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Note

The $L(d, 1)$-number of powers of paths

Anja Kohl

Institute of Discrete Mathematics and Algebra, TU Bergakademie Freiberg, 09596 Freiberg, Germany

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ABSTRACT

Given a graph $G = (V, E)$ and a positive integer d, an $L(d, 1)$-*labelling* of G is a function $f : V \to \{0, 1, \ldots\}$ such that if two vertices x and y are adjacent, then $|f(x) - f(y)| \geq d$; if they are at distance 2, then $|f(x) - f(y)| \geq 1$. The $L(d, 1)$-*number* of G, denoted by $\lambda_{d,1}(G)$, is the smallest number m such that G has an $L(d, 1)$-labelling with $m = \max\{f(x) \mid x \in V\}$. We correct the result on the $L(d, 1)$-number of powers of paths given by Chang et al. in [G.J. Chang, W.-T. Ke, D. Kuo, D.D.-F. Liu, R.K. Yeh, On $L(d, 1)$-labelings of graphs, Discrete Math. 220 (2000) 57–66].

1. Introduction

Let $G = (V, E)$ be a simple, undirected graph with vertex set V and edge set E. For $x, y \in V$ we denote by $d(x, y)$ the distance between x and y, which is the number of edges in a shortest (x, y)-path. The rth power of a graph G, written G^r, is a graph on the same vertex set such that two vertices are joined by an edge if and only if their distance in G is at most r. Let P_n^r, $r \geq 1$, denote the rth power of a path with n vertices.

Given a graph $G = (V, E)$ and a positive integer d, an $L(d, 1)$-*labelling* of G is a function $f : V \to \{0, 1, \ldots\}$ such that for any two vertices $x, y \in V$

1. $|f(x) - f(y)| \geq d$, if $d(x, y) = 1$ and
2. $|f(x) - f(y)| \geq 1$, if $d(x, y) = 2$.

The $L(d, 1)$-*number* of G, denoted by $\lambda_{d,1}(G)$, is the smallest number m such that G has an $L(d, 1)$-labelling with $m = \max\{f(x)|x \in V\}$.

$L(d, 1)$-labellings arose from a variation of the frequency assignment problem introduced by Hale [7]. There has been a high interest in such distance constrained labellings in recent years; see e.g. [1–6,8–10].

In this paper we determine $\lambda_{d,1}(P_n^r)$ for all $d, r, n \in \mathbb{N}_+ := \mathbb{N} \setminus \{0\}$, for which partially wrong values were presented by Chang et al. [3].

2. Main result

Theorem 1. *Let $d, r, n \in \mathbb{N}_+$ and $a := \min\{d, r + 1\}$. Then $\lambda_{1,1}(P_n^r) = \min\{n - 1, 2r\}$, and for $d \geq 2$,*

$$\lambda_{d,1}(P_n^r) = \begin{cases} (n-1)d, & \text{if } n \leq r+1 \\ \left\lceil \dfrac{n}{r+1} \right\rceil - 1 + rd, & \text{if } r+1 < n \leq a(r+1) \\ a + rd, & \text{if } a(r+1) < n. \end{cases} \tag{1}$$

E-mail address: kohl@math.tu-freiberg.de.

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Note that the results on $\lambda_{d,1}(P_n^r)$ given by Chang et al. are wrong for any case where $\min\{d, r+1\} \geq 3$ and $n > 3(r+1)$.

The next two subsections are devoted to the proof of this theorem.

2.1. Preliminaries

Let $n > r+1$. We define *color levels* $C_j := \{jd, 1+jd, \dots, d-1+jd\}$ for $j = 0, 1, \dots r+1$. Since two labels within the same color level have difference at most $d-1$, any $r+1$ consecutive vertices in P_n^r have to get labels from pairwise distinct color levels. For simplification we partition the vertex set of P_n^r into parts of $r+1$ consecutive vertices

$$V(P_n^r) = \{v_0^0, \dots, v_r^0; v_0^1, \dots, v_r^1; \dots; v_0^{q-1}, \dots, v_r^{q-1}; v_0^q, \dots, v_{p-1}^q\}$$

such that $n = q(r+1) + p$, $p \in \{1, 2, \dots, r+1\}$. Note that $q = \lceil \frac{n}{r+1} \rceil - 1 \geq 1$.

Assume that there exists a proper $L(d,1)$-labelling f of P_n^r such that $f(v) \in \bigcup_{j=0}^r C_j$ for all $v \in V(P_n^r)$. Let π be that permutation of the numbers $0, 1, \dots, r$ for which $f(v_{\pi(j)}^0) \in C_j$ for every $j \in \{0, 1, \dots, r\}$. Set $J_i := \{\pi^{-1}(l) \mid 0 \leq l \leq r\} = \{0, 1, \dots, r\}$ for $i \in \{0, 1, \dots, q-1\}$, and $J_q := \{\pi^{-1}(l) \mid 0 \leq l \leq p-1\}$. Then we note the following assertions:

Claim 2. *$f(v_{\pi(j)}^i) \in C_j$ for $i \in \{1, 2, \dots, q\}$; $j \in J_i$.*

Proof. For $i \in \{1, 2, \dots, q\}$ and $j \in J_i$ there exist r vertices that are adjacent both to $v_{\pi(j)}^{i-1}$ and $v_{\pi(j)}^i$. Hence for both vertices the same r color levels are forbidden. Since f uses only $r+1$ color levels the two vertices $v_{\pi(j)}^{i-1}$ and $v_{\pi(j)}^i$ have to receive labels from the same color level, which is C_j because of $f(v_{\pi(j)}^0) \in C_j$. #

Claim 3. *$f(v_{\pi(j)}^{i-1}) \neq f(v_{\pi(j)}^i)$ for $i \in \{1, 2, \dots q\}$; $j \in J_i$.*

Proof. For $i \in \{1, 2, \dots q\}$ and $j \in J_i$ the two vertices $v_{\pi(j)}^{i-1}$ and $v_{\pi(j)}^i$ have distance 2; therefore they have to receive distinct labels. #

Claim 4. *For $i \in \{0, 1, \dots, q-1\}$ and $j \in J_i$ let $a_j := f(v_{\pi(j)}^i) - jd$. Then $a_j \in \{0, 1, \dots, d-1\}$ and $a_r \geq a_{r-1} \geq \dots \geq a_0$.*

Proof. Since $f(v_{\pi(j)}^i) \in C_j$, it follows that $a_j \in \{0, 1, \dots, d-1\}$. Let $k \in \{0, 1, \dots, r-1\}$. By the definition of C_k and C_{k+1}, $f(v_{\pi(k+1)}^i) > f(v_{\pi(k)}^i)$, and calling in the distance 1 condition, we conclude $f(v_{\pi(k+1)}^i) - f(v_{\pi(k)}^i) \geq d$. This implies $a_{k+1} \geq a_k$ for any $k \in \{0, 1, \dots, r-1\}$. #

Claim 5. *If $\pi(j) < \pi(j+1)$ then $f(v_{\pi(j+1)}^i) \geq \max\left\{f(v_{\pi(j)}^i), f(v_{\pi(j)}^{i+1})\right\} + d$ for $i \in \{0, 1, \dots q-1\}$; $j \in J_{i+1} \wedge j \leq r-1$. If $\pi(j) > \pi(j+1)$ then $f(v_{\pi(j+1)}^i) \geq \max\left\{f(v_{\pi(j)}^{i-1}), f(v_{\pi(j)}^i)\right\} + d$ for $i \in \{1, 2 \dots q\}$; $j \in J_i \wedge j \leq r-1$.*

Proof. If $i \leq q-1$, $\pi(j) < \pi(j+1)$, $j \leq r-1$, and $j \in J_{i+1}$ then the vertices $v_{\pi(j)}^i$, $v_{\pi(j)}^{i+1}$, $v_{\pi(j+1)}^i$ exist and $v_{\pi(j+1)}^i$ is adjacent to both $v_{\pi(j)}^i$ and $v_{\pi(j)}^{i+1}$. If $i \geq 1$, $\pi(j) > \pi(j+1)$, $j \leq r-1$, and $j \in J_i$ then the vertices $v_{\pi(j)}^{i-1}$, $v_{\pi(j)}^i$, $v_{\pi(j+1)}^i$ exist and $v_{\pi(j+1)}^i$ is adjacent to both $v_{\pi(j)}^{i-1}$ and $v_{\pi(j)}^i$. Applying the distance 1 condition yields the desired inequalities. #

For $k \in \{0, 1, \dots, r\}$ let i_k be the number of integers j, $j \in \{0, 1, \dots, k-1\}$, such that $\pi(j+1) < \pi(j)$.

Lemma 6. *Let $k \in \{0, 1, \dots, r\}$.*

If $\pi(0) = 0$ then $\max\{f(v_{\pi(k)}^i), f(v_{\pi(k)}^{i+1})\} \geq k+1+kd$ for $i = i_k, \dots, i_k+q-k-1$. If $\pi(0) > 0$ then this inequality holds only for $i = i_k, \dots, i_k+q-k-2$.

Proof. Suppose $\pi(0) = 0$. We apply induction on k.

Let $k = 0$. Then $i_0 = 0$ and $f(v_0^i) \geq 0$ for $i \in \{0, 1, \dots, q\}$. By Claim 3, $\max\{f(v_{\pi(0)}^i), f(v_{\pi(0)}^{i+1})\} \geq 1$ for $i \in \{0, 1, \dots, q-1\}$.

Assuming the statement holds for k we prove it for $k+1$ ($k \geq 0$).

Case 1. $\pi(k) < \pi(k+1)$, i.e. $i_{k+1} = i_k$.

By the induction hypothesis and Claim 5, $f(v_{\pi(k+1)}^i) \geq k+1+(k+1)d$ for $i \in \{i_k, \dots, i_k+q-k-1\}$, and according to Claim 3, $\max\left\{f(v_{\pi(k+1)}^i), f(v_{\pi(k+1)}^{i+1})\right\} \geq k+2+(k+1)d$ for $i \in \{i_k, \dots, i_k+q-(k+1)-1\}$.

Case 2. $\pi(k) > \pi(k+1)$, i.e. $i_{k+1} = i_k+1$.

By the induction hypothesis and Claim 5, $f(v_{\pi(k+1)}^i) \geq k+1+(k+1)d$ for $i \in \{i_k+1, \dots, i_k+q-k\}$. Using Claim 3, we obtain $\max\left\{f(v_{\pi(k+1)}^i), f(v_{\pi(k+1)}^{i+1})\right\} \geq k+2+(k+1)d$ for $i \in \{i_k+1, \dots, i_k+1+q-(k+1)-1\}$.

If $\pi(0) > 0$ then in the induction basis we can just guarantee $\max\{f(v_{\pi(0)}^i), f(v_{\pi(0)}^{i+1})\} \geq 1$ for $i \in \{0, 1, \dots, q-2\}$ because the vertex $v_{\pi(0)}^q$ may not exist. The induction step is analogous to that for the case $\pi(0) = 0$. Hence, we only have to reduce the upper bound for the variable i by 1. □

Lemma 7. *There exists an integer i, $i \in \{0, 1, \dots, q\}$, such that $f(v^i_{\pi(r)}) \geq r + 1 + rd$ if $q > r + 1$ and $f(v^i_{\pi(r)}) \geq q + rd$ if $q \leq r + 1$.*

Proof. Let $q > r + 1$. The two vertices $v^{i_r}_{\pi(r)}$ and $v^{i_r+1}_{\pi(r)}$ exist because of $i_r \geq 0$ and $i_r + 1 \leq r + 1 \leq q - 1$. By Lemma 6, $\max\{f(v^{i_r}_{\pi(r)}), f(v^{i_r+1}_{\pi(r)})\} \geq r + 1 + rd$. Hence, there is a vertex with label at least $r + 1 + rd$.

Let $q \leq r + 1$ and $t := \pi^{-1}(0)$.

Case 1. $t = 0$. According to Lemma 6 there exists $i' \in \{i_{q-1}, i_{q-1} + 1\}$ such that $f(v^{i'}_{\pi(q-1)}) \geq q + (q-1)d$. Since $i_{q-1} \geq 0$ and $i_{q-1} + 1 \leq q - 1$ it follows that $0 \leq i' \leq q - 1$. Hence, $f(v^{i'}_{\pi(r)}) \geq q + rd$, by Claim 4.

Case 2. $t > 0 \wedge q = 1$.

By Claim 3, $\max\{f(v^0_{\pi(t)}), f(v^1_{\pi(t)})\} \geq 1 + td$. Hence, if $t = r$ then the vertex with label at least $1 + rd = q + rd$ exists. If $t < r$ then the vertex $v^0_{\pi(t+1)}$ is adjacent to $v^0_{\pi(t)}$ and $v^1_{\pi(t)}$, such that we can conclude $f(v^0_{\pi(t+1)}) \geq 1 + (t+1)d$, by Claim 5. According to Claim 4 it follows that $f(v^0_{\pi(r)}) \geq 1 + rd = q + rd$.

Case 3. $t > 0 \wedge 2 \leq q \leq r + 1$.

According to Lemma 6 there exists $i^* \in \{i_{q-2}, i_{q-2} + 1\}$ such that $f(v^{i^*}_{\pi(q-2)}) \geq q - 1 + (q-2)d$.

Subcase 3.1. $t > q - 2$.

Since $i_{q-2} \geq 0$ and $i_{q-2} + 1 \leq q - 1$ it follows that $0 \leq i^* \leq q - 1$. Hence, we can apply Claim 4 to obtain $f(v^{i^*}_{\pi(t-1)}) \geq q - 1 + (t-1)d$. The vertices $v^{i^*}_{\pi(t)}$, $v^{i^*+1}_{\pi(t)}$ exist and they are adjacent to $v^{i^*}_{\pi(t-1)}$. Therefore $\max\{f(v^{i^*}_{\pi(t)}), f(v^{i^*+1}_{\pi(t)})\} \geq q + td$, by Claim 3. By an argument similar to that for Claim 4, we get $f(v^{i^*}_{\pi(r)}) \geq q + rd$.

Subcase 3.2. $t \leq q - 2$.

From $0 < t \leq q - 2$ we know that $i_{q-2} \geq 1$ and $q \geq 3$.

Suppose $i_{q-2} = q - 2$, i.e. $\pi(q-2) < \pi(q-3) < \dots < \pi(0)$. Then $t = q - 2$ holds. Because of $1 \leq i_{q-2} \leq i^* \leq q - 1$, the vertices $v^{i^*-1}_{\pi(q-1)}$, $v^{i^*}_{\pi(q-1)}$ exist and they are adjacent to $v^{i^*}_{\pi(q-2)} = v^{i^*}_{\pi(t)}$. Hence, it follows that $\max\{f(v^{i^*-1}_{\pi(q-1)}), f(v^{i^*}_{\pi(q-1)})\} \geq q + (q-1)d$, by Claim 3. Applying Claim 4 we obtain $\max\{f(v^{i^*-1}_{\pi(r)}), f(v^{i^*}_{\pi(r)})\} \geq q + rd$. Therefore a vertex with label at least $q + rd$ exists.

Suppose $i_{q-2} \leq q - 3$. Hence, $1 \leq i^* \leq q - 2$. If $\pi(q-2) < \pi(q-1)$ then $\max\{f(v^{i^*-1}_{\pi(q-1)}), f(v^{i^*}_{\pi(q-1)})\} \geq q + (q-1)d$, according to Claim 5. By Claim 4, $\max\{f(v^{i^*-1}_{\pi(r)}), f(v^{i^*}_{\pi(r)})\} \geq q + rd$. If $\pi(q-2) > \pi(q-1)$ we use an analogous argument to show $\max\{f(v^{i^*}_{\pi(r)}), f(v^{i^*+1}_{\pi(r)})\} \geq q + rd$. This proves the existence of a vertex with label at least $q + rd$. $\square$

2.2. *Proof of* Theorem 1

If $d = 1$ then $\lambda_{1,1}(P^r_n) = \lambda_{1,0}(P^{2r}_n) = \chi(P^{2r}_n) - 1 = \min\{n-1, 2r\}$.

Now suppose $d \geq 2$. If $n \leq r+1$ then $P^r_n \cong K_n$, and therefore $\lambda_{d,1}(P^r_n) = \lambda_{d,0}(P^r_n) = d \cdot \lambda_{1,0}(P^r_n) = d(\chi(P^r_n) - 1) = d(n-1)$.

Let $n > r + 1$. Obviously, $\lambda_{d,1}(P^r_n) \geq \lambda_{d,1}(P^r_{r+1}) = rd$. If we label the first vertices using the sequence

$$\underbrace{0, d, \dots, (r+1)d}_{r+2 \text{ terms}}, \qquad \underbrace{1, 1+d, \dots, 1+rd}_{r+1 \text{ terms,}}$$

and repeat this pattern if necessary for the remaining vertices, then we obtain a proper $L(d, 1)$-labelling with maximum label at most $(r+1)d$. Hence, $\lambda_{d,1}(P^r_n) \leq (r+1)d$.

Assume that there exists an $L(d, 1)$-labelling f of P^r_n with $\max_{v \in V(P^r_n)} f(v) < (r+1)d$, i.e. $\forall v \in V(P^r_n) : f(v) \in \bigcup_{j=0}^{r} C_j$. By Lemma 7,

$$\max_{v \in V(P^r_n)} f(v) \geq \begin{cases} q + rd, & \text{for } q \leq r + 1 \\ r + 1 + rd, & \text{for } q > r + 1. \end{cases} \tag{2}$$

This contradicts the assumption $\max_{v \in V(P^r_n)} f(v) < (r+1)d$ for the case $d \leq \min\{q, r+1\}$. Here the labelling scheme specified above is optimal and yields $\lambda_{d,1}(P^r_n) = (r+1)d$.

Now let $d > \min\{q, r+1\}$. By inequality (2), $\lambda_{d,1}(P^r_n) \geq q + rd$ for $q \leq r + 1$ and $\lambda_{d,1}(P^r_n) \geq r + 1 + rd$ for $q > r + 1$. For both cases we show that the lower bound is sharp by construction of a proper $L(d, 1)$-labelling with maximum label $q + rd$ or $r + 1 + rd$, respectively.

If $q \leq r + 1$ we use the following *decreasing labelling scheme*:

$$\underbrace{q, q+d, \dots, q+rd}_{r+1 \text{ terms}}, \underbrace{q-1, q-1+d, \dots, q-1+rd}_{r+1 \text{ terms}}, \dots, \underbrace{0, d, \dots, (p-1)d}_{p \text{ terms}}.$$

So we obtain a proper $L(d, 1)$-labelling of P^r_n with maximum label $q + rd = \left\lceil \frac{n}{r+1} \right\rceil - 1 + rd$. In the case of $q > r + 1$ we apply an *alternating labelling scheme*:

$$\underbrace{0, 1+d, \dots, r+rd}_{r+1 \text{ terms}}, \qquad \underbrace{1, 2+d, \dots, (r+1)+rd}_{r+1 \text{ terms}}$$

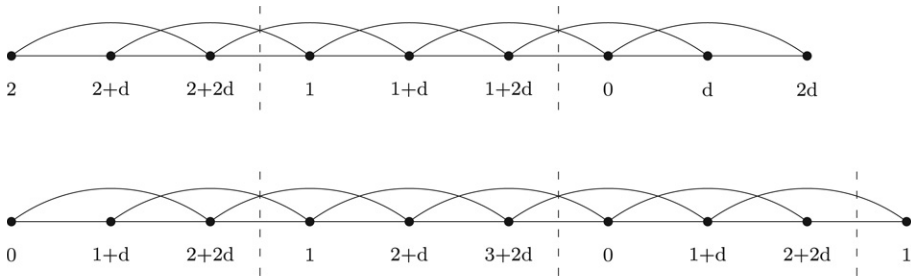

Fig. 1. $L(d, 1)$-labellings of the graphs P_9^2 and P_{10}^2.

and repeat it until all vertices have received a label. This provides a proper $L(d, 1)$-labelling of P_n^r with maximum label $r + 1 + rd$.

Sorting the values for $\lambda_{d,1}(P_n^r)$ depending on n yields the result (1). $\square$

Example 8. Let $d \geq 3$. We consider the two graphs P_9^2 and P_{10}^2. According to the proof of Theorem 1, we label the vertices of P_9^2 using the decreasing labelling scheme and those of P_{10}^2 using an alternating labelling scheme (see Fig. 1).

3. Concluding remarks

Distance constrained labellings can be generalized to an arbitrary number k of distance constraints.

Let $p_1, \ldots, p_k \in \mathbb{N}$. An *$L(p_1, \ldots, p_k)$-labelling* of a graph $G = (V, E)$ is a function $f : V \to \{0, 1, \ldots\}$ such that $|f(x) - f(y)| \geq p_i$ for any two vertices $x, y \in V$ with distance $d(x, y) \leq i \leq r$. $\lambda_{p_1,\ldots,p_k}(G)$ is the smallest number m for which an $L(p_1, \ldots, p_k)$-labelling of G with maximum label m exists.

Obviously, an $L(d, 1)$-labelling of a graph power G^r is equivalent to an $L(d, \ldots, d, 1 \ldots, 1)$-labelling of G (with $2r$ distance constraints). Hence, the determination of $\lambda_{d,1}$ for a graph power provides several bounds and information for other distance constrained labellings. Since paths and/or cycles occur as subgraphs in any graph G it is advisable to consider $L(d, 1)$-labellings for powers of paths and cycles. In this paper we established $\lambda_{d,1}(P_n^r)$ for all $d, n, r \in \mathbb{N}_+$.

Let C_n^r be the rth power of a cycle with n vertices. In [9] we determined $\lambda_{d,1}(C_n^r)$ for all $n, r \in \mathbb{N}$ and $d \geq 3$ as well as bounds for $\lambda_{2,1}(C_n^r)$. The calculation of these values is very extensive and needs a lot of case analysis; therefore we will not present it here.

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