{X} \rightarrow X$ and all other modifications are “intermediate”, $\tilde{X} \rightarrow X' \rightarrow X$.

Definition 2.19. A determinantal representation \mathcal{M} of a hypersurface is called X'/X -saturated if every entry of \mathcal{M}^\vee belongs to the relative adjoint ideal

$$Adj_{X'/X} = \{g \in \mathcal{O}_X | \nu^*(g)\mathcal{O}_{X'} \subset \nu^{-1}\mathcal{O}_X\}.$$

Any determinantal representation is X'/X -saturated for the identity morphism $X' \xrightarrow{\sim} X$. A determinantal representation is X'/X -saturated iff its kernel is a X'/X -saturated module, i.e. $E = \nu_*(\nu^*E/Torsion)$.

As in the maximally generated case, the X'/X -saturated determinantal representations possess various excellent properties, in particular tend to be decomposable.

Property 2.20 ([34]).

1. Consider the modification

$$(X', 0) = (X_1, 0) \coprod (X_2, 0) \xrightarrow{\nu} (X, 0) = (X_1, 0) \cup (X_2, 0) \tag{21}$$

which is the separation of the components. Then \mathcal{M} is decomposable, $\mathcal{M} \sim \mathcal{M}_1 \oplus \mathcal{M}_2$, iff it is X'/X -saturated.

2. In particular, if $(X, 0) = \cup_{\alpha}(X_{\alpha}, 0)$ is the union of smooth hypersurface germs and $(X', 0) = \coprod_{\alpha}(X_{\alpha}, 0)$, then any X'/X -saturated determinantal representation of $(X, 0)$ is equivalent to the diagonal matrix (in particular it is maximally generated).
3. If the determinantal representation \mathcal{M} of a plane curve is \tilde{C}/C -saturated, where $\tilde{C} \xrightarrow{\nu} C$ is the normalization, then \mathcal{M} is maximally generated and the entries of \mathcal{M}^{\vee} generate the adjoint ideal $\text{Adj}_{\tilde{C}/C}$.

Finally we state an additional decomposability criterion from [34].

Property 2.21. Let $(X, 0) = \cup(X_{\alpha}, 0) \subset (\mathbb{k}^n, 0)$ be a collection of reduced, smooth hypersurfaces. The determinantal representation \mathcal{M} of $(X, 0)$ is completely decomposable iff the geometric fibers $\{E_{\alpha}|_0\}$ are linearly independent: $\text{Span}(\cup E_{\alpha}|_0) = \oplus E_{\alpha}|_0$.

3. Global decomposability

The local decomposability at each point implies the global one.

Theorem 3.1. Let $X = X_1 \cup X_2 \subset \mathbb{P}^n$ be a global decomposition of the hypersurface. Here X_1, X_2 can be further reducible, non-reduced, but without common components. Then \mathcal{M} is globally decomposable, i.e. $\mathcal{M} \stackrel{\text{globally}}{\sim} \mathcal{M}_1 \oplus \mathcal{M}_2$, iff it is locally decomposable at each point $pt \in X$, i.e. $\mathcal{M} \stackrel{\text{locally}}{\sim} \mathbb{1} \oplus \mathcal{M}_1|_{(\mathbb{P}^n, pt)} \oplus \mathcal{M}_2|_{(\mathbb{P}^n, pt)}$. Here $\mathcal{M}_{\alpha}|_{(\mathbb{P}^n, pt)}$ are the local determinantal representations near $pt \in \mathbb{P}^n$, one works over $\mathcal{O}_{(\mathbb{k}^n, 0)} = \mathbb{k}[x_1, \dots, x_n]_{(m)}$.

Proof. \Leftarrow Let $f = f_1 f_2$ be the homogeneous polynomials defining X, X_1, X_2 . Here f_{α} can be reducible, non-reduced, but mutually prime.

Part 1. By the assumption at each point $\mathcal{M}^{\vee} \stackrel{\text{locally}}{\sim} f\mathbb{1} \oplus f_2\mathcal{M}^{\vee}_1 \oplus f_1\mathcal{M}^{\vee}_2$. The local ideal of $\mathcal{O}_{(\mathbb{k}^n, 0)}$ generated by the entries of \mathcal{M}^{\vee} is invariant under local equivalence. Hence we get: any entry of \mathcal{M}^{\vee} at any point $pt \in \mathbb{P}^n$ belongs to the local ideal $\langle f_1, f_2 \rangle \subset \mathcal{O}_{(\mathbb{k}^n, 0)}$.

Now use Noether’s $AF + BG$ theorem [1, p. 139]: Given some homogeneous polynomials $F_1 \dots F_k$, whose zeros define a subscheme of \mathbb{P}^n of dimension $(n - k)$. Suppose at each point of \mathbb{P}^n the homogeneous polynomial G belongs to the local ideal generated by $F_1 \dots F_k$. Then G belongs to the global ideal in $\mathbb{k}[x_0, \dots, x_n]$ generated by $F_1 \dots F_k$.

Apply this to each entry of \mathcal{M}^{\vee} . Hence we get, in the matrix notation:

$$\mathcal{M}^{\vee} = f_2\mathcal{N}^{\vee}_1 + f_1\mathcal{N}^{\vee}_2, \quad \mathcal{N}^{\vee}_{\alpha} \in \text{Mat}\left(d \times d, H^0(\mathcal{O}_{\mathbb{P}^n}(deg(f_{\alpha}) - 1))\right). \tag{22}$$

Part 2. From the last equation we get: $f_1 f_2 \mathbb{1} = \mathcal{M}\mathcal{M}^{\vee} = f_2\mathcal{M}\mathcal{N}^{\vee}_1 + f_1\mathcal{M}\mathcal{N}^{\vee}_2$. Note that f_1, f_2 are relatively prime, thus: $\mathcal{M}\mathcal{N}^{\vee}_{\alpha} = f_{\alpha}A_{\alpha}$. Here A_{α} is a $d \times d$ matrix whose entries are forms of degree zero, i.e. constants. Similarly, by considering $\mathcal{M}^{\vee}\mathcal{M}$ we get $\mathcal{N}^{\vee}_{\alpha}\mathcal{M} = f_{\alpha}B_{\alpha}$.

Note that $A_1 + A_2 = \mathbb{1} = B_1 + B_2$, by their definition. In addition $A_\alpha A_\beta = \mathbb{0} = B_\alpha B_\beta$ for $\alpha \neq \beta$. Indeed: $f_\alpha B_\alpha \mathcal{N}^\vee_\beta = \mathcal{N}^\vee_\alpha \mathcal{M} \mathcal{N}^\vee_\beta = f_\beta \mathcal{N}^\vee_\alpha A_\beta$. So $B_\alpha \mathcal{N}^\vee_\beta$ is divisible by f_β and $\mathcal{N}^\vee_\alpha A_\beta$ is divisible by f_α . But $\{f_\alpha\}$ are mutually prime and the degree of the entries in \mathcal{N}^\vee_α is $d_\alpha - 1$. Therefore: $B_\alpha \mathcal{N}^\vee_\beta = 0 = \mathcal{N}^\vee_\alpha A_\beta$ (for $\alpha \neq \beta$). And this causes $B_\alpha (\mathcal{N}^\vee_\beta \mathcal{M}) = 0 = (\mathcal{M} \mathcal{N}^\vee_\alpha A_\beta)$.

Thus $\{A_\alpha\}$ and $\{B_\alpha\}$ form a partition of identity, i.e. $\oplus A_\alpha = \mathbb{1} = \oplus B_\alpha$. So, one can bring the collections $\{A_\alpha\}, \{B_\alpha\}$ to the block-diagonal form:

$$\begin{pmatrix} \tilde{A}_1 & 0 \\ 0 & \tilde{A}_2 \end{pmatrix} = \mathbb{1} = \begin{pmatrix} \tilde{B}_1 & 0 \\ 0 & \tilde{B}_2 \end{pmatrix}. \tag{23}$$

This is done by the multiplication $\mathcal{M} \rightarrow U_1 \mathcal{M} U_2$ (and accordingly $\mathcal{M}^\vee \rightarrow U_2^{-1} \mathcal{M}^\vee U_1^{-1}$), which acts on A, B as $A_\alpha \sim \mathcal{M}^\vee_\alpha \mathcal{M} \rightarrow U_2^{-1} A_\alpha U_2$ and $B_\alpha \sim \mathcal{M} \mathcal{M}^\vee_\alpha \rightarrow U_1 B_\alpha U_1^{-1}$.

Then from the definitions $\mathcal{N}^\vee_\alpha \mathcal{M} \sim B_\alpha$ and $\mathcal{M} \mathcal{N}^\vee_\alpha \sim A_\alpha$ one gets: $\mathcal{N}^\vee_\alpha \sim B_\alpha \mathcal{M}^\vee$ and $\mathcal{N}^\vee_\alpha \sim \mathcal{M}^\vee A_\alpha$. So, \mathcal{N}^\vee_α is just one block on the diagonal. Thus \mathcal{M}^\vee is block diagonal.

Finally note that from $\mathcal{N}^\vee_\alpha \mathcal{M} = f_\alpha A_\alpha$ it follows that $\det(\mathcal{N}^\vee_\alpha) f_\alpha = f_\alpha^{d_\alpha} \times \text{const}$. So the multiplicities are determined uniquely. \square

If we combine the theorem with [Property 2.18](#), and [Property 2.20](#) we get the following.

Corollary 3.2. *Let $X = \cup p_\alpha X_\alpha \subset \mathbb{P}^n$ be the global decomposition into distinct irreducible reduced projective hypersurfaces.*

1. *Let $X' = \coprod p_\alpha X_\alpha \xrightarrow{v} \cup p_\alpha X_\alpha$ be the separation of components (a finite modification). Any X'/X -saturated determinantal representation is globally completely decomposable, $\mathcal{M} \sim \oplus \mathcal{M}_\alpha$, where \mathcal{M}_α is a determinantal representation of $p_\alpha X_\alpha$.*
2. *Suppose for each point $pt \in X$ any two components passing through this point $pt \in X_\alpha \cap X_\beta$ intersect generically transverse, i.e. $(X_\alpha \cap X_\beta)$ is reduced. Then any maximally generated determinantal representation of X is globally completely decomposable, $\mathcal{M} \sim \oplus \mathcal{M}_\alpha$.*

It is interesting that this local-to-global theorem, a non-trivial result in linear algebra, is immediate if viewed as a statement about the kernel sheaves.

Proposition 3.3. *Let $X = X_1 \cup X_2 \subset \mathbb{P}^n$, here X_α can be further reducible, non-reduced, but with no common components. Let \mathcal{M} be a global determinantal representation of X .*

1. *Suppose at each point $pt \in X_1 \cap X_2$ the determinantal representation is locally decomposable, i.e. $\mathcal{M} \overset{loc}{\sim} \mathcal{M}_1 \oplus \mathcal{M}_2$ with \mathcal{M}_α the local determinantal representation of X_α . Then \mathcal{M} is globally decomposable.*
2. *Suppose at each point $pt \in X_1 \cap X_2$ the determinantal representation can be brought locally to the upper block-triangular form, i.e. $\mathcal{M} \overset{loc}{\sim} \begin{pmatrix} \mathcal{M}_1 & \dots \\ \mathbb{0} & \mathcal{M}_2 \end{pmatrix}$ with \mathcal{M}_α the local determinantal representation of X_α . Then \mathcal{M} is globally equivalent to an upper-block-triangular matrix.*

Proof. For $n = 1$ any determinantal representation is completely decomposable; hence we assume $n > 1$ and X is connected.

1. Let E be the kernel sheaf of \mathcal{M}_X , let $X_\alpha \xrightarrow{i_\alpha} X$ be the natural embeddings. Define the restriction to the component: $E_\alpha := i_\alpha^*(E)/Torsion$.

Consider the new sheaf on X : $E' := (i_1)_*(E_1) \oplus (i_2)_*(E_2)$. It is a coherent sheaf and by construction there is a natural globally defined map: $E \rightarrow E'$, the direct sum of two restrictions. We claim that it is isomorphism on all the stalks. It is an isomorphism on $X \setminus (X_1 \cap X_2)$, so the kernel of this map would be a torsion subsheaf of E . But E is torsion-free; hence the map is injective. So, one only need to check the surjectivity (over $X_1 \cap X_2$), which holds by construction.

Therefore by the axiom of sheaf the global map is an isomorphism too:

$$E \xrightarrow{\sim} E' = i_1^*(E_1) \oplus i_2^*(E_2). \tag{24}$$

So, by the global automorphism of the sheaf, i.e. a linear operator with constant coefficient, \mathcal{M} is brought to the block diagonal form; see Proposition 2.14.

2. Define the sheaf E' on X as the extension

$$0 \rightarrow (i_1)_*(E_1) \rightarrow E' \rightarrow (i_2)_*(E_2) \rightarrow 0, \tag{25}$$

where at each intersection point of $X_1 \cap X_2$ the stalk of E' is defined by its extension class: $[E'] = [E] \in Ext((i_2)_*(E_2), (i_1)_*(E_1))$. Again one has the natural map $E \rightarrow E'$, which is an isomorphism on stalks; hence a global isomorphism. \square

Example 3.4. Consider a reducible (reduced) curve $X = X_1 \cup X_2 \subset \mathbb{P}^2$, suppose all the intersections of X_1, X_2 are transverse, i.e. the corresponding singularities are nodes. Let \mathcal{M} be a determinantal representation of X . Then it either splits, $\mathcal{M} \sim \mathcal{M}_1 \oplus \mathcal{M}_2$, or for at least one intersection point, $pt \in X_1 \cap X_2$, the corank is minimal: $corank(\mathcal{M}|_{pt}) = 1$.

4. (Co-)kernel sheaves of determinantal representations

4.1. Properties and classification of kernels on the hypersurface

Recall (from Section 2.4) that the (co)kernel sheaves are defined by

$$\begin{aligned} 0 \rightarrow E_X \rightarrow \mathcal{O}_X^{\oplus d}(d-1) \xrightarrow{\mathcal{M}} \mathcal{O}_X^{\oplus d}(d) \rightarrow Coker(\mathcal{M})_X \rightarrow 0, \\ 0 \rightarrow E_X^l \rightarrow \mathcal{O}_X^{\oplus d}(d-1) \xrightarrow{\mathcal{M}^T} \mathcal{O}_X^{\oplus d}(d) \rightarrow Coker(\mathcal{M}^T)_X \rightarrow 0. \end{aligned} \tag{26}$$

Theorem 4.1. 1. The sheaf E_X has periodic resolutions:

$$\begin{aligned} \dots \xrightarrow{\mathcal{M}} \mathcal{O}_X^{\oplus d}(-d) \xrightarrow{\mathcal{M}^\vee} \mathcal{O}_X^{\oplus d}(-1) \xrightarrow{\mathcal{M}} \mathcal{O}_X^{\oplus d} \xrightarrow{\mathcal{M}^\vee} E_X \rightarrow 0, \\ 0 \rightarrow E_X \rightarrow \mathcal{O}_X^{\oplus d}(d-1) \xrightarrow{\mathcal{M}} \mathcal{O}_X^{\oplus d}(d) \xrightarrow{\mathcal{M}^\vee} \mathcal{O}_X^{\oplus d}(2d-1) \dots \end{aligned}$$

In particular, for any point $pt \in X$, the stalk $E_{(X,pt)}$ is a Cohen–Macaulay module over $\mathcal{O}_{(X,pt)}$.

2. $h^0(E_X(-1)) = 0$. $h^i(E_X(j)) = 0$, for $0 < i < n - 1$ and $j \in \mathbb{Z}$. $h^{n-1}(E_X(-j)) = 0$, for $j < n$. In particular $\chi(E(-j)) = 0$ for $j = 1, \dots, n - 1$.
3. The sheaves E_X and $Coker(\mathcal{M})_X$ are torsion free.
4. The sheaf E_X is quasi-locally free iff $dim(Ker \mathcal{M}|_{pt}) = const$ on X . The sheaf E_X is locally free iff $dim(Ker \mathcal{M}|_{pt}) = 1 = const$ on X . In particular if the hypersurface is smooth the kernel sheaf is invertible.

5. Suppose $X \subset \mathbb{P}^n$ is reduced and E_X is locally free. Then $n < 4$ and

$$c_1(E_X)L^{n-2} = \frac{(d-1)}{2}L^{n-1}, \quad c_1^2(E_X)L^{n-3} = \frac{(d-1)(d-2)}{6}L^{n-1},$$

where $[L] \in H^2(X, \mathbb{Z})$ is the hyperplane class and $\deg(L^{n-1}) = d$, the degree of the zero-dimensional cycle.

All the properties hold for E_X^l (is generated by the columns of $(\mathcal{M}^\vee|_X)^T$ etc.).

Proof. 1. Follows immediately from Proposition 2.14.

2. Let $I_X \subset \mathcal{O}_{\mathbb{P}^n}$ be the defining ideal of the hypersurface, i.e. $0 \rightarrow I_X \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_X \rightarrow 0$. Note that $I_X \approx \mathcal{O}_{\mathbb{P}^n}(-d)$ and $h^{i>0}(\mathcal{O}_{\mathbb{P}^n}(j)) = 0$ for $i \neq n$ or $j + n + 1 > 0$. Therefore the long exact sequence of cohomology gives

$$\begin{aligned} h^i(\mathcal{O}_X(j)) &= 0 \quad \text{for } 0 < i < n - 1, j \in \mathbb{Z}, \quad \text{and} \\ h^{n-1}(\mathcal{O}_X(j)) &= 0 \quad \text{for } j > d - n - 1. \end{aligned} \tag{27}$$

In addition $H^0(\mathcal{O}_{\mathbb{P}^n}(j)) \rightarrow H^0(\mathcal{O}_X(j)) \rightarrow 0$ for $n > 2$ or for $n = 2, j > d - 3$.

Now we prove that $h^0(E_X(-1)) = 0$. Let $0 \neq s \in H^0(E_X(-1))$, i.e. s is a d -tuple whose entries are sections of $\mathcal{O}_X(d - 2)$. As $H^0(\mathcal{O}_{\mathbb{P}^n}(d - 2)) \rightarrow H^0(\mathcal{O}_X(d - 2)) \rightarrow 0$, we get a global section $S \in H^0(\mathcal{O}_{\mathbb{P}^n}^{\oplus d}(d - 2))$ that restricts to s on the hypersurface. Thus $\mathcal{M}_{\mathbb{P}^n} S$ is proportional to f . But the total degree of any entry of $\mathcal{M}_{\mathbb{P}^n} S$ is $(d - 1)$, which is less than $\deg(f) = d$. So $\mathcal{M}_{\mathbb{P}^n} S = 0$, contradicting the global non-degeneracy of \mathcal{M} .

The vanishing $h^{i>0}(E_X(j)) = 0$. Use the exact sequence

$$0 \rightarrow E_X(-1) \rightarrow \mathcal{O}_X^{\oplus d}(d - 2) \xrightarrow{\mathcal{M}} \text{Im}(\mathcal{M}) \rightarrow 0. \tag{28}$$

Twist it by $\mathcal{O}(-j)$, for $j \geq 0$, and note that $h^0(E_X(-1 - j)) = 0$. Further, note that the map $H^0(\mathcal{O}_X^{\oplus d}(d - 2 - j)) \rightarrow H^0(\text{Im}(\mathcal{M})(-j))$ is surjective. Indeed, $\text{Im}(\mathcal{M}) \subset \mathcal{O}_X^{\oplus d}(d - 1)$; hence any global section of $\text{Im}(\mathcal{M})(-j)$ is the restriction of some section of $\mathcal{O}_{\mathbb{P}^n}^{\oplus d}(d - j - 1)$. The sequence of cohomologies consists of the parts:

$$\begin{aligned} H^i(\mathcal{O}_X^{\oplus d}(d - 2 - j)) &\rightarrow H^i(\text{Im}(\mathcal{M})(-j)) \rightarrow H^{i+1}(E_X(-1 - j)) \\ &\rightarrow H^{i+1}(\mathcal{O}_X^{\oplus d}(d - 2 - j)). \end{aligned} \tag{29}$$

For $n = 2$ the case $j = 0 = i$ gives $h^1(E_X(-1)) = 0$, proving the statement.

Suppose $n > 2$. Then the last sequence for $i = 0$ and $j \geq 0$ gives: $H^1(E_X(-1 - j)) = 0$. Further, the sheaf $\text{Im}(\mathcal{M})$ can be considered as the kernel sheaf of $\mathcal{O}_X^{\oplus d}(d - 1) \xrightarrow{\mathcal{M}^\vee} \mathcal{O}_X^{\oplus d}(2d - 2)$. Therefore we have the additional exact sequence

$$0 \rightarrow \text{Im}(\mathcal{M}) \rightarrow \mathcal{O}_X^{\oplus d}(d - 1) \xrightarrow{\mathcal{M}^\vee} E_X(d - 1) \rightarrow 0. \tag{30}$$

Twist it and take cohomology:

$$\begin{aligned} H^i(\mathcal{O}_X^{\oplus d}(-j)) &\rightarrow H^i(E_X(-j)) \rightarrow H^{i+1}(\text{Im}(\mathcal{M})(1 - d - j)) \\ &\rightarrow H^{i+1}(\mathcal{O}_X^{\oplus d}(-j)). \end{aligned} \tag{31}$$

Similarly to $H^1(E_X(-1 - j)) = 0$ one gets $H^1(\text{Im}(\mathcal{M})(-j)) = 0$. These are the ‘‘initial conditions’’. Combine now Eqs. (29) and (31) with the initial conditions to get

$$H^i(E_X(-j)) = 0, \quad H^i(\text{Im}(\mathcal{M})(-j)) = 0, \quad \text{for } 0 < i < n - 1, j \geq 0 \tag{32}$$

while for $i = n - 1$ one has:

$$\begin{aligned} 0 \rightarrow H^{n-1}(E_X(-1 - j)) \rightarrow H^{n-1}(\mathcal{O}_X^{\oplus d}(d - 2 - j)) \\ \rightarrow H^{n-1}(Im(\mathcal{M})(-j)) \rightarrow 0 \end{aligned} \tag{33}$$

giving $H^{n-1}(E_X(-1 - j)) = 0$ for $j < n - 1$.

3. The sheaves E, E^l are torsion-free as sub-sheaves of the torsion-free sheaf $\mathcal{O}_X^{\oplus d}(d - 1)$.

To check that $Coker(\mathcal{M}|_X), Coker(\mathcal{M}^T|_X)$ are torsion free it is enough to consider their stalks at a point. Then for $s \in \mathcal{O}_{(X,0)}^{\oplus d} / \mathcal{M}\mathcal{O}_{(X,0)}^{\oplus d}$ and $g \in \mathcal{O}_{(X,0)}$ not a zero divisor we should prove: if $gs = 0 \in \mathcal{O}_{(X,0)}^{\oplus d} / \mathcal{M}\mathcal{O}_{(X,0)}^{\oplus d}$ then $s \in \mathcal{M}\mathcal{O}_{(X,0)}^{\oplus d}$.

But if $gs \in \mathcal{M}\mathcal{O}_{(X,0)}^{\oplus d}$ then $\mathcal{M}^\vee gs \in \mathcal{M}^\vee \mathcal{M}\mathcal{O}_{(X,0)}^{\oplus d} = \det(\mathcal{M})\mathcal{O}_{(X,0)}^{\oplus d} \equiv 0$. So $\mathcal{M}^\vee_X s = 0$; hence $s = \mathcal{M}s_1$.

4. Suppose E_X is quasi-locally free at $0 \in (X, 0)$, i.e. the stalk of its associated graded sheaf (Section 2.1) is a free $\mathcal{O}_{(X_{red},0)}$ module. We can assume $\mathcal{M}|_0 = \mathbb{O}$. Let s_1, \dots, s_k be the generators of $E_{(X,0)}$; hence if $\sum g_i s_i = 0$ for some $\{g_i \in \mathcal{O}_{(X_{red},0)}\}$ then $\{g_i = 0 \in \mathcal{O}_{(X_{red},0)}\}$.

But $\{s_1, \dots, s_k\}$ are columns of \mathcal{M}^\vee , so $(s_1, \dots, s_k)\mathcal{M} = \det(\mathcal{M})\mathbb{1} = \mathbb{O}$. Hence $\mathcal{M}|_{(X_{red},0)} = \mathbb{O}$. In particular $corank(\mathcal{M}|_X) = const$.

If E_X is locally free then by the same argument get $\mathcal{M}|_{(X,0)} = \mathbb{O}$, which is possible only if \mathcal{M} is a 1×1 matrix.

5. If the kernel sheaf is locally free then its fiber at each point of X is of dimension one. Hence at the points where $corank \mathcal{M} > 1$ the kernel cannot be locally free. We claim that the locus of such point is of codimension at most 4 in \mathbb{P}^n . This locus is defined by the vanishing of all the maximal minors of \mathcal{M} . Hence we can use the standard fact: inside the parameter space of all the $d \times d$ matrices (with constant coefficients) the locus of matrices of corank at least two has codimension four.

Assume local freeness of E_X , and hence $n < 4$. As was proved above $\chi(E(-j)) = 0$ for $j = 1, \dots, n - 1$. Apply the Hirzebruch–Riemann–Roch theorem; see Section 2.1. For $n = 2$ we get

$$0 = \chi(E(-1)) = deg(E(-1)) - (p_a - 1), \quad \rightsquigarrow deg(E) = \frac{d(d - 1)}{2}. \tag{34}$$

For $n = 3$ we get $(ch(E(-1 - j))Td(X))_{top.dim.} = 0$ for $j = 0, 1$. Note that $ch(E(-1 - j)) = ch(E)ch((-1 - j)L)$. Hence we get the system of two equations:

$$\left(\left(\sum_{i \geq 0} \frac{((-1 - j)L)^i}{i!} \right) ch(E)Td(X) \right)_{top.dim.} = 0, \quad j = 0, 1. \tag{35}$$

The first bracket produces the matrix: $\begin{pmatrix} 1 & -L & \frac{(-L)^2}{2!} \\ 1 & -2L & \frac{(-2L)^2}{2!} \end{pmatrix}$. Apply row operations to bring this

matrix to the form: $\begin{pmatrix} 1 & 0 & \frac{-L^2}{2} \\ 0 & -L & \frac{3L^2}{2} \end{pmatrix}$. Now substitute this to the initial equations to get

$$\begin{aligned} L \left(c_1(E_X) + \frac{c_1(T_X)}{2} \right) &= \frac{3L^2}{2}, \\ \left(\frac{c_1^2(E_X)}{2} + c_1(E_X) \frac{c_1(T_X)}{2} + \frac{c_1^2(T_X) + c_2(T_X)}{12} \right) &= L^2. \end{aligned} \tag{36}$$

Finally, use Example 2.5. \square

Remark 4.2. The vector bundles satisfying $h^{0 < i < \dim(X)}(E_X(j)) = 0$ are called ACM (arithmetically Cohen–Macaulay) bundles. For example on smooth quadrics in \mathbb{P}^n there are only two such bundles (up to a twist): the trivial and the spinor bundle. For a recent review cf. [45].

It is possible to give a simple criterion for a torsion free sheaf to be the kernel of some determinantal representation.

Theorem 4.3. *Let $X = \cup p_\alpha X_\alpha \subset \mathbb{P}^n$ be a hypersurface of degree d . Let E_X be a torsion-free sheaf of multi-rank $\{p_\alpha\}_\alpha$, suppose at each point of X the stalk $E|_{(X,pt)}$ is a maximally Cohen–Macaulay module over $\mathcal{O}_{(X,pt)}$. Assume $h^0(E_X(-1)) = 0$, $h^i(E_X(j)) = 0$ for $0 < i < n - 1$ and $j \in \mathbb{Z}$, and $h^{n-1}(E_X(1 - n)) = 0$. Then E_X is the kernel sheaf of a determinantal representation of X , i.e.*

$$0 \rightarrow E_X \rightarrow \mathcal{O}_X^{\oplus d}(d - 1) \xrightarrow{\mathcal{M}} \mathcal{O}_X^{\oplus d}(d) \rightarrow \dots, \tag{37}$$

where \mathcal{M} is a determinantal representation of $X \subset \mathbb{P}^n$.

Proof. *Step 1.* The preparation. Define the auxiliary sheaf: $E_X^l := E_X^* \otimes w_X(n)$. As X is Gorenstein, the dualizing sheaf w_X is invertible; hence E_X^l is torsion-free. By Serre duality: $h^i(E_X^l(j)) = h^{n-1-i}(E_X(-n - j))$. Hence $h^i(E_X^l(j)) = 0$ for $0 < i < n - 1$, $j \in \mathbb{Z}$ and $h^0(E_X^l(-1)) = h^{n-1}(E_X(1 - n)) = 0$ and $h^{n-1}(E_X^l(1 - n)) = h^0(E_X(-1)) = 0$.

Now we prove that $h^0(E_X^l) = d = h^0(E_X)$. If $n = 2$, i.e. X is a curve, and E_X is invertible, then this follows from Riemann–Roch:

$$\chi(E_X) - \chi(E_X(-1)) = \deg(E_X) - \deg(E_X(-1)) = \deg(\mathcal{O}_X(1)) = d. \tag{38}$$

In higher dimensions one can argue as follows. Let $L \subset \mathbb{P}^n$ be a line. We assume L is generic enough, such that it intersects the reduced hypersurface X_{red} at smooth points only and the stalks of E at $Z = L \cap X$ are quasi-locally free (cf. Section 2.1). Let $I_Z \subset \mathcal{O}_X$ be the defining sheaf of ideals:

$$0 \rightarrow I_Z \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Z \rightarrow 0. \tag{39}$$

The line L is the intersection of $(n - 1)$ hyperplanes; hence I_Z admits Koszul resolution. Tensor the exact sequence above with E_X :

$$\begin{aligned} Tor_{\mathcal{O}_X}^1(I_Z, E_X) &\rightarrow \underbrace{Tor_{\mathcal{O}_X}^1(\mathcal{O}_X, E_X)}_{=0} \rightarrow Tor_{\mathcal{O}_X}^1(\mathcal{O}_Z, E_X) \\ &\rightarrow E_X \otimes I_Z \rightarrow E_X \rightarrow E_X \otimes \mathcal{O}_Z \rightarrow 0. \end{aligned} \tag{40}$$

We claim that $Tor_{\mathcal{O}_X}^1(\mathcal{O}_Z, E_X) = 0$. Indeed, let $\rightarrow P_2 \rightarrow P_1 \rightarrow E_X \rightarrow 0$ be a projective resolution, then $Tor_{\mathcal{O}_X}^1(\mathcal{O}_Z, E_X)$ is the cohomology of the tensored complex at the place $\rightarrow P_2 \otimes \mathcal{O}_Z \rightarrow$. As Z consists of several generic points of X , the latter complex is exact, hence the cohomology vanishing. Then from the exact sequence above we also get $Tor_{\mathcal{O}_X}^1(I_Z, E_X) = 0$.

Finally, take the cohomology of the sequence $0 \rightarrow E_X \otimes I_Z \rightarrow E_X \rightarrow E_X \otimes \mathcal{O}_Z \rightarrow 0$. We claim that $h^1(E_X \otimes I_Z) = 0$. Indeed, the resolution of I_Z is $0 \rightarrow Syz \rightarrow \mathcal{O}_X^{\oplus d}(-1) \rightarrow I_Z \rightarrow 0$, where Syz is the syzygy module. Multiply this by E_X , as mentioned above $Tor_{\mathcal{O}_X}^1(I_Z, E_X) = 0$; thus $0 \rightarrow Syz \otimes E_X \rightarrow \mathcal{O}_X^{\oplus d}(-1) \otimes E_X \rightarrow I_Z \otimes E_X \rightarrow 0$. Now take the cohomology,

and use $h^i(E_X(-1)) = 0$ to get $h^i(I_Z \otimes E_X) = h^{i+1}(Syz \otimes E_X)$. As I_Z has Koszul resolution the resolution of syzygy module is Koszul too (and shorter). Hence we get a reduction in codimension. Finally, using $h^i(E_X^{\oplus(n-1)}(-1)) = (n-1)h^i(E_X(-1)) = 0$ we have $h^1(I_Z \otimes E_X) = 0$.

Therefore:

$$h^0(E_X) = h^0(E_X \otimes \mathcal{O}_Z) = \text{length}(E_X \otimes \mathcal{O}_Z) = \sum_{pt \in \text{Supp}(Z)} \text{mult}(X, pt) = d. \tag{41}$$

Here we used Proposition 2.4: the local rank of E at the smooth point of X_α is p_α .

Similarly for $h^0(E_X^l) = d$.

Step 2. Construction of the candidate for \mathcal{M}^\vee . Consider the global sections $H^0(E_X, X) = \text{Span}(s_1 \dots s_d)$ and $H^0(E_X^l, X) = \text{Span}(s_1^l \dots s_d^l)$. By the construction of E_X, E_X^l one has the pairing:

$$\left(H^0(E_X), H^0(E_X^l) \right) \rightarrow H^0(w_X(n)) \approx H^0(\mathcal{O}_X(d-1)). \tag{42}$$

The later isomorphism is due to the adjunction for hypersurface in \mathbb{P}^n : $w_X \approx \mathcal{O}_X(d-n-1)$.

By the surjection $H^0(\mathcal{O}_{\mathbb{P}^n}(d-1)) \rightarrow H^0(\mathcal{O}_X(d-1)) \rightarrow 0$, the chosen basis of global sections defines a matrix of homogeneous polynomials:

$$\langle s_i, s_j^l \rangle \rightarrow \{A_{ij}\} \in \text{Mat} \left(d \times d, H^0(\mathcal{O}_{\mathbb{P}^n}(d-1)) \right). \tag{43}$$

The entries A_{ij} are defined up to the global sections of $I_X(d-1) \subset \mathcal{O}_{\mathbb{P}^n}(d-1)$. As $I_X \approx \mathcal{O}_{\mathbb{P}^n}(-d)$ we get $h^0(I_X(d-1)) = h^0(\mathcal{O}_{\mathbb{P}^n}(-1)) = 0$. Hence the entries A_{ij} are defined uniquely.

Step 3. We prove that the matrix A is globally non-degenerate. First consider the case of reduced hypersurface, so E, E^l are of constant rank 1. It is enough to show that the restriction of A to a line in \mathbb{P}^n is non-degenerate.

Let $L \subset \mathbb{P}^n$ be the generic line, such that $L \cap X = \{pt_1 \dots pt_d\}$ is the set of distinct reduced points. Consider the restriction $A|_L$, i.e. a determinantal representation of $L \cap X$ in \mathbb{P}^1 .

By linear transformations applied to $\text{Span}(s_1 \dots s_d)$ and $\text{Span}(s_1^l \dots s_d^l)$ we can choose the basis of sections satisfying the conditions:

$$L \cap X \supset \text{div}(s_i) \geq \sum_{j \neq i} pt_j \not\geq pt_i, \quad L \cap X \supset \text{div}(s_i^l) \geq \sum_{j \neq i} pt_j \not\geq pt_i. \tag{44}$$

Indeed, suppose $s_1|_{pt_1} \neq 0$, then we can assume $s_{i>1}|_{pt_1} = 0$ and continue with the remaining points $\{pt_{i>1}\}$ and sections $\{s_{i>1}\}$. At each step there is at least one section that does not vanish at the given point. Otherwise at the end we get a section vanishing at all the points, thus producing a section of $E_{L \cap X}(-1)$, contrary to $h^0(E_{L \cap X}(-1)) = 0$.

By the choice of sections, one has $\forall k, \forall i \neq j: A_{ij}|_{pt_k} = 0$. So, $\text{deg}(A_{ij}|_L) \geq d$. But the entries of A are polynomials on \mathbb{P}^n of degree $(d-1)$. Hence for $i \neq j: A_{ij}|_L \equiv 0$. On the contrary: $A_{ii}|_{pt_i} \neq 0$, so $A_{ii}|_L \neq 0$. Therefore, $A|_L$ is a diagonal matrix, none of whose diagonal entries vanishes identically on the hypersurface $L \cap X$. Thus $\det(A|_L) \neq 0$, so $\det(A) \neq 0$.

Now consider the non-reduced case: $X = \cup p_\alpha X_\alpha$ and the rank of E_X on X_α is p_α . The generic line L intersects the hypersurface along the scheme $\sum_{X_\alpha \subset X} (p_\alpha \sum_j pt_{\alpha,j})$. Here

$pt_{\alpha,j} \in L \cap X_{\alpha}$ and by genericity X_{red} is smooth at $pt_{\alpha,j}$ and the stalk E_X is quasi-locally free at $pt_{\alpha,j}$, of rank p_{α} .

Consider the values of the sections at the point $p_1pt_{1,1}$, i.e. the p_1 'th infinitesimal neighborhood of $pt_{1,1}$. By linear transformation we can assume: $s_{i>p_1}|_{p_1pt_{1,1}} = 0$. In other words these sections are locally divisible by the defining equation of the germ $(p_1X_1, pt_{1,1})$.

Similarly for $p_1pt_{1,2}$: $rankSpan\{s_{i>p_1}|_{p_1pt_{1,1}}\} \leq r_1p_1$ etc., going over all the points of the intersections $L \cap X$.

If at least for one point the inequality $rank(Span(\cdot)) \leq p_{\alpha}$ is strict then at the end we get a non-zero section that vanishes at all the points $p_{\alpha}pt_{\alpha,j}$. But this is a section of $E(-1)$, contrary to the initial assumptions. Hence at each of the above steps we have equality: $rank(Span(\cdot)) = p_{\alpha}$.

Repeating this process we can assume that s_1, \dots, s_{p_1} vanish at all the points $p_{\alpha}pt_{\alpha,i}$ except for $p_1pt_{1,1}$. Similarly for all other sections and for the sections of E_X^l .

Therefore in the chosen basis, $A|_L$ has a block structure, with diagonal blocks corresponding to the intersection points $L \cap X$. As in the reduced case one gets that the off diagonal blocks vanish on L . Hence for non-degeneracy of $A|_L$ we need to check the non-degeneracy of all the diagonal blocks. But each such block has rank p_{α} at the point $p_{\alpha}pt_{\alpha,j}$, by the construction above. So, $A|_L$ is non-degenerate on L ; hence A is non-degenerate on \mathbb{P}^n .

Step 4. The vanishing orders of the entries of $A|_L$.

Let $p_{\alpha}pt_{\alpha,i} \in L \cap X$. Let $A_{\alpha,i}$ be the diagonal block corresponding to this point, i.e. the only block whose entries do not all vanish on $p_{\alpha}pt_{\alpha,i}$. By the choice of the basic sections as above we have: $(A_{\alpha,i})_{1,1}$ does not vanish at $pt_{\alpha,i}$. $(A_{\alpha,i})_{1,2}$ and $(A_{\alpha,i})_{2,1}$ vanish to the first order at $pt_{\alpha,i}$. The entries of the next anti-diagonal vanish to the second order etc. Finally the entries below the main (longest) anti-diagonal, that goes from $(X_{\alpha,i})_{1,p_{\alpha}}$ to $(A_{\alpha,i})_{p_{\alpha},1}$, vanish on $p_{\alpha}pt_{\alpha,i}$.

Hence, any entry of $A^{\vee}|_L$, i.e. any principal minor of $A|_L$, vanishes at $pt_{\alpha,i}$ to the order at least

$$(d - 1 - p_{\alpha})p_{\alpha} + p_{\alpha}(p_{\alpha} - 1) = p_{\alpha}(d - 2) \tag{45}$$

Here the first summand corresponds to all the blocks except for $X_{\alpha,i}$, the second summand corresponds precisely to the main anti-diagonal of the block $A_{\alpha,i}$.

By going over all the points of $L \cap X$ one has: any entry of $A^{\vee}|_L$ vanishes on $(d - 2)X \cap L$. As this is true for any generic line L we get: any entry of A^{\vee} vanishes on $(d - 2)X$, i.e. is divisible by f^{d-2} , where $X = \{f = 0\} \subset \mathbb{P}^n$.

Step 5. Construction of \mathcal{M} . As any entry of A^{\vee} is a polynomial of degree $(d - 1)^2$ and is divisible by f^{d-2} , we define $\mathcal{M} := \frac{A^{\vee}}{f^{d-2}} \in Mat(d \times d, H^0(\mathcal{O}_{\mathbb{P}^n}(1)))$. Then

$$\det(\mathcal{M}) = \frac{\det(A^{\vee})}{f^{d(d-2)}} = \left(\frac{\det(A)}{f^{d-1}}\right)^{d-1} f. \tag{46}$$

As the left hand side is a polynomial we get that $\det(A)$ is divisible by a power of $\prod f_{\alpha}$. The direct check gives the minimal possible power: $\det(A)$ is divisible by f^{d-1} . But entries of X are of degree $(d - 1)$ thus, after scaling by a constant: $\det(A) = f^{d-1}$ and $\det(\mathcal{M}) = f$.

Finally, by construction $MA = f\mathbb{1}$, while $A = \{(e_i, e'_j)\}$. Thus $\mathcal{M}(e_i, e'_j) \equiv f\mathbb{0}$, causing: $\mathcal{M}e_i \equiv 0(f)$. Therefore $Span(e_1 \dots e_d) \subset Ker(\mathcal{M}|_X)$ and $E = Ker(\mathcal{M}|_X)$ (by the equality of dimensions). \square

4.2. Kernels on the reduced curves/hypersurfaces

Suppose the hypersurface is non-reduced: $X = \cup p_\alpha X_\alpha \subset \mathbb{P}^n$, where each X_α is irreducible reduced. Then the kernel sheaf on X can be restricted to the sheaf $E_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X_{red}}$ on $X_{red} = \cup X_\alpha$, corresponding to the natural embedding of closed points $X_{red} \xrightarrow{i} X$. In other words one considers E as the sheaf of modules over $\mathcal{O}_{X_{red}}$. This is a “non-embedded” restriction.

One could consider an “embedded” restriction, induced by the projection $\mathcal{O}_X^{\oplus d}(d-1) \rightarrow \mathcal{O}_{X_{red}}^{\oplus d}(d-1)$. Namely, take any section of $E_X \subset \mathcal{O}_X^{\oplus d}(d-1)$ and consider its image in $\mathcal{O}_{X_{red}}^{\oplus d}(d-1)$. Then consider the sheaf generated by such sections. This naive restriction is far from being injective. For example, any kernel sheaf of a generically maximally generated determinantal representation will go to zero (as the sheaf is spanned by the columns of \mathcal{M}^\vee and each of them is divisible by $\prod f_\alpha^{p_\alpha-1}$, see Property 2.18). However, for generically maximally generated determinantal representations one can define a natural injective reduction.

Definition–Proposition 4.4. *Let E_X be the kernel sheaf of the generically maximally generated determinantal representation \mathcal{M} . Let E_X^{red} be the sheaf of $\mathcal{O}_{X_{red}}$ modules, generated by the columns of $\frac{1}{\prod_\alpha f^{p_\alpha-1}} \mathcal{M}^\vee$. Then E_X^{red} depends on E_X only and has the multi-rank (p_1, \dots, p_k) on $\cup X_\alpha$. The reduction $E_X \rightarrow E_X^{red}$ is injective on generically maximally generated determinantal representations. The pair $(\mathcal{M}, \frac{1}{\prod_\alpha f^{p_\alpha-1}} \mathcal{M}^\vee)$ is a matrix factorization of $\prod f_\alpha$.*

Proof. By the remark above $\frac{1}{\prod_\alpha f^{p_\alpha-1}} \mathcal{M}^\vee$ is a matrix of polynomials. Further, any equivalence of \mathcal{M} induces that of \mathcal{M}^\vee , preserving the isomorphism class of E_X^{red} .

Finally, given E_X^{red} , generated by the columns of \mathcal{N}^\vee , the kernel E_X must be generated by the columns of $\prod_\alpha f^{p_\alpha-1} \mathcal{N}^\vee$, i.e. is determined uniquely. \square

Similarly the reduction for left kernel $E_X^l \rightarrow (E_X^l)^{red}$ is defined. Note that $E_X^{red}, (E_X^l)^{red}$ are naturally embedded into $\mathcal{O}_{X_{red}}^{\oplus d}(d_{red}-1)$, where $d_{red} := \sum d_\alpha$.

These reductions have properties similar to the original kernels, in particular they fix the determinantal representations uniquely, in the sense of Proposition 2.14.

Remark 4.5. The kernel sheaf of such a reduction can be defined geometrically. First, suppose that E_X^{red} has a constant rank on the hypersurface. Then it defines the rational map:

$$X_{red} \xrightarrow{\phi} Gr(\mathbb{P}^{rank(E_X^{red})-1}, \mathbb{P}^{d-1}), \quad Smooth(X_{red}) \ni pt \rightarrow Ker(\mathcal{M}|_{pt}). \tag{47}$$

As in the reduced case, let $\tilde{X} \xrightarrow{\nu} X_{red}$ be a birational morphism such that $\nu^* \phi$ extends to the morphism on the whole \tilde{X} . Let τ be the tautological bundle on $Gr(\mathbb{P}^{rank(E_X^{red})-1}, \mathbb{P}^{d-1})$. Then $\nu^* E_X^{red}(1-d_{red})/Torsion = \phi^* \tau$, as their fibers coincide. And, moreover, $\phi^* \tau$ determines $E_X^{red}(1-d_{red})$ uniquely.

More generally, let the decomposition $X = \cup X_\alpha$ satisfy: X_α, X_β have no common components for $\alpha \neq \beta$, and E_X^{red} has a constant rank on each X_α . Then, as previously, each $E_X^{red}|_{X_{\alpha,red}}$ is determined uniquely from the corresponding tautological bundle.

Theorem 4.1 is translated almost verbatim.

Proposition 4.6. *Let $X = \cup p_\alpha X_\alpha$ be a hypersurface, $\text{deg}(X) = d$ and \mathcal{M} a generically maximally generated determinantal representation. Let E_X^{red} be the reduction of E_X . Then*

$$0 \rightarrow E_X^{\text{red}} \rightarrow \mathcal{O}_{X_{\text{red}}}^{\oplus d}(d_{\text{red}} - 1) \xrightarrow{\mathcal{M}} \mathcal{O}_{X_{\text{red}}}^{\oplus d}(d_{\text{red}}) \rightarrow \text{Coker}(\mathcal{M})_{X_{\text{red}}}^{\text{red}} \rightarrow 0, \tag{48}$$

1. The sheaf E_X^{red} is generated by the columns of $\frac{1}{\prod_\alpha f_\alpha^{p_\alpha-1}} \mathcal{M}^\vee$ as an $\mathcal{O}_{X_{\text{red}}}$ module. In particular $h^0(E_X^{\text{red}}) = d$ and $h^0(E_X^{\text{red}}(-1)) = 0$ and $h^i(E_X^{\text{red}}(j)) = 0$ for $0 < i < n - 1$, $j \in \mathbb{Z}$.
2. The sheaves E_X^{red} , $\text{Coker}(\mathcal{M})_{X_{\text{red}}}^{\text{red}}$ are torsion free.
3. Similar statements hold for $(E_X^l)^{\text{red}}$.

Proof. The exactness of the sequences follows immediately from $\frac{\mathcal{M}^\vee}{\prod_\alpha f_\alpha^{p_\alpha-1}} \mathcal{M} = \prod f_\alpha \mathbb{1}$.

1. The proof is identical to that in Theorem 4.1.
2. E_X^{red} and $(E_X^l)^{\text{red}}$ are torsion free as subsheaves of torsion free sheaves. For $\text{Coker}(\mathcal{M})_{X_{\text{red}}}^{\text{red}}$, $\text{Coker}(\mathcal{M}^T)_{X_{\text{red}}}^{\text{red}}$ the proof is similar to that in 4.1. If $gs = 0 \in \mathcal{O}_{X_{\text{red}}}$ then $(\frac{1}{\prod_\alpha f_\alpha^{p_\alpha-1}} \mathcal{M}^\vee)gs = 0$; hence $s = \mathcal{M}s_1$, etc. \square

The characterization of kernel sheaves is also translated immediately from Theorem 4.3.

Proposition 4.7. *Let E_X^{red} be a torsion free sheaf on the reduction of the hypersurface $X_{\text{red}} = \cup X_\alpha$, satisfying $h^0(E_X^{\text{red}}(-1)) = 0$ and $h^i(E_X^{\text{red}}(j)) = 0$ for $0 < i < n - 1$, $j \in \mathbb{Z}$ and $h^{n-1}(E_X^{\text{red}}(1 - n)) = 0$. Suppose the multirank of E_X^{red} is (p_1, \dots, p_k) . Then E_X^{red} is the restriction of a kernel from $X = \cup p_\alpha X_\alpha$, i.e. $0 \rightarrow E_X^{\text{red}} \rightarrow \mathcal{O}_{X_{\text{red}}}^{\oplus d}(d_{\text{red}} - 1) \xrightarrow{\mathcal{M}} \dots$*

Proof. E_X^{red} is a module over $\mathcal{O}_{X_{\text{red}}} = \mathcal{O}_{\mathbb{P}^n} / \prod f_\alpha$. Let $\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^n} / \prod f_\alpha^{p_\alpha} \xrightarrow{\pi} \mathcal{O}_{X_{\text{red}}}$ be the natural projection.

Construct an \mathcal{O}_X module by defining the action of \mathcal{O}_X on E_X^{red} :

$$\text{for } g \in \mathcal{O}_X \text{ and } s \in E_X^{\text{red}}, \quad gs := \prod_\alpha f_\alpha^{p_\alpha-1} \pi(g)s. \tag{49}$$

Denote this module by E_X . It is torsion free by construction and all the needed cohomologies vanish.

Hence by Theorem 4.3 E_X is the kernel of a determinantal representation, whose reduction is E_X^{red} . \square

4.3. Kernels on modifications of curves

It is useful to have the classification of kernels in terms of locally free sheaves on the normalization $\tilde{C} \xrightarrow{\nu} C$ or, more generally, torsion free sheaves on an intermediate modification $C' \xrightarrow{\nu} C \subset \mathbb{P}^2$.

4.3.1. When is the pushforward $v_*(E)_C$ the kernel sheaf?

Recall (Section 2.4.5) that a torsion free sheaf E_C is C'/C -saturated if $E_C = v_*(v^*E_C/Torsion)$. Equivalently, the kernel E_C of \mathcal{M} is C'/C -saturated iff every element of \mathcal{M}^\vee belongs to the relative adjoint ideal $Adj_{C'/C}$.

In particular, the cohomology dimensions of these sheaves are preserved under pullbacks and pushforwards. Hence Theorem 4.3 implies immediately the following.

Corollary 4.8. *Given a modification $C' \xrightarrow{v} C = \cup p_\alpha C_\alpha \subset \mathbb{P}^2$ and a torsion free sheaf $E_{C'}$, whose pushforward $v_*(E_{C'})$ is of multi-rank (p_1, \dots, p_k) . The pushforward $v_*(E_{C'})$ is the kernel sheaf of a determinantal representation \mathcal{M}_C iff all the relevant cohomologies vanish: $h^0(E_{C'}(-1)) = 0 = h^1(E_{C'}(-1))$. In this case \mathcal{M}_C is the unique C'/C -saturated determinantal representation whose kernel pulls-back to $E_{C'}$.*

(Here $E_{C'}(-1) = E_{C'} \otimes v^*\mathcal{O}_C(-1)$.)

Example 4.9. 1. For the normalization $\tilde{C} \xrightarrow{v} C$ any torsion free sheaf $E_{\tilde{C}}$ is locally free. Hence we get the characterization of the locally free sheaves on \tilde{C} arising as kernels of determinantal representations.

2. In particular, let $C_d \subset \mathbb{P}^2$ be an irreducible curve, and let $\tilde{C}_g \xrightarrow{v} C_d$ be the normalization. Then $v^*\mathcal{O}_C(-1)$ is a line bundle of degree d . The \tilde{C}/C -saturated determinantal representations of C_d correspond to line bundles of degree $d + g - 1$ on \tilde{C}_g satisfying: $h^0(\mathcal{L}_{\tilde{C}_g} \otimes v^*\mathcal{O}_C(-1)) = 0$. If $g = 0$ then the unique such bundle on $\tilde{C} = \mathbb{P}^1$ is $\mathcal{O}_{\mathbb{P}^1}(d - 1)$. Hence there exists unique determinantal representation of $C \subset \mathbb{P}^2$ that is \tilde{C}/C saturated at each point. More generally, such line bundles are parameterized by an open dense subset of Jacobian of C_g , the complement of the Brill–Noether locus.

4.3.2. When is a torsion free sheaf $E_{C'}$ the pullback of some kernel sheaf on C ?

Usually, many torsion free sheaves on C are not pushforwards of locally free sheaves, i.e. are not of the form $v_*E_{C'}$ for any modification $C' \xrightarrow{v} C$; cf. [34].

Thus, while the kernel sheaf has no global sections, $h^0(E_C(-1)) = 0$, its pull-back can have them. Let $E_{C'}$ be a torsion free sheaf and $pt \in C'$. Consider the stalk $E_{(C', pt)}$, a module over the local ring $\mathcal{O}_{(C', pt)}$. Suppose it is minimally generated by the elements a_1, \dots, a_k . Let $v^{-1}(\mathcal{O}_{(C, v(pt))})\langle a_1, \dots, a_k \rangle$ be a vector subspace of $E_{(C', pt)}$.

For generic point $pt \in C'$ one has $v^{-1}(\mathcal{O}_{(C, v(pt))}) = \mathcal{O}_{(C', pt)}$; hence $v^{-1}(\mathcal{O}_{(C, v(pt))})\langle a_1, \dots, a_k \rangle = E_{(C', pt)}$. The natural object is the quotient vector space $E_{(C', pt)}/v^{-1}(\mathcal{O}_{(C, v(pt))})\langle a_1, \dots, a_k \rangle$. The dimension of this vector space is independent of the choice of generators $\{a_i\}$. We denote this dimension by $length_{pt} E_{C'}/\mathcal{O}_C$.

Definition 4.10. The global section $s \in H^0(E_{C'})$ is said to descend to C locally if for any point $pt \in C'$ there is a choice of the local generators of $E_{(C', pt)}$ as above, such that $s \in v^{-1}(\mathcal{O}_{(C, v(pt))})\langle a_1, \dots, a_k \rangle$.

Proposition 4.11. *The torsion free sheaf $E_{C'}$ is the pull-back of a kernel sheaf on C iff $h^1(E_{C'}(-1)) = 0$ and $h^0(E_{C'}(-1)) \leq \sum_{pt \in C'} length_{pt} E_{C'}/\mathcal{O}_C$ and no non-zero global section of $E_{C'}$ descends to C locally.*

Proof. \Rightarrow Suppose E_C is the kernel module, let $E_{C'} = v^*(E_C)/Torsion$. Then $0 \rightarrow E_C \rightarrow v_*(E_{C'}) \rightarrow sky \rightarrow 0$, where sky is a skyscraper sheaf supported at a finite number of points, at which $E_{C,pt} \subsetneq v_*E_{C',pt}$. Take the cohomology:

$$\begin{aligned} h^0(E_C(-1)) = 0 &\rightarrow h^0(v_*(E_{C'})(-1)) \rightarrow length(sky) \\ &\rightarrow h^1(E_C(-1)) = 0 \rightarrow h^1(v_*(E_{C'})(-1)) \rightarrow 0. \end{aligned} \tag{50}$$

Hence, $h^1(v_*(E_{C'})(-1)) = 0$ and moreover:

$$h^0(v_*(E_{C'})(-1)) = length(sky) = \sum_{pt \in C} v_*E_{(C',v^{-1}(pt))}/E_{(C,pt)} \leq \sum_{pt \in C'} E_{C'}/\mathcal{O}_C. \tag{51}$$

As $h^0(E_C(-1)) = 0$ no global section of $h^0(v_*(E_{C'})(-1))$ descends to C .

\Rightarrow Suppose $E_{C'}$ is given. Take a torsion sheaf, sky_C , with $Supp(sky) \subset \cup_{pt \in C} Supp(v_*E/v^{-1}\mathcal{O}_{(C,pt)})$ and $length(sky) = h^0(E_{C'}(-1))$ and $length_{pt}(sky) \leq length v_*E/v^{-1}\mathcal{O}_{(C,pt)}$. It exists by the assumption $h^0(E(-1)) \leq \sum_{pt \in C'} length_{pt}/E_{C'}\mathcal{O}_C$.

Take a surjection $v_*E_{C'}(-1) \rightarrow sky \rightarrow 0$. Let E_C be the kernel of this map,

$$0 \rightarrow E_C \rightarrow v_*(E_{C'}) \rightarrow sky \rightarrow 0. \tag{52}$$

Then, by construction, $\chi(E_C(-1)) = 0$. And by the assumption on non-descent of the global sections of $E_{C'}$: $h^0(E_C(-1)) = 0$. Hence $h^1(E_C(-1)) = 0$. \square

4.4. Families of determinantal representations

Theorems 4.1 and 4.3 translate the study of determinantal representations into the study of the sheaves with specific properties. Many related questions are open. We make only a few remarks.

- On a smooth curve the generic line bundle of $deg(\mathcal{L}(-1)) = g - 1$ has $h^0(\mathcal{L}(-1)) = 0 = h^1(\mathcal{L}(-1))$. Hence the kernel bundles correspond to an open dense locus on the Jacobian of all the line bundles of the given degree.
- For a singular, reducible curve $\cup_i C_i$ the \tilde{C}/C -saturated determinantal representations correspond to line bundles $\mathcal{L}_{\sqcup \tilde{c}_i}$ with $h^0(\mathcal{L}_{\sqcup \tilde{c}_i} \otimes v^*\mathcal{O}_C(-1)) = 0 = h^1(\mathcal{L}_{\sqcup \tilde{c}_i} v^*\mathcal{O}_C(-1))$. Hence the multi-degree: $deg(\mathcal{L}(-1)) = \{g(\tilde{C}_i) - 1\}_i$. Again, the generic line bundle with such a multi-degree has no global sections and hence corresponds to the kernel bundle.
- More generally, the families of C'/C -saturated determinantal representations correspond to some families of torsion free sheaves on C' , with vanishing cohomology.
- (Continuing Example 2.11.) A quadric with one A_1 singularity is the cone over a smooth plane conic. The kernel bundle corresponds to a line on the quadric passing through the singular point. The corresponding determinantal representation is equivalent to a symmetric one.
- The determinantal representations of smooth cubics in \mathbb{P}^3 are studied e.g. in [9,12], for singular cubics; cf. [17].

5. Symmetric determinantal representations

Here we consider *symmetric* local/global determinantal representations, i.e.

$$\mathcal{M} = \mathcal{M}^T \in Mat(d \times d, R) \quad \text{for } R = \mathcal{O}_{(\mathbb{k}^n, 0)} \text{ or } R = |\mathcal{O}_{\mathbb{P}^n}(1)| \tag{53}$$

In this case the local/global symmetric equivalence is: $\mathcal{M} \stackrel{s}{\sim} A\mathcal{M}A^T$, for A -(locally) invertible matrices with entries in $\mathcal{O}_{(\mathbb{k}^n, 0)}$ or with constant entries.

As in the ordinary case (Proposition 2.6) the symmetric reduction of determinantal representations exists. Choose the local coordinates (x_1, \dots, x_n) on $(\mathbb{k}^n, 0)$.

Property 5.1. 1. Any symmetric matrix $\mathcal{M} \in \text{Mat}(d \times d, \mathcal{O}_{(\mathbb{k}^n, 0)})$ is symmetrically equivalent to $\mathbb{1} \oplus \mathcal{N}$, where $\mathcal{N} = \mathcal{N}^T$ and $\mathcal{N}|_{(0,0)} = \mathbb{O}$. The symmetrically reduced matrix, i.e. \mathcal{N} , is unique up to the local symmetric equivalence.

2. Any local symmetric determinantal representation with rational entries is a symmetric reduction of some global symmetric determinantal representation. If $\mathcal{M}_1 \stackrel{s}{\sim} \mathcal{M}_2$ locally and \mathcal{M}_1 is the reduction of $\mathcal{M}_{\text{global}} = \mathcal{M}_{\text{global}}^T$ then \mathcal{M}_2 is also the symmetric reduction of $\mathcal{M}_{\text{global}}$.

Proof. 1. (For a part of this statement, see also [43, Lemma 1.7].) Let $\mathcal{M} \rightarrow A\mathcal{M}A^T$ be a transformation diagonalizing $\text{jet}_0\mathcal{M}$. By further symmetric scaling and permutation of the rows/columns one can assume $\text{jet}_0(A\mathcal{M}A^T) = \mathbb{1} \oplus \mathbb{O}$. Now, in the transformed matrix $A\mathcal{M}A^T$ kill all the non-constant entries of the first row/column. This is done symmetrically and decomposes \mathcal{M} as $\mathbb{1}_{1 \times 1} \oplus N_1$, where $N_1 = N_1^T$. Do the same for N_1 etc. The uniqueness of the localization is proved as in Proposition 2.6.

2. Let \mathcal{M} have rational entries, let g be the product of the denominators in all the entries of \mathcal{M} . Then g is invertible at the origin and $(g\mathbb{1})\mathcal{M}(g\mathbb{1})$ has polynomial entries.

Given $\mathcal{M} = \mathcal{M}^T$ with polynomial entries we should find $\mathcal{N} = \mathcal{N}^T$, whose entries are polynomials of degree at most one, such that $\mathcal{N} \stackrel{s}{\sim} \mathbb{1} \oplus \mathcal{M}$.

As in Lemma 2.8 consider the monomials, appearing in \mathcal{M} , with the maximal total degree $\text{deg } \mathcal{M}$. It is enough to show that $\mathbb{1}_{k \times k} \oplus \mathcal{M}$ is symmetrically equivalent to a matrix $\mathcal{N} = \mathcal{N}^T$, such that $\text{deg } \mathcal{N} \leq \text{deg } \mathcal{M}$ and the number of monomials in \mathcal{N} , whose total degree is $\text{deg } \mathcal{M}$, is strictly less than in \mathcal{M} . Let $x_1^{a_1} \dots x_n^{a_n}$ be one such monomial, we can assume that it is on the diagonal of \mathcal{M} . Then the step of induction is

$$\begin{aligned}
 & \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & x_1^{a_1} \dots x_n^{a_n} + \dots & \dots \\ 0 & 0 & \dots & \dots \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & \dots & 0 \\ 1 & 0 & & 0 \\ 0 & 0 & x_1^{a_1} \dots x_n^{a_n} + \dots & \dots \\ 0 & 0 & \dots & \dots \end{pmatrix} \\
 & \rightsquigarrow \begin{pmatrix} 0 & 1 & x_1^{a_1} \dots x_i^{a_i-1} \dots x_n^{a_n} & 0 \\ 1 & 0 & 0 & 0 \\ x_1^{a_1} \dots x_i^{a_i-1} \dots x_n^{a_n} & 0 & x_1^{a_1} \dots x_n^{a_n} + \dots & \dots \\ 0 & 0 & \dots & \dots \end{pmatrix} \\
 & \rightsquigarrow \begin{pmatrix} 0 & 1 & x_1^{a_1} \dots x_i^{a_i-1} \dots x_n^{a_n} & 0 \\ 1 & 0 & -\frac{x}{2} & 0 \\ x_1^{a_1} \dots x_i^{a_i-1} \dots x_n^{a_n} & -\frac{x}{2} & 0 + \dots & \dots \\ 0 & 0 & \dots & \dots \end{pmatrix}. \quad \square \tag{54}
 \end{aligned}$$

A determinantal representation is symmetric iff its left and right kernels coincide.

Proposition 5.2. Let $\mathcal{M}_{d \times d}$ be a local/global determinantal representation and let $E, E^{(l)} \subset R^{\oplus d}$ be its right and left kernel modules/sheaves. Here $R = \mathcal{O}_{(\mathbb{k}^n, 0)}$ in the local case or $R = |\mathcal{O}_{\mathbb{P}^n}(d-1)|$ in the global case.

1. $E \approx E^{(l)}$ iff \mathcal{M} is equivalent to a symmetric matrix.
2. $E = E^{(l)} \subset R^{\oplus d}$ iff $\mathcal{M} = \mathcal{M}^T$.

Proof. The parts \Leftarrow are obvious. The converse follows immediately from the uniqueness of minimal free resolution:

$$\begin{array}{ccccccc}
 0 & \rightarrow & E & \rightarrow & R^{\oplus d} & \xrightarrow{\mathcal{M}} & R^{\oplus d} \\
 & & \downarrow \phi & & \downarrow A & & \downarrow B \\
 0 & \rightarrow & E^{(l)} & \rightarrow & R^{\oplus d} & \xrightarrow{\mathcal{M}^T} & R^{\oplus d}
 \end{array} \tag{55}$$

An isomorphism ϕ induces the isomorphisms A, B , by Proposition 2.14. And if ϕ is identity then A, B are identities too. \square

In the symmetric case the ordinary equivalence implies the symmetric one.

- Proposition 5.3.**
1. If two (local or global) symmetric determinantal representations of hypersurfaces are (locally or globally) equivalent then they are symmetrically equivalent.
 2. Suppose $\mathcal{M} = \mathcal{M}^T$ is locally or globally decomposable: $\mathcal{M} \sim \begin{pmatrix} \mathcal{M}_1 & \mathbb{0} \\ \mathbb{0} & \mathcal{M}_2 \end{pmatrix}$. Then there exists a symmetric decomposition: $\mathcal{M} \overset{s}{\sim} \begin{pmatrix} \mathcal{N}_1 & \mathbb{0} \\ \mathbb{0} & \mathcal{N}_2 \end{pmatrix}$, with $\mathcal{N}_\alpha^T = \mathcal{N}_\alpha \sim \mathcal{M}_\alpha$.
 3. Global symmetric determinantal representation decomposes (symmetrically) iff it decomposes locally at the relevant points, as in Theorem 3.1.

Proof. Part 1 in the global case is a classical fact e.g. for two variables, see [40, Chapter VI, Section 23, Theorem 3]. We (re-)prove it both in the global and local cases.

Let $\mathcal{M}_1 = \mathcal{M}_1^T$ and $\mathcal{M}_2 = \mathcal{M}_2^T$ be equivalent, i.e. $\mathcal{M}_1 = A\mathcal{M}_2B$. We can get rid of B by replacing \mathcal{M}_2 by $(B^T)^{-1}\mathcal{M}_2B^{-1}$, so \mathcal{M}_2 stays symmetric. Now one has $\mathcal{M}_1 = A\mathcal{M}_2 = (A\mathcal{M}_2)^T = \mathcal{M}_2A^T$. Note that at each $pt \in X$: $Ker(A\mathcal{M}_2|_{pt}) = Ker(\mathcal{M}_2|_{pt})$; thus the comparison of the left and right parts gives

$$\forall pt \in X : A^T Ker(\mathcal{M}_2|_{pt}) = Ker(\mathcal{M}_2|_{pt}) \subset \mathbb{k}^d. \tag{56}$$

Global case. Suppose for the generic point of X that the vector space $Ker(\mathcal{M}_2|_{pt})$ is one-dimensional. Then we get a constant matrix A^T acting on $\mathbb{P}(\mathbb{k}^d)$, preserving the image $\phi(X) \subset \mathbb{P}(\mathbb{k}^d)$; cf. Section 2.4.2. As $Span(\phi(X))$ is the whole ambient space, this implies that A^T is diagonal.

In general, for the decomposition $X = \cup p_\alpha X_\alpha$, let d_α be the generic dimension of $Ker(\mathcal{M}_2)$ on X_α . Then $A^T \circ Gr(\mathbb{P}^{d_\alpha-1}, \mathbb{P}^{d-1})$ and A^T preserves $\phi(p_\alpha X_\alpha) \subset Gr(\mathbb{P}^{d_\alpha-1}, \mathbb{P}^{d-1})$. Combining this for all the components we get again: A^T acts diagonally.

Finally, if A is diagonal we get, $A\mathcal{M}_2 = \mathcal{M}_2A$. Thus $A = \tilde{A}^2$ with $\mathcal{M}_1 = A\mathcal{M}_2 = \tilde{A}\mathcal{M}_2\tilde{A}$.

Local case. Expand A in powers of local coordinates, then the constant part, $jet_0(A)$, is invertible and satisfies: $jet_0(A)\mathcal{M}_2 = \mathcal{M}_2 jet_0(A)^T + \text{higher.order.terms}$. Thus, arguing as above we get that $jet_0(A)$ is diagonal and further, is symmetrically equivalent to the identity. So, we assume $jet_0(A) = \mathbb{1}$. Now define \sqrt{A} as follows. Consider the Taylor series $\sqrt{1+x} := g(x)$ and define $\sqrt{A} = \sqrt{\mathbb{1} + (A - \mathbb{1})} = g(A - \mathbb{1})$. As $jet_0(A - \mathbb{1}) = \mathbb{0}$ the series $g(A - \mathbb{1})$ is well

defined (at least as a formal series). From $A\mathcal{M}_2 = \mathcal{M}_2A^T$ we get $A^j\mathcal{M}_2 = \mathcal{M}_2(A^T)^j$, thus $\sqrt{A}\mathcal{M}_2 = \mathcal{M}_2\sqrt{A^T}$. Therefore, we get

$$\mathcal{M}_1 = A\mathcal{M}_2 = \sqrt{A}\mathcal{M}_2\sqrt{A^T}. \tag{57}$$

Part 2. As $E = E^{(l)}$ their restrictions to the components $(X_\alpha, 0)$ coincide too. Hence $\mathcal{M} \sim \mathcal{N}_1 \oplus \mathcal{N}_2$ where $\mathcal{N}_\alpha^T = \mathcal{N}_\alpha$. Now, by the first statement we get that the equivalence can be chosen symmetric.

Part 3 follows from the ordinary decomposability and part 2. \square

Finally we characterize the sheaves that are kernels of symmetric determinantal representations.

Proposition 5.4. *Let E_X be a torsion-free sheaf on the hypersurface $X \subset \mathbb{P}^n$. Assume that all the relevant cohomologies vanish (as in Theorem 4.3) and $E_X \approx E_X^* \otimes w_X(n) = \text{Hom}(E_X, w_X(n))$. Then there exists $\mathcal{M}^T = \mathcal{M} \in \text{Mat}(d \times d, H^0(\mathcal{O}_{\mathbb{P}^n}(1)))$ such that*

$$0 \rightarrow E_X \rightarrow \mathcal{O}_X^{\oplus d}(d-1) \xrightarrow{\mathcal{M}} \mathcal{O}_X^{\oplus d}(d) \rightarrow \dots \tag{58}$$

If moreover X is reduced and E_X is locally free then $n \leq 2$.

Proof. The main construction is done in the proof of Theorem 4.3, it realizes E_X and $E_X^l = E_X^* \otimes w_X(n)$ as the right and left kernels of \mathcal{M} . Hence, as $E_X \approx E_X^l$ we get from Proposition 5.2 that \mathcal{M} is equivalent to a symmetric matrix.

The last statement is proved as the last statement of Theorem 4.1. Local freeness implies that corank of \mathcal{M}_X is one at any point. But in the parameter space of all the $d \times d$ symmetric matrices the subset of matrices of corank at least two is of codimension three. Hence if $n \geq 3$ there will be always a point $pt \in X$ such that $\dim(\text{Ker } \mathcal{M}|_{pt}) > 1$. \square

Remark 5.5. 1. Note that even if the hypersurface is reduced and the kernel is of rank one and generically locally free, we ask for $E_X \approx E_X^* \otimes w_X(n)$ rather than $E_X \otimes E_X \approx w_X(n)$ as $E_X \otimes E_X$ is not a torsion free sheaf, unless E_X is locally free. In fact, even with the torsion factored out, the map $E_X \otimes E_X / \text{Torsion} \rightarrow w_X(n)$ is injective (as its kernel would be a torsion subsheaf) but is not surjective.

2. In the case of curves the "self-dual" sheaves of the proposition are the well known theta characteristics; see [28] for the smooth case and [44] for reduced case. In particular, for any collection of the "local types", i.e. the stalks of E_X at the points where E_X is not locally free, there exists a theta characteristic with this collection of types.

6. Self-adjoint determinantal representations

In this section we work over \mathbb{C} , i.e. $\mathbb{P}^n = \mathbb{P}_{\mathbb{C}}^n$. Let $\mathbb{P}_{\mathbb{R}}^n$ be the real projective space. For a subscheme $X \subset \mathbb{P}^n$ we denote the set of its real points by $X_{\mathbb{R}}$.

6.1. Setup

Let the hypersurface $X \subset \mathbb{P}^n$ be defined over \mathbb{R} , i.e. its defining polynomial has real coefficients. Let $\tau \circ \mathbb{C}$ be the complex conjugation, so τ acts on \mathbb{P}^n and on X . Thus τ acts on the twisting sheaves $\tau \circ \mathcal{O}_{\mathbb{P}^n}(d)$ and $\tau \circ \mathcal{O}_X(d)$. The complex conjugation acts on the set of all the sheaves of embedded modules.

Definition 6.1. Let $E_X \subset \bigoplus_\alpha \mathcal{O}_X(d_\alpha)$ be a sheaf of modules. The induced conjugation, $E_X^\tau := \tau(E_X)$, is defined by the action on sections, i.e. each section is sent to its conjugate.

For any matrix define $A^\tau := \bar{A}^T$, i.e. both conjugated and transposed.

Definition 6.2. A local or global determinantal representation is called self-adjoint if for any $pt \in \mathbb{P}^n_{\mathbb{R}}$ one has: $\mathcal{M}|_{pt} = \mathcal{M}|_{pt}^\tau$. Self-adjoint determinantal representations are considered up to Hermitian equivalence: $\mathcal{M} \overset{\tau}{\sim} A\mathcal{M}A^\tau$.

As in the symmetric case, the self-adjointness can be expressed in terms of kernels and the ordinary equivalence implies an almost Hermitian equivalence, generalizing [53, Section 9, Theorem 8].

Proposition 6.3. 1. $\mathcal{M} = \mathcal{M}^\tau$ iff $E^\tau = E^{(l)} \subset \mathcal{O}_X^{\oplus d}(d - 1)$. And $\mathcal{M} \sim \mathcal{M}^\tau$ iff $E^\tau \approx E^{(l)}$.
 2. Let $\mathcal{M}, \mathcal{M}'$ be (local or global) determinantal representations of the same hypersurface. If $\mathcal{M} \sim \mathcal{M}'$ and both are self-adjoint then either $\mathcal{M} \overset{\tau}{\sim} \mathcal{M}'$ or $\mathcal{M} \overset{\tau}{\sim} -\mathcal{M}'$ or

$$\mathcal{M} \overset{\tau}{\sim} \begin{pmatrix} \mathcal{M}_1 & \mathbb{O} \\ \mathbb{O} & \mathcal{M}_2 \end{pmatrix} \quad \text{and} \quad \mathcal{M}' \overset{\tau}{\sim} \begin{pmatrix} \mathcal{M}_1 & \mathbb{O} \\ \mathbb{O} & -\mathcal{M}_2 \end{pmatrix}. \tag{59}$$

Proof. Part one is obvious.

Part two. As in Proposition 5.3, starting from $\mathcal{M} = A\mathcal{M}' = (A\mathcal{M}')^\tau = \mathcal{M}'A^\tau$ we get: A preserves the embedded kernel at each point of X . If A is constant (e.g. in the global case) then A is diagonal, so after reshuffling the rows/columns and rescaling (over reals!) we can assume $A = \begin{pmatrix} \mathbb{1} & \mathbb{O} \\ \mathbb{O} & -\mathbb{1} \end{pmatrix}$. This implies that \mathcal{M}' is block diagonal and gives the statement in the global case.

In the local case we get $jet_0(A) = \begin{pmatrix} \mathbb{1} & \mathbb{O} \\ \mathbb{O} & -\mathbb{1} \end{pmatrix}$; hence the expansion of $jet_0(A) \times A$ in local coordinates begins with the constant term $\mathbb{1}$. Therefore we can use the Taylor expansion of $\sqrt{jet_0(A) \times A} = \sqrt{\mathbb{1} + (jet_0(A) \times A - \mathbb{1})}$ and proceed as in the proof of Proposition 5.3. \square

6.2. Classification

The classification of self-adjoint determinantal representations of smooth plane curves was done in [53, Section 9, Theorem 7]. As in the ordinary/symmetric cases we generalize to an arbitrary hypersurface.

Theorem 6.4. Let $X \subset \mathbb{P}^n$ be an arbitrary hypersurface, defined over \mathbb{R} . A torsion free sheaf E_X is the kernel sheaf of a self-adjoint determinantal representation iff: all the relevant cohomologies vanish (as in Theorem 4.3) and $E_X^{(l)} = E_X^* \otimes w_X(n) \approx E_X^\tau$.

Proof. The statement on vanishing cohomologies was proved in Theorems 4.1 and 4.3.

For the self-conjugacy of the kernel sheaf, the direction \Rightarrow is obvious. The inverse direction is proved exactly as in Proposition 5.4. We should only check Step 3 of the original construction.

Let $E_X \xrightarrow{\sim} (E_X^{(l)})^\tau$ be the isomorphism of sheaves. By Proposition 2.14 it extends to the global automorphism $\phi \circ \mathcal{O}_X^{\oplus d}(d - 1)$. As ϕ is a global automorphism it is presented by a constant invertible matrix. Choose the basis of $\mathcal{O}_X^{\oplus d}(d - 1)$ so that ϕ becomes identity. Namely E_X and $(E_X^{(l)})^\tau$ coincide as embedded sheaves. Finally, once the sheaves are identified, choose the same bases: $E_X, (s_1, \dots, s_d) = (E_X^{(l)})^\tau, (s_1, \dots, s_d)$. \square

6.3. An application to hyperbolic polynomials

Let $\mathcal{M} \in \text{Mat} \left(d \times d, H^0(\mathcal{O}_{\mathbb{P}^n_{\mathbb{R}}}(1)) \right)$ be a self-adjoint positive definite determinantal representation of a real projective hypersurface. Namely, \mathcal{M} is a matrix of linear forms in homogeneous coordinates on $\mathbb{P}^n_{\mathbb{R}}$ and for any $pt \in \mathbb{P}^n_{\mathbb{R}}$ one has: $\mathcal{M}|_{pt} = \mathcal{M}|^t_{pt}$ and at least for one point $\mathcal{M}|_{pt}$ is positive definite.

Proposition 6.5. *Let \mathcal{M} be positive definite at least at one point. Then for some choice of coordinates on $\mathbb{P}^n_{\mathbb{R}}$: $\mathcal{M} = \sum_{i=0}^n \mathcal{M}_i x_i$, where all the matrices \mathcal{M}_i are positive definite.*

Proof. If \mathcal{M} is positive definite at $pt \in \mathbb{P}^n_{\mathbb{R}}$ then it is positive definite on some open neighborhood of this point. Let $pt_0, \dots, pt_n \in \mathbb{P}^n_{\mathbb{R}}$ be some points close to pt , such that \mathcal{M} is positive definite at these points and the points do not lie in a hyperplane, i.e. they span \mathbb{R}^n locally.

Consider $\mathbb{P}^n_{\mathbb{R}}$ as $\text{Proj}(\mathbb{R}^{n+1})$, so one can choose the coordinate axes $\hat{x}_0, \dots, \hat{x}_n$ of \mathbb{R}^{n+1} , corresponding to these points. Hence $\mathcal{M}|_{pt_j} = x_j \mathcal{M}_j$ for $x_j > 0$. Hence in this coordinate system all \mathcal{M}_i are positive definite. \square

Definition 6.6. A homogeneous polynomial $f \in \mathbb{R}[x_0, \dots, x_n]$ is called hyperbolic if there exists a point $pt \in \mathbb{P}^n_{\mathbb{R}}$ with the property: for any line L through pt all the complex roots of $f|_L = 0$ are real. (In other words the line intersects the hypersurface $\{f = 0\} \subset \mathbb{P}^n_{\mathbb{R}}$ at $\text{deg}(f)$ real points, counted with multiplicities.)

For the general introduction to the theory of hyperbolic polynomials cf. [24,25,36,27,8,48]. For a given hyperbolic polynomial the union of all points satisfying the definition above is called *the region of hyperbolicity*. It is a convex set in $\mathbb{P}^n_{\mathbb{R}}$ under projection $\mathbb{R}^{n+1} \rightarrow \mathbb{P}^n_{\mathbb{R}}$, i.e. its preimage in \mathbb{R}^{n+1} is the disjoint union of two convex sets. Suppose the real homogeneous polynomial in three variables defines a smooth plane curve. Then the polynomial is hyperbolic iff the corresponding curve has the maximal possible number of nested ovals.

A self-adjoint positive-definite determinantal representation defines a hyperbolic polynomial (see [54, Section 6] for curves and [32] for hypersurfaces). In this case the hyperbolicity region consists of all points $pt \in \mathbb{P}^n_{\mathbb{R}}$ such that $\mathcal{M}|_{pt}$ is (positive or negative) semi-definite (see the citations above).

A naive converse statement could be: if $f(x_0, \dots, x_n)$ is a hyperbolic polynomial with the hyperbolicity region $\mathbb{R}^n_{\geq 0}$ then f^N has a positive-definite determinantal representation for $N \gg 0$. This turns to be wrong in higher dimensions; see [10]. But for $n = 2$ this is true even for f itself, not only for its higher multiples, [19,54,32]. A weaker statement, *there exists a hyperbolic polynomial $g(x_0, \dots, x_n)$ with the hyperbolicity region containing that of f , such that gf is determinantal*, has not been checked yet.

Theorem 6.7. 1. *Let $X = \cup p_{\alpha} X_{\alpha} \subset \mathbb{P}^n$ be a hypersurface defined over \mathbb{R} by a hyperbolic polynomial. The real part of its reduced locus, $\cup X_{\alpha}^{\mathbb{R}}$, can have at most one (real) singular point with a non-smooth locally irreducible component. In the latter case the region of hyperbolicity degenerates to this singular point.*

2. *In particular, if the hypersurface X possesses a self-adjoint positive definite determinantal representation then the germ of its reduced locus at each of its (real) singular point is the union of smooth hypersurfaces. Hence there exists a finite modification $\tilde{X} = \coprod p_{\alpha} \tilde{X}_{\alpha} \rightarrow X = \cup X_{\alpha}$ such that the reduced (real) locus $\coprod \tilde{X}_{\alpha}^{\mathbb{R}}$ is smooth.*

3. Let \mathcal{M} be a self-adjoint positive-definite representation of the hypersurface $X \subset \mathbb{P}^n$. Let $\tilde{X} \rightarrow X$ be the finite modification as above. The representation is \tilde{X}/X -saturated at all the real points of X .

For the definition of \tilde{X}/X -saturated see Section 2.4.5.

Proof. 1. Let $pt \in \mathbb{P}_{\mathbb{R}}^n$ be a hyperbolic point. Suppose $X_{\mathbb{R}}$ has a singular locally irreducible component $(X_{\alpha}^{\mathbb{R}}, 0) \subset (X_{\mathbb{R}}, 0)$, i.e. the multiplicity $\text{mult}(X_{\alpha}^{\mathbb{R}}, 0) > 1$. Suppose $pt \neq 0$. We prove that there exists a family of real lines $L_1(t)$, all passing through the point pt and satisfying the following.

- ★ $L_1(0) \ni 0$; hence $\text{deg}(L_1(0) \cap (X_{\alpha}^{\mathbb{R}}, 0)) \geq \text{mult}(X_{\alpha}^{\mathbb{R}}, 0)$.
- ★ There exists a small neighborhood of $0 \in \mathbb{P}_{\mathbb{R}}^n$, in the classical topology, such that for any $t \neq 0$ the total intersection degree $\text{deg}(L_1(t) \cap X_{\alpha}^{\mathbb{R}})$ in this neighborhood is less than $\text{mult}(X_{\alpha}^{\mathbb{R}}, 0)$.

This will contradict hyperbolicity of the polynomial, implying that either $0 = pt$ or $(X_{\alpha}^{\mathbb{R}}, 0)$ is smooth.

First, the problem can be reduced to the planar case, i.e. $n = 2$. Indeed, let $L_2 \subset (\mathbb{R}^n, 0)$ be the generic real two-dimensional plane through the origin. Then $(L_2 \cap X_{\alpha}^{\mathbb{R}}, 0)$ is locally irreducible and singular. Hence, it is enough to present a family of lines in L_2 , with the needed properties relative to the curve $(L_2 \cap X_{\alpha}^{\mathbb{R}}, 0)$.

Now, observe that a locally irreducible real plane curve $(L_2 \cap X_{\alpha}^{\mathbb{R}}, 0)$ divides the small neighborhood (in classical topology) of $0 \in L_2$ into two parts. Hence, if $\text{mult}(L_2 \cap X_{\alpha}^{\mathbb{R}}, 0) = \text{mult}(X_{\alpha}^{\mathbb{R}}, 0) > 2$ then any family of lines $L_1(t) \subset L_2$, whose generic fiber does not pass through the origin and $0 \in L_1(0)$, has the needed properties.

If $\text{mult}(L_2 \cap X_{\alpha}^{\mathbb{R}}, 0) = \text{mult}(X_{\alpha}^{\mathbb{R}}, 0) = 2$ then the singularity is of type A_k , i.e. after a (real-analytic) change of coordinates the curve is defined by $y^2 = \pm x^{k+1}$. Now the family is constructed directly.

2. If the hypersurface $X \subset \mathbb{P}^n$ has a self adjoint positive definite determinantal representation (and not just semi-definite) then the hyperbolic region is a non-empty open set. Hence by the previous part, the hypersurface has no singular points with singular locally irreducible components. Hence, if $X = \cup p_{\alpha} X_{\alpha}$, there exists the finite modification $\coprod p_{\alpha} \tilde{X}_{\alpha} \rightarrow \cup p_{\alpha} X_{\alpha}$, where each $\tilde{X}_{\alpha}^{\mathbb{R}} \rightarrow X_{\alpha}^{\mathbb{R}}$ is the normalization and $\tilde{X}_{\alpha}^{\mathbb{R}}$ is smooth.

3. As the hypersurface has only smooth multiple local components, it is enough to prove that the representation is locally completely decomposable at every real singular point of X , according to the decomposition into the distinct multiple components; see Property 2.20.

Let $0 \in X_{\mathbb{R}}$ be a real singular point. Let $pt \in \mathbb{P}_{\mathbb{R}}^n \setminus X_{\mathbb{R}}$ be a point in the hyperbolic region. We can assume that the line $\overline{(pt, 0)}$ is not tangent to $X_{\mathbb{R}}$ at any point. Let $\epsilon \in \mathbb{P}_{\mathbb{R}}^n$ be a point near $0 \in X_{\mathbb{R}}$. Consider the line $\overline{(pt, \epsilon)}$. We can assume that this line does not lie on $X_{\mathbb{R}}$.

Restrict the original determinantal representation to this line. So, we get a one-dimensional family of matrices:

$$(1 - t)\mathcal{M}|_{\epsilon} + t\mathcal{M}|_{pt}. \tag{60}$$

By construction $\det((1 - t)\mathcal{M}|_{\epsilon} + t\mathcal{M}|_{pt})$ vanishes precisely for those values of t where the line intersects $X_{\mathbb{R}}$. By hyperbolicity there are $\text{deg}(X)$ such points (counted with multiplicities).

As $\mathcal{M}|_{pt}$ is self-adjoint and positive definite, it can be presented as $\mathcal{M}|_{pt} = U_{pt}U_{pt}^{\tau}$. As $pt \in \mathbb{P}_{\mathbb{R}}^n \setminus X$ one gets that U_{pt} is invertible. Hence the equation above can be presented as

$$\det(t\mathbb{1} + (1 - t)U_{pt}^{-1}\mathcal{M}|_{\epsilon}U_{pt}^{-\tau}) = 0 \tag{61}$$

i.e. as the equation for the eigenvalues of a matrix. As the matrix is self-adjoint, the corresponding eigenvectors are orthogonal. As $\epsilon \rightarrow 0$ they converge to the mutually orthogonal eigenvectors of $U_{pt}^{-1} \mathcal{M}|_0 U_{pt}^{-\tau}$; see e.g. [7].

Hence we obtain: in the limit $\epsilon \rightarrow 0$ the normalized sections of the kernel of $\mathcal{M}|_\epsilon$ have linearly independent limits. Hence, by [Property 2.21](#), the determinantal representation \mathcal{M} is completely decomposable near $0 \in X_{\mathbb{R}}$. Thus it is \tilde{X}/X -saturated. \square

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