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Path Integral Quantization of Constrained Systems

BY

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0.1 Dedication

To my parents

Tahany Riziq Dalloul

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0.2 Acknowledgements

First of all, I would like to express my gratitude to Allah who guided and aided me to bring this thesis to light.
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0.3 Abstract

In this thesis some singular physical systems are quantized using the canonical formulation and the Dirac's method. The two methods represent the Hamiltonian treatment of the constrained systems. Dirac's method introduced the primary constraints, then constructing the total Hamiltonian. The consistency conditions are checked on the primary constraints. The equations of motion in this method are in ordinary differential equation form. In the canonical method the equations of motion are total differential equations in many variables. These equations are integrable if the integrability conditions are identically satisfied.

Path integral quantization of three different systems, are studied, free relativistic spinless particle, relativistic spinless particle in an external electromagnetic field and a charged particle moving in a constant magnetic field. In the study, the integrability conditions are satisfied, so the systems are integrable. Consequently the path integral quantization is obtained directly as an integration over the canonical reduced phase space coordinates. This makes the canonical method simpler than Dirac's method.

0.4 Arabic Abstract

Chapter 1

Introduction

This work is mainly concerned with the path integral quantization of singular systems (constrained systems), which are characterized by singular Lagrangian. In this chapter we will make a brief review for the singular systems, important basic definitions, primary and secondary constraints, first and second class constraints, and consistency conditions. The main objective of this thesis is to study the path integral quantization of some constrained systems using two different approach, Dirac's approach and the canonical approach.

1.1 Historical Background

The study of constrained systems for the purpose of quantization was initiated by Dirac [1,2], where he sets up the formalism for treating singular systems and constraints. He showed that in the presence of the constraints, the numbers of degrees of freedom of the dynamical system was reduced. His approach are subsequently extended to continuous systems [3].

The presence of constraints in the singular Lagrangian theories makes one be careful when applying Dirac's method, especially when first class-constraints arise. Dirac showed that the algebra of Poisson's brackets determine a division of constraints into two classes: the first-class constraints and the second-class ones. The first-class constraints which have zero Poisson's brackets with all other constraints in the subspace of phase space in which constraints hold. Constraints, which are not first-class, are by definition second-class. In the case of second-class constraints Dirac introduced a new Poisson brackets, the Dirac brackets, to attain self-consistency. However, whenever we adopt the Dirac method, we frequently encounter the problem of the operating ordering ambiguity. In order to avoid this problem, Batalin, Fradkin and Tyutin (BFT) developed a method by enlarging the phase space with some extra variables such that the second-class constraints being converted into first-class ones, which are considered as generators of gauge transformations, this will lead to the gauge freedom. In other words, the equations of motion are still degenerate and depend on the functional arbitrariness, one has to impose external gauge fixing constraint for each first-class constraint, which is not always an easy task [4].

Folloing Dirac, there is another approach for quantizing constrained systems of classical singular theories which is the path integral approach given by Faddeev [5]. This approach has applied when only first-class constraints are present. It was shown by Faddeev that gauge-fixing condition should be imposed for each first-class constraint in order to convert the system into second-class constraints. By this is meant, the introduction of some constraints, $\chi_\alpha = 0$, are supposed to have vanishing Poisson bracket with the canonical Hamiltonian H_0 In order to eliminate the unphysical variable. However Faddeev's Hamiltonian path integral method for a singular Lagrangian is generalized to the case when the second-class constraints appear in the theory by Senjanovic [6]. Moreover Fradkin [7] considered quantization of bosonic theories with the first and second-class constraints and its extension to include fermions in the canonical gauges. More, Gitman and Tyutin [8] discussed the canonical quantization of singular theories as well as the Hamiltonian formalism of gauge theories in an arbitrary gauge. Recently, an alternative approach was developed by Buekenhout, Sprague and Faddeev [9,10] without following Dirac step by step. In this formalism there is no need to distinguish between first and second-class or primary and secondary constraints. Where the primary constraint is a set of relations connected between the momenta and the coordinates.

The general formalism is then applied to several problems, quantization of the massive Yang-Mills field theory, Light-Cone quantization of the self interacting scalar field theory, and quantization of a local field theory of magnetic monopolies, etc.

A most powerful approach for treating constrained systems is the canonical approach [11,12]. The Hamilton-Jacobi approach which is called canonical method has been developed to investigate the constrained systems. Several constrained systems were investigated by using the canonical method [13-17]. The equivalent Lagrangian method is used to obtain a set of Hamilton-Jacobi Partial Differential Equations (HJPDE). In this approach, the distinction between the first and second-class constraints is not necessary. The equations of motion are written as total differential equations in many variables which require integrability conditions. In other words, the integrability conditions may lead to a new constraint. Moreover, it is shown that gauge fixing, which is an essential procedure to study singular systems by Dirac's method, is not necessary if the canonical method is used [18-20]. Simultaneous solutions of the canonical equations with all these constraints provide to obtain the set of canonical phase space coordinates, besides the canonical action integral is obtained in terms of the canonical coordinates.

1.2 Constrained Systems

The singular Lagrangian system represents a special case of a more general dynamics called constrained system [2]. The dynamics of the physical system is encoded by the Lagrangian, a function of positions and velocities of all degrees of freedoms which comprise the system [21]. The singular Lagrangian can be achieved by two formulations, the Lagrangian and the Hamiltonian formulations. The Lagrangian formulation of classical physics requires the configuration space formed by n generalized coordinates q_i, n generalized velocities $\dot{q}_i$ and parameter τ, defined as

$$L \equiv L(q_i, \dot{q}_i; \tau), \qquad i = 1, \ldots, n. \tag{1.1}$$

where τ is a parameter which henceforth will be the time on which the coordinates q_i depend.

For a system characterized by this Lagrangian, the action which is a function of path in configuration space reads as

$$S = \int L(q_i, \dot{q}_i; \tau)\, dt. \tag{1.2}$$

The action principle asserts that the path which satisfies the classical equation is the one which brings the action to extremes

$$\delta S = \delta \int L(q_i, \dot{q}_i; \tau)\, dt.$$
$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i}\, \delta q_i + \frac{\partial L}{\partial \dot{q}_i}\, \delta \dot{q}_i + \frac{\partial L}{\partial t}\, \delta t \right). \tag{1.3}$$

In deriving (1.3), it was assumed that $\dot{q}_i$ is dependent of q_i, so that $\delta \dot{q}_i = \frac{d}{dt}\, \delta q_i$. Imposing $\delta S = 0$, we obtain the Euler-Lagrange equations of motion

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0. \tag{1.4}$$

So, the Lagrangian equations are of second order.

To go over the Hamiltonian formalism, defining a generalized momentum p_i conjugate to q_i as [21,22]

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \tag{1.5}$$

then the momentum is function of q_j and $\dot{q}_j$ such that,

$$p_i = p_i(q_j, \dot{q}_j) \qquad j = 1, \ldots, n. \tag{1.6}$$

The canonical Hamiltonian H_0 is defined by

$$H_0 = \sum_{i=1}^{n} \dot{q}_i p_i - L. \tag{1.7}$$

Consider the differential of the Lagrange function (1.1) and using eqs. (1.4), (1.5) and (1.7), then we read off the Hamilton's equations of motion as

$$\dot{q}_i = \frac{\partial H_0}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H_0}{\partial q_i}. \tag{1.8}$$

It is standard national practice to define the poisson bracket of two functions f and g on phase space by [20]

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right), \tag{1.9}$$

thus the Hamilton's equation may be written as

$$\dot{q}_i = \{q_i, H_0\}, \qquad \dot{p}_i = \{p_i, H_0\}. \tag{1.10}$$

So, the time evolution of any function of positions and momenta is given by

$$\frac{dF}{dt} = \{F, H_0\} + \frac{\partial F}{\partial t}. \tag{1.11}$$

In order to characterize the constrained systems; one evaluates the time derivative of the momentum as

$$\frac{dp_i}{dt} = \frac{\partial p_i}{\partial q_j} \dot{q}_j + \frac{\partial p_i}{\partial \dot{q}_j} \ddot{q}_j. \tag{1.12}$$

But we can write the Lagrangian equation of motion (1.4) as

$$\frac{\partial L}{\partial q_i} - \frac{dp_i}{dt} = 0, \tag{1.13}$$

then by using the definition (1.5) and the Lagrangian equation of motion (1.4), we get

$$\frac{\partial L}{\partial q_i} = \frac{\partial p_i}{\partial q_j} \dot{q}_j + \frac{\partial p_i}{\partial \dot{q}_j} \ddot{q}_j. \tag{1.14}$$

$$\frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial q_i\,\partial \dot{q}_j}\,\dot{q}_j - \frac{\partial^2 L}{\partial \dot{q}_i\,\partial \dot{q}_j}\,\ddot{q}_j = 0. \tag{1.15}$$

Defining Hessian matrix elements A_{ij} of second derivatives of the Lagrangian with respect to velocities as

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \tag{1.16}$$

so we can solve $\ddot{q}_j$ as

$$\ddot{q}_j = A_{ij}^{-1}\left[\frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial q_i\,\partial \dot{q}_j}\,\dot{q}_j\right]. \tag{1.17}$$

A valid phase space is formed if the rank of the Hessian matrix is n. Systems, which posses this property, are called regular and their treatments are found in a standard mechanics books. Systems, which have the rank less than n are called singular systems. Thus, by definition we have [2]

$$Hessian = det\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) = \begin{cases} \neq 0 & regular\ system, \\ = 0 & singular\ system. \end{cases} \tag{1.18}$$

To clarify the situation of singular systems, it can be investigated by two different approach of quantization.

1.3 Dirac Approach

The standard quantization methods can't be applied directly to the singular Lagrangian theories. However, the basic idea of the classical treatment and the quantization of such systems were presented along time by Dirac [1,2]. And is now widely used in investigating the theoretical models in a contemporary elementary particle physics and applied in high energy physics, especially in the gauge theories [8].

The presence of constraints in such theories makes one careful on applying Dirac's method, especially when first-class constraints arise. This is because the first-class constraints are generators of gauge transformation which lead to the gauge freedom [16].

Let us consider a system which is described by the Lagrangian (1.1) such that the rank of the Hessian matrix is $(n - r)$, $r < n$.

The singular system characterized by the fact that all velocities $\dot{q}_i$ are not uniquely determined in terms of the coordinates and momenta only. In other words, not all momenta are independent, and there must exist a certain set of relations among them, of the form

$$\phi_m(p_i, q_i) = 0, \tag{1.19}$$

The q's and the p's are the dynamical variables of the Hamiltonian theory. They are connected by the relations (1.19) which are called primary constraints of the Hamiltonian formalism. Since the rank of the Hessian matrix is $(n-r)$, the momenta components will be functionally dependent. The first $(n-r)$ equations of (1.5) can be solved for the $(n-r)$ components of $\dot{q}_i$ in terms of q_i as well as the first $(n-r)$ components of p_i and the last r components of $\dot{q}_i$.

In other words

$$\dot{q}_a = \omega_a(q_i, p_a, \dot{q}_\mu), \tag{1.20}$$

$$a = 1, \ldots, n-r, \qquad \mu = 1, \ldots, r, \qquad i = 1, \ldots, n$$

If these expressions for the $\dot{q}_a$ are substituted into the last r equation of (1.5), the resulting equations will yield r relations of the form

$$p_\mu = \left.\frac{\partial L}{\partial \dot{q}_\mu}\right|_{\dot{q}_a = \omega_a} \equiv -H_\mu(q_i, p_a, \dot{q}_\nu). \tag{1.21}$$

These relations indicate that the generalized momenta p_μ are dependent of p_a, which is natural result of the singular nature of the Lagrangian. Eq (1.21) can be written in the form

$$H'_\mu(q_i, p_a, \dot{q}_\nu) \equiv p_\mu + H_\mu \approx 0, \tag{1.22}$$

which are called primary constraints [1,2].

Now the usual Hamiltonian H_0 for any dynamical system is defined as

$$H_0(p_i, q_i) = p_i\,\dot{q}_i - L \tag{1.23}$$

(Here the Einstein summation rule is used which is a convention when repeated indices are implicitly summed over).

H_0 will not be uniquely determined, since we may add to it any linear combinations of the primary constraints H'_μ's which are zero, so that the total Hamiltonian is [2, 3, 7]

$$H_T = H_0 + \lambda_\mu H'_\mu \tag{1.24}$$

where $\lambda_\mu(q, p)$ being some unknown coefficients, they are simply Lagrange's undetermined multipliers. Making use of the Poisson brackets, one can write the total time derivative of any function $g(q, p)$ as

$$\dot{g} \equiv \frac{dg}{d\tau} \approx \{g, H_T\} = \{g, H_0\} + \lambda_\mu \{g, H'_\mu\} \tag{1.25}$$

where Dirac's symbol ($\approx$) for weak equality has been used in the sense that one can't consider $H'_\mu = 0$ identically before working out the Poisson brackets. Thus the equations of motion can be written as

$$\dot{q}_i \approx \{q_i, H_T\} = \{q_i, H_0\} + \lambda_\mu \{q_i, H'_\mu\} \tag{1.26}$$

$$\dot{p}_i \approx \{p_i, H_T\} = \{p_i, H_0\} + \lambda_\mu \{p_i, H'_\mu\} \tag{1.27}$$

subject to the so-called consistency conditions. This means that the total time derivative of the primary constraints should be zero;

$$\begin{aligned} \dot{H}'_\mu \equiv \frac{dH'_\mu}{d\tau} &\approx \{H'_\mu, H_T\} \\ &= \{H'_\mu, H_0\} + \lambda_\nu \{H'_\mu, H'_\nu\} \approx 0, \qquad \mu, \nu = 1, \ldots, r. \end{aligned} \tag{1.28}$$

These equations may be reduced to $0 = 0$, where it is identically satisfied as a result of primary constraints, else they will be lead to new conditions which are called secondary constraints. Repeating this procedure as many times as needed, one arrives at a final set of constraints or/and specifies some of λ_μ. Such constraints are classified into two types, a) First-class constraints which have vanishing Poisson brackets with all other constraints. b) Second-class constraints which have non-vanishing Poisson brackets. The second-class constraints could be used to eliminate conjugated pairs of the p's and q's from the theory by expressing them as functions of the remaining p's and q's. The total Hamiltonian for the remaining variable is then the canonical Hamiltonian plus the primary constraints H'_μ of the first type as in eq. (1.24), where H'_μ are all the independent remaining first-class constraints.

The first-class constraints are the generators of the gauge transformations. This will lead to the gauge freedom. Besides, λ_μ are still undetermined. To remove this arbitrariness, one has to impose external gauge constraints for each first-class constraints. Such a gauge fixing,

$$\chi = 0, \tag{1.29}$$

which is a set of constraints independent of H'_μ and equal in number to all first-class constraints H'_μ. Such a choice makes the whole set of constraints $\{H'_\mu, H'_\nu\}$ to be second-class constraints, with

$$det\{H'_\mu, H'_\nu\} \neq 0, \qquad \mu, \nu = n - r + 1, \dots, n. \tag{1.30}$$

This is a canonical physical gauge if it does not violate the equation of motion [3,16].

Fixing any gauge is not an easy task, since we fix it by hand and there is no basic rule to select it.

1.4 Canonical Path Integral Quantization

In this section, we study the constrained systems by using the canonical method and demonstrate the fact that the gauge fixing problem is solved naturally.

Let us consider a system which is described by the Lagrangian (1.1), such that the rank of the Hessian matrix defined in (1.16) of rank $(n-r)$, $r < n$. The generalized momenta p_i corresponding to the generalized coordinates q_i are defined as

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \qquad a = 1, 2, \ldots, n-r, \tag{1.31}$$

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \qquad \mu = n-r+1, \ldots, n. \tag{1.32}$$

Since the rank of the Hessian matrix is $(n-r)$, one may solve eq. (1.31) for $\dot{q}_a$ as

$$\dot{q}_a = \dot{q}_a(q_i, p_a, \dot{q}_\mu; \tau) \equiv \omega_a \tag{1.33}$$

Substituting eq. (1.33), into eq. (1.32), we get

$$p_\mu = \left.\frac{\partial L}{\partial \dot{q}_\mu}\right|_{\dot{q}_a = \omega_a} \equiv -H_\mu(q_i, \dot{q}_\nu, p_a; \tau). \tag{1.34}$$

Relations (1.34) indicate the fact that the generalized momenta p_μ are not independent of p_a which is a natural result of the singular nature of the lagrangian.

The canonical Hamiltonian H_o is defined as

$$H_0 = -L(q_i, \dot{q_\nu}, \dot{q_a} \equiv \omega_a, \tau) + p_a\omega_a + p_\mu\dot{q_\mu}\Big|_{p_\nu=-H_\nu}. \qquad (1.35)$$

The set of the Hamilton-Jacobi Partial Differential Equations (HJPDE) is expressed as

$$H_0'\left(\tau,\ q_\nu,\ q_a,\ p_i = \frac{\partial S}{\partial q_i},\ p_0 = \frac{\partial S}{\partial \tau}\right) = 0, \qquad (1.36)$$

$$H_\mu'\left(\tau,\ q_\nu,\ q_a,\ p_i = \frac{\partial S}{\partial q_i},\ p_0 = \frac{\partial S}{\partial \tau}\right) = 0, \qquad (1.37)$$

Eqs (1.36) and (1.37) may be expressed in a compact form as

$$H_\alpha'\left(\tau,\ q_\nu,\ q_a,\ p_i = \frac{\partial S}{\partial q_i},\ p_0 = \frac{\partial S}{\partial \tau}\right) = 0, \qquad (1.38)$$

$$\alpha = 0, n-r+1, \ldots, n.$$

where

$$H_0' = p_0 + H_0 = 0, \qquad H_\mu' = p_\mu + H_\mu = 0. \qquad (1.39)$$

Here H_0' can be interpreted as the generator of time evolution while H_μ' are the generators of gauge transformation.

The fundamental equations of the equivalent Lagrangian method are

$$p_0 = \frac{\partial S}{\partial \tau} \equiv -H_0(q_i, \dot{q_\nu}, p_a; \tau), \quad p_a = \frac{\partial S}{\partial q_a}, \quad p_\mu = \frac{\partial S}{\partial q_\mu} \equiv -H_\mu, \qquad (1.40)$$

with $q_0 = \tau$, and S being the action.

The equations of motion are obtained as total differential equations and take the form [11].

$$dq_r = \frac{\partial H'_\alpha}{\partial p_r}\, dt_\alpha, \qquad r = 0, 1, \ldots, n\,, \tag{1.41}$$

$$dp_a = -\frac{\partial H'_\alpha}{\partial q_a}\, dt_\alpha, \qquad a = 1, \ldots, n - r, \tag{1.42}$$

$$dp_\mu = -\frac{\partial H'_\alpha}{\partial q_\mu}\, dt_\alpha, \qquad \mu = n - r + 1, \ldots, n, \tag{1.43}$$

$$\alpha = 0, n - r + 1, \ldots, n.$$

Defining

$$Z = S(t_\alpha, q_a), \tag{1.44}$$

and making use of eq. (1.41) and the definitions of generalized momenta in (1.40) we obtain,

$$dZ = \frac{\partial S}{\partial t_\alpha}\, dt_\alpha + \frac{\partial S}{\partial q_a}\, dq_a = (-H_\alpha dt_\alpha + p_a dq_a)$$

$$dZ = \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right)\, dt_\alpha. \tag{1.45}$$

Eqs (1.41-1.43) and (1.45) are called the total differential equations for the characteristics.

Now, we will discuss the integrability conditions for Eqs (1.41-1.43) and the action function (1.44), to obtain the necessary and sufficient conditions that the system of total differential equations (1.41-1.43) and (1.45) be completely integrable in order to obtain the the path integral quantization of the constrained system.

To any set of total differential equations [23,24].

$$dx_i = b_{i\alpha}(t_\beta, x_j)\, dt_\alpha. \tag{1.46}$$

$$i, j = 1, \ldots, n+1, \qquad \alpha, \beta = 0, 1 \ldots, r < n.$$

These corresponds a set of partial differential equations of the form

$$b_{i\alpha}\frac{\partial f}{\partial x_i} = 0. \tag{1.47}$$

To solve the set (1.47), we introduce the following linear operator:

$$X_\alpha f = b_{i\alpha}\frac{\partial f}{\partial x_i}. \tag{1.48}$$

Equation (1.46) are integrable if the corresponding set of partial differential equations (1.48) is a Jacobi system: " Complete system". In other words, the following relations should hold

$$(X_\alpha, X_\beta)f = (X_\alpha X_\beta - X_\beta X_\alpha)f = 0, \qquad \forall\ \alpha, \beta. \tag{1.49}$$

Those relations which cannot be expressed in the form

$$(X_{\alpha'}, X_{\beta'}) = C^J_{\alpha'\beta'} X_J f, \tag{1.50}$$

may be added as new equations. Thus, either one obtains a complete system or a trivial solution

$$f = const. \tag{1.51}$$

Now let us investigate the integrability conditions of equations (1.41-1.43) and (1.45). To achieve this goal we define the linear operator X_α which correspond to total differential equations (1.41-1.43) and (1.45) as

$$\begin{aligned} X_\alpha f(t_\beta, q_a, p_a, z) &= \frac{\partial f}{\partial t_\alpha} + \frac{\partial H'_\alpha}{\partial p_a}\frac{\partial f}{\partial q_a} - \frac{\partial H'_\alpha}{\partial q_a}\frac{\partial f}{\partial p_a} \\ &+ \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a}\right)\frac{\partial f}{\partial z}, \\ &= [H'_\alpha, f]\,\frac{\partial f}{\partial z}\,H'_\alpha, \end{aligned} \tag{1.52}$$

$$\alpha, \beta = 0, 1\ldots, r < n, \qquad a = 1\ldots, n-r.$$

Lemma. A system of total differential equations (1.41-1.43) and (1.45) is integrable if and only if

$$[H'_\alpha, H'_\beta] = 0, \qquad \forall\ \alpha, \beta. \tag{1.53}$$

Proof. Suppose that eq. (1.53) is satisfied, then

$$\begin{aligned} (X_\alpha, X_\beta)f &= (X_\alpha X_\beta - X_\beta X_\alpha)f \\ &= [H'_\alpha, [H'_\beta, f]] - [H'_\beta, [H'_\alpha, f]] - 2\frac{\partial f}{\partial z}[H_\alpha, H'_\beta]. \end{aligned} \tag{1.54}$$

Now we apply the Jacobi relation

$$[f, [g, h]] = [g, [h, f]] + [h, [f, g]] \tag{1.55}$$

to the right of formula (1.54), we find

$$(X_\alpha, X_\beta)f = [[H'_\alpha, H'_\beta], f] - \frac{\partial f}{\partial z}[H'_\alpha, H'_\beta]. \tag{1.56}$$

From (1.53) we conclude that

$$(X_\alpha, X_\beta)f = 0. \tag{1.57}$$

Conversely, if the system is jacobi (integrable), then (1.57) is satisfied for any α and β and we get

$$[H'_\alpha, H'_\beta] = 0. \tag{1.58}$$

Now the total differential for any function $F(t_\beta, q_a, q_a)$ can be written as

$$\begin{aligned} dF &= \frac{\partial F}{\partial q_a}\, dq_a + \frac{\partial F}{\partial p_a}\, dp_a + \frac{\partial F}{\partial t_\alpha}\, dt_\alpha \\ &= \left(\frac{\partial F}{\partial q_a}\, \frac{\partial H'_\alpha}{\partial p_a} \; - \; \frac{\partial F}{\partial p_a}\, \frac{\partial H'_\alpha}{\partial q_a} \; + \frac{\partial F}{\partial t_\alpha} \right) dt_\alpha \\ &= [F, H'_\alpha]\, dt_\alpha. \end{aligned} \tag{1.59}$$

Using this result, we have

$$dH'_\beta = [H'_\beta, H'_\alpha]\, dt_\alpha, \tag{1.60}$$

and, consequently, the integrability condition (1.53) reduces to

$$dH'_\alpha = 0 \qquad \forall\ \alpha. \tag{1.61}$$

This is necessary and sufficient condition that the system (1.41-1.43) and (1.45) of total differential equations be completely integrable.

If conditions (1.61) are not satisfied identically, one may consider them as a new constraint and again test the integrability conditions, then repeating this procedure, a set of conditions may be obtained.

The simultaneous solutions of eqs. (1.41-1.43) and (1.45) give us trajectories of the motion in the canonical phase space as

$$q_a \equiv q_a(t, q_\mu), \qquad p_a \equiv p_a(t, q_\mu), \qquad \mu = 1, \ldots, r. \tag{1.62}$$

In this case, the path integral representation may be written as [18-20]

$$\langle Out \mid S \mid In \rangle = \int \prod_{a=1}^{n-r} dq^a dp_a \exp\left[i \int_{t_\alpha}^{t'_\alpha} \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right) dt_\alpha \right]. \tag{1.63}$$

One should notice that the integral (1.63) is an integration over the canonical phase space coordinates q^a, p_a.

1.5 Path Integral Methods for Quantized Constrained Systems

1.5.1 Faddeev Popov Method

The classical dynamics of an n-dimensional system is determined by the Lagrangian, a function of the n coordinates and their time derivatives. From the Lagrangian, we can construct the Hamiltonian, which is a function of the phase space. In canonical quantization, the Hamiltonian becomes an operator which acts on Hilbert space which built from the n coordinates. The Hamiltonian is a generator of time translations and thus determine quantum dynamics.

For a system with n degrees of freedom and having α first-class constraints ϕ_a, but no second-class constraints, Faddeeve has formulated the transition amplitude as [5]

$$\langle Out \mid S \mid In \rangle = \int exp\left[i \int_{-\infty}^{\infty} (p_i \dot{q}_i - H_0)\, dt \right] \prod_t d\mu(q_i(t), p_i(t)), \tag{1.64}$$

Where H_0 is the Hamiltonian of the system. The measure of integration is defined by

$$d\mu(q,p) = \left(\prod_{a=1}^{\alpha} \delta(\chi_a)\delta(\phi_a) \right) det||\{\chi_a, \phi_a\}|| \prod_{i=1}^{n} dp_i\, dq^i. \tag{1.65}$$

and $\chi_a(p_i, q_i)$ are the gauge-fixing condition with

1. $\{\chi_a, \chi_{a'}\} = 0$,
2. $det||\{\chi_a, \phi_a\}|| \neq 0$.

1.5.2 Senjanovic Method

In this section we shall generalize Faddeeve's method to the case when second-class constraints are present. This generalization is called Senjanovic method.

Consider a mechanical system with α first-class constraints ϕ_a, β second-class constraints θ_b, and the gauge conditions associated with the first-class constraints χ_a. Let the χ_a be chosen in such a way that $\{\chi_a, \chi_b\} = 0$.

Then the expression for the S-matrix element is [6]

$$\langle Out \mid S \mid In \rangle = \int exp\left[i \int_{-\infty}^{\infty} (p_i \dot{q}_i - H_0)\, dt\right] \prod_t d\mu(q(t), p(t)), \tag{1.66}$$

and

$$d\mu(q,p) = \left(\prod_{a=1}^{\alpha} \delta(\chi_a)\delta(\phi_a)\right) det||\{\chi_a, \phi_a\}|| \times \prod_{b=1}^{\beta} \delta(\theta_b)\, det||\{\theta_a, \theta_b\}||^{\frac{1}{2}} \prod_{i=1}^{n} dp_i\, dq^i. \tag{1.67}$$

where H_0 is the Hamiltonian of the system and $d\mu(q,p)$ is the measure of integration.

Chapter 2

The Relativistic Spinless Particle

In this chapter we will study the path integral quantization of the actual physical systems, which illustrate the basic concepts of the proceeding chapter.

In Section **2.1** there are essential definitions and notations included, and we will apply the canonical method and Dirac's method in Section **2.2** to study the path integral quantization of the free relativistic spinless particle.

2.1 Preliminaries

The dynamics of the continuous systems is described by a function $Q(x)$ of space-time, rather than functions of time $q_i(t)$ in discrete systems. The discrete label i is replaced by the continuous label $x \equiv (ct, \overline{x})$. Further, in continuous systems the function of coordinate $f(q)$ becomes a functional $F[Q]$ of fields. The most general form of the Lagrangian in the field theory is the functional of fields as well as their time and space derivatives, that is [21],

$$L = \int \mathcal{L} d^3x, \tag{2.1}$$

where

$$\mathcal{L} = \mathcal{L}(Q_r,\ \partial^\mu Q_r), \qquad r = 1,\ 2,\ 3, \qquad \mu = 0,\ 1,\ 2,\ 3. \tag{2.2}$$

is the corresponding Lagrangian density with

$$\partial^\mu Q_r \equiv \frac{\partial Q_r}{\partial x_\mu}. \tag{2.3}$$

At this point we must decide on a metric convention for treating covariant vectors in four-dimensional space-time. The relation between the covariant vector A_μ and its contravariant partner A^μ is defined as [21, 25, 26]

$$A_\mu = g_{\mu\nu}\ A^\nu \qquad \mu, \nu = 0, 1, 2, 3. \tag{2.4}$$

where its inverse is defined as

$$A^\mu = g^{\mu\nu}\ A_\nu, \tag{2.5}$$

where $g_{\mu\nu}$ is the metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.6}$$

2.2 The Relativistic Spinless Particle System

As an example, let us consider the action of the free relativistic spinles particle of mass $m > 0$ moving in 4-dimensional Minkowski space x_μ [3,8,16].

$$S = -m \int (\dot{x_\mu} \dot{x^\mu})^{\frac{1}{2}} \, d\tau, \qquad \mu = 0, 1, 2, 3. \tag{2.7}$$

Here x^μ are functions of arbitrary parameter τ describing the displacement of the particle along its world line, and the lagrangian is given by

$$L = -m(\dot{x_\mu} \dot{x^\mu})^{\frac{1}{2}}, \tag{2.8}$$

and the metric $g_{\mu\nu} = diag\,(1, -1, -1, -1)$ is used.

The Lagrangian is singular since the rank of the Hessian matrix

$$A_{\mu\nu} = \frac{\partial^2 L}{\partial \dot{x^\mu} \partial \dot{x^\nu}}, \qquad \mu, \nu = 0, 1, 2, 3 \tag{2.9}$$

is three.

This system will be investigated by using the canonical method, as illustrated in chapter one.

The generalized momenta p_μ conjugated to the coordinates x_μ according to eqs. (1.31) and (1.32) are given as

$$\begin{aligned} p_\mu &= \frac{\partial L}{\partial \dot{x_\mu}} \\ &= -m \frac{\dot{x^\mu}}{(\dot{x_\nu}\dot{x^\nu})^{\frac{1}{2}}}. \end{aligned} \tag{2.10}$$

Therefore the zeroth component is

$$p_0 = -m \frac{\dot{x_0}}{(\dot{x_\mu}\dot{x^\mu})^{\frac{1}{2}}}, \tag{2.11}$$

and the a^{th} components are

$$p_a = m \frac{\dot{x_a}}{(\dot{x_\mu}\dot{x^\mu})^{\frac{1}{2}}}, \tag{2.12}$$

where

$$a = 1, 2, 3, \qquad \mu = 0, 1, 2, 3.$$

Since the rank of the Hessian matrix is three; one may solve eq. (2.12) for $\dot{x_a}$ in terms of p_a and $\dot{x_0}$. as

$$\dot{x_a} = \frac{\dot{x_0}\, p_a}{(m^2 + |\mathbf{p}|^2)^{\frac{1}{2}}} \equiv \omega_a. \tag{2.13}$$

Substituting eq. (2.13) in eq. (2.11) one gets

$$p_0 = -(m^2 + |\mathbf{p}|^2)^{\frac{1}{2}} \equiv -H_0. \tag{2.14}$$

The canonical Hamiltonian H is defined as

$$
\begin{aligned}
H &= p_a\dot{x_a} + p_0\,\dot{x_0} - L \\
&= p_a\omega_a - H_0\,\dot{x_0} - L|_{\dot{x_a}\equiv\omega_a}.
\end{aligned}
\tag{2.15}
$$

Making use of eqs. (2.8) and (2.11 - 2.13), eq. (2.15) becomes

$$
\begin{aligned}
H = \dot{x_0}&\left\{\frac{p_a p_a}{(m^2+|\mathbf{p}|^2)^{\frac{1}{2}}} - (m^2+|\mathbf{p}|^2)^{\frac{1}{2}} + m\left(1-\frac{p_a p_a}{m^2+|\mathbf{p}|^2}\right)^{\frac{1}{2}}\right\} \\
= \dot{x_0}&\left\{\frac{|\mathbf{p}|^2}{(m^2+|\mathbf{p}|^2)^{\frac{1}{2}}} - (m^2+|\mathbf{p}|^2)^{\frac{1}{2}} + \left(\frac{m^2}{(m^2+|\mathbf{p}|^2)^{\frac{1}{2}}}\right)\right\} \\
= \dot{x_0}&\left\{\frac{(m^2+|\mathbf{p}|^2)}{(m^2+|\mathbf{p}|^2)^{\frac{1}{2}}} - (m^2+|\mathbf{p}|^2)^{\frac{1}{2}}\right\} \\
= 0.&
\end{aligned}
\tag{2.16}
$$

Calculations show that the canonical Hamiltonian H vanishes identically.

Now let us quantize this system by using the canonical approach.

Using eqs. (1.36) and (1.37) the set of (HJPDE) reads as

$$H' = p + H = p = 0, \tag{2.17}$$

$$H'_0 = p_0 + H_0 = p_0 + (m^2+|\mathbf{p}|^2)^{\frac{1}{2}} = 0. \tag{2.18}$$

The equations of motion are obtained as total differential equations in many variables as follows

$$dx_a = \frac{\partial H'}{\partial p_a}\, d\tau + \frac{\partial H'_0}{\partial p_a}\, dx_0 = \frac{p_a}{(m^2 + |\mathbf{p}|^2)^{\frac{1}{2}}}\, dx_0, \tag{2.19}$$

$$dp_a = -\frac{\partial H'}{\partial x_a}\, d\tau - \frac{\partial H'_0}{\partial x_a}\, dx_0 = 0, \tag{2.20}$$

$$dp_0 = -\frac{\partial H'}{\partial x_0}\, d\tau - \frac{\partial H'_0}{\partial x_0}\, dx_0 = 0. \tag{2.21}$$

To check whether this set of equations (2.19 - 2.21) is integrable or not, let us consider the total variation of equation (2.17) and (2.18). In fact

$$dH' = 0, \tag{2.22}$$

and

$$\begin{aligned} dH'_0 &= dp_0 + dH_0 \\ &= dp_0 + \frac{\partial H_0}{\partial x_0}\, dx_0 + \frac{\partial H_0}{\partial x_a}\, dx_a + \frac{\partial H_0}{dp_a}\, dp_a \equiv 0. \end{aligned} \tag{2.23}$$

Then the variation of equations (2.22) and (2.23) vanishes identically, hence, the equations of motion (2.19 - 2.21) are integrable.

Because H_0 is independent of x_μ, the solutions of eq. (2.19) are

$$x_a = \frac{p_a}{(m^2 + |\mathbf{p}|^2)^{\frac{1}{2}}}\, x_0 + C_a, \tag{2.24}$$

where C_a's are constant.

Using equation (1.44) the action integral $Z = S(x_0, x_a)$ is calculated as

$$dZ = \left(-H + p_a \frac{\partial H'}{\partial p_a}\right) d\tau + \left(-H_0 + p_a \frac{\partial H'_0}{\partial p_a}\right) dx_0,$$
$$= \left(-H_0 + p_a \frac{\partial H'_0}{\partial p_a}\right) dx_0, \qquad (2.25)$$

so we can write

$$Z = \int_{x'_0}^{x''_0} \left(-H_0 + p_a \frac{\partial H'_0}{\partial p_a}\right) dx_0,$$
$$= \int_{x'_0}^{x''_0} \left(-(m^2 + |\mathbf{p}|^2)^{\frac{1}{2}} + p_a \dot{x_a}\right) dx_0. \qquad (2.26)$$

Now we turn to the problem of the path integral quantization, the S-matrix element is given as [18 - 20]

$$\langle Out \mid S \mid In \rangle = \int \prod_a dx^a dp_a \exp\left[i \int_{x'_0}^{x''_0} \left(-H_0 + p_a \frac{\partial H'_0}{\partial p_a}\right) dx_0 \right]. \qquad (2.27)$$

or

$$\langle x'_a, x'_0 \mid x''_a, x''_0 \rangle = \int \prod_a dx^a dp_a \exp\left[i \int_{x'_0}^{x''_0} \left(-H_0 + p_a \frac{\partial H'_0}{\partial p_a}\right) dx_0 \right]$$
$$= \int \prod_a dx^a dp_a \exp\left[i \int_{x'_0}^{x''_0} \left(-(m^2 + |\mathbf{p}|^2)^{\frac{1}{2}} + p_a \dot{x_a}\right) dx_0 \right]. \qquad (2.28)$$

To check the results obtained using the canonical approach, we will study the problem using Dirac's method.

The total Hamiltonian (1.24) reads as

$$H_T = H + \lambda H_0'. \tag{2.29}$$

The canonical Hamiltonian (2.16) and the constraints (2.18), which is called the primary constraints in Dirac's method, lead to

$$H_T = \lambda H_o' = \lambda \left\{ p_0 + (m^2 + |\mathbf{p}|^2)^{\frac{1}{2}} \right\}. \tag{2.30}$$

From the consistency conditions (1.28), the time derivative of the primary constraint should be zero, that is

$$\dot{H}_0' = \{H_0', H_T\} = \{H_0', \lambda H_0'\} \approx 0. \tag{2.31}$$

It is obvious that the above consistency condition is identically zero. Therefore, no further constraints is arise.

The constraint H_0' has been derived by distinguishing $x_0(\tau)$. This is of course artificial and any $x_\mu(\tau)$ could have been chosen instead. If we prefer manifest covariant quantities we may use in place of $H_0' \approx 0$ the constraint

$$H_0' = (p^2 + m^2) \approx 0 \tag{2.32}$$

with $p^2 = p_\mu p^\mu$, and the total Hamiltonian is

$$H_T = \lambda H_0' = \lambda \left(p^2 + m^2\right) \tag{2.33}$$

Equations of motion (1.26) and (1.27) read as

$$\dot{x_\mu} = \{x_\mu, H\} + \lambda\{x_\mu, H_0'\}, \tag{2.34}$$

$$\dot{p_\mu} = \{p_\mu, H\} + \lambda\{p_\mu, H_\mu'\}, \qquad \mu = 0, 1, 2, 3. \tag{2.35}$$

Since the Hamiltonian H, given by eq. (2.16), is zero; eqs. (2.34) and (2.35) take the following forms:

$$\dot{x_0} = \lambda\{x_0, H_0'\} = \lambda\left\{x_0, +(m^2 + p^2)\right\}, \tag{2.36}$$

$$\dot{x_a} = \lambda\{x_a, H_0'\} = \lambda\left\{x_a, +(m^2 + p^2)\right\}, \tag{2.37}$$

$$\dot{p_0} = \lambda\{p_0, H_0'\} = \lambda\left\{p_0, +(m^2 + p^2)\right\}, \tag{2.38}$$

$$\dot{p_a} = \lambda\{p_a, H_0'\} = \lambda\left\{p_a, +(m^2 + p^2)\right\}. \tag{2.39}$$

Making use of eq. (1.9), eqs. (2.36 - 2.39) may be written as

$$\dot{x_0} = 2\,\lambda\,p_0, \tag{2.40}$$

$$\dot{x_a} = 2\,\lambda\,p_a, \tag{2.41}$$

$$\dot{p_0} = 0, \tag{2.42}$$

$$\dot{p_a} = 0. \tag{2.43}$$

To determine λ, we introduce a gauge fixing condition χ. Since the constraint is first-class (there is only one constraint; the primary); one may determine the gauge fixing (1.29) as [3, 8, 16]

$$\chi = x_0 - \tau = 0. \tag{2.44}$$

Differentiating eq. (2.44) with respect to the time τ, one gets

$$\dot{x_0} = 1. \tag{2.45}$$

Eqs. (2.40) and (2.45) lead to

$$\lambda = \frac{1}{2p_0}. \tag{2.46}$$

Therefore the equations of motion (2.41 - 2.43) become

$$\dot{x_1} = \frac{p_1}{(m^2 + |\mathbf{p}|^2)^{\frac{1}{2}}}, \tag{2.47}$$

$$\dot{x_2} = \frac{p_2}{(m^2 + |\mathbf{p}|^2)^{\frac{1}{2}}}, \tag{2.48}$$

$$\dot{x_3} = \frac{p_3}{(m^2 + |\mathbf{p}|^2)^{\frac{1}{2}}}, \tag{2.49}$$

$$\dot{p_0} = 0, \tag{2.50}$$

$$\dot{p_a} = 0. \tag{2.51}$$

The above equations of motion are in exact agreement with those obtained by using the canonical method (2.19 - 2.21)

To obtain the path integral quantization, talking into our consideration that we have one constraint (primary constraint), we can substitute into the Faddeeve method which given in eq.

(1.64) as

$$\langle Out \mid S \mid In \rangle = \int exp\left[i\int_{-\infty}^{\infty}(p_0\,\dot{x_0}+p_a\dot{x_a})\,dx_0\right]\delta(x_0-\tau)$$
$$\times\ \delta\left(\frac{p^2+m^2}{2p_0}\right)\left|\left\{(x_0-\tau),\left(\frac{p^2+m^2}{2p_0}\right)\right\}\right|dx_o\,dp_0$$
$$\times\prod_a dx_a\,dp_a. \quad (2.52)$$

Integrating over x_0 and p_0, we get

$$\langle Out \mid S \mid In \rangle = \int exp\left[i\int_{-\infty}^{\infty}\left(-(|\mathbf{p}|^2+m^2)^{\frac{1}{2}}+p_a\dot{x_a}\right)dx_0\right]$$
$$\times\prod_a dx_a\,dp_a. \quad (2.53)$$

This result is in exact agreement with eq. (2.28), which is calculated by the canonical method.

In this chapter the path integral quantization of the relativistic spinless particle is done by the two methods, the canonical method and the Dirac method.

The results obtained were the same in the two methods, but there are many advantages of the canonical method such that, no need to use the gauge fixing condition which is not an easy task, and no need to use the Dirac delta function. This makes the canonical method simpler than Dirac.

Chapter 3

Charged Particle in a Constant Magnetic Field

In this chapter, the charged particle in a constant magnetic field will be studied by using the canonical method discussed in chapter one. Here we consider a charged particle moving in a constant magnetic field whose Lagrangian takes the form [27]

$$L = \frac{1}{2}\big(\dot{q}_1 - q_3 q_2\big)^2 + \frac{1}{2}\big(\dot{q}_2 + q_3 q_1\big)^2. \tag{3.1}$$

This system came up in a study of Chern-Simons quantum mechanics. The Lagrangian function (3.1) is singular, since the rank of the Hessian matrix (1.16) is two.

The generalized momenta (1.31) and (1.32) are written as

$$\begin{aligned} p_1 &= \frac{\partial L}{\partial \dot{q}_1} \\ &= \dot{q}_1 - q_3\, q_2, \end{aligned} \tag{3.2}$$

$$p_2 = \frac{\partial L}{\partial \dot{q}_2}$$

$$= \dot{q}_2 + q_3\, q_1, \tag{3.3}$$

$$p_3 = \frac{\partial L}{\partial \dot{q}_3}$$

$$= 0 = -H_3. \tag{3.4}$$

Since the rank of the Hessian matrix is two, one solve (3.2) and (3.3) for $\dot{q}_1$ and $\dot{q}_2$ in terms of p_1 and p_2 as

$$\dot{q}_1 = p_1 + q_3\, q_2 \equiv \omega_1, \tag{3.5}$$

$$\dot{q}_2 = p_2 - q_3\, q_1 \equiv \omega_2. \tag{3.6}$$

The canonical Hamiltonian H_0 is

$$H_0 = -L|_{\dot{q}_a \equiv \omega_a} + p_a \dot{q}_a + p_\mu \dot{q}_\mu,$$

$$H_0 = -L + p_1\, \omega_1 + p_2\, \omega_2 + p_3\, \dot{q}_3. \tag{3.7}$$

Making use of eqs. (3.1) and (3.4 - 3.6) then eq. (3.7) becomes

$$H_0 = \frac{1}{2}\, p_1^2 + \frac{1}{2}\, p_2^2 + p_1\, q_3\, q_2 - p_2\, q_3\, q_1. \tag{3.8}$$

Following the canonical approach, the corresponding set of (HJPDE) according to eqs. (1.36) and (1.37), is

$$H_0' = p_0 + \frac{1}{2}\, p_1^2 + \frac{1}{2}\, p_2^2 + p_1\, q_3\, q_2 - p_2\, q_3\, q_1 = 0, \tag{3.9}$$

$$H_3' = p_3 + H_3 = p_3 = 0. \tag{3.10}$$

The above equations are the constraints restricting the system. The total differential equations (1.41), (1.42) and (1.43) are written as

$$dq_1 = \frac{\partial H_0'}{\partial p_1}\, d\tau + \frac{\partial H_3'}{\partial p_1}\, dq_3, \tag{3.11}$$

$$dq_2 = \frac{\partial H_0'}{\partial p_2}\, d\tau + \frac{\partial H_3'}{\partial p_2}\, dq_3, \tag{3.12}$$

$$dq_3 = \frac{\partial H_0'}{\partial p_3}\, d\tau + \frac{\partial H_3'}{\partial p_3}\, dq_3, \tag{3.13}$$

$$dp_1 = -\frac{\partial H_0'}{\partial q_1}\, d\tau - \frac{\partial H_3'}{\partial q_1}\, dq_3, \tag{3.14}$$

$$dp_2 = -\frac{\partial H_0'}{\partial q_2}\, d\tau - \frac{\partial H_3'}{\partial q_2}\, dq_3, \tag{3.15}$$

$$dp_3 = -\frac{\partial H_0'}{\partial q_3}\, d\tau - \frac{\partial H_3'}{\partial q_3}\, dq_3, . \tag{3.16}$$

Substituting eqs. (3.9), (3.10) in eqs. (3.11 - 3.16), we obtain the total differential equations of motion as

$$dq_1 = [p_1 + q_3\, q_2]\; d\tau, \tag{3.17}$$

$$dq_2 = [p_2 - q_3\, q_1]\; d\tau, \tag{3.18}$$

$$dq_3 = dq_3, \tag{3.19}$$

$$dp_1 = p_2\, q_3\, d\tau, \tag{3.20}$$

$$dp_2 = -p_1\, q_3\, d\tau, \tag{3.21}$$

$$dp_3 = -\left[p_1\, q_2 - p_2\, q_1\right] d\tau. \tag{3.22}$$

To check whether the above set of equations is integrable or not, let us consider the total variations of H'_0 and H'_3.
In fact

$$dH'_0 = 0, \tag{3.23}$$

and

$$dH'_3 = dp_3 = [-p_1\, q_2 + p_2\, q_1]\, d\tau \tag{3.24}$$

Since dH'_3 is not identically zero, we have a new constraint H'_4,

$$H'_4 = [p_1\, q_2 - p_2\, q_1] \equiv 0. \tag{3.25}$$

Thus for a valid theory, the total differential of H'_4 is identically zero,

$$dH'_4 = p_1\, dq_2 + q_2\, dp_1 - p_2\, dq_1 - q_1\, dp_2 = 0. \tag{3.26}$$

So the system of eqs. (3.17 - 3.22) together with eq. (3.25) is integrable.
Since the equations of motion are integrable, the canonical phase space coordinates (q_1, q_2) and (p_1, p_2) are obtained in terms of parameters (τ, q_3). Now eq. (1.45) reads as

$$\begin{aligned} dZ &= \left(p_1\, \dot{q}_1 + p_2\, \dot{q}_2 - \frac{1}{2}\, p_1^2 - \frac{1}{2}\, p_2^2 - p_1\, q_2\, q_3 + p_2\, q_1\, q_3\right)\, d\tau, \\ &= \frac{1}{2}\left(\, p_1^2 + p_2^2\, \right) d\tau. \end{aligned} \tag{3.27}$$

Making use of eqs. (3.27) and (1.63) the canonical path integral quantization for the system (3.1) is obtained as

$$\langle q_1, q_2, \tau\,;\, q_1', q_2', \tau' \rangle = \int dq_1\, dq_2\, dp_1\, dp_2 \exp\left[i \int \frac{1}{2} \left(p_1^2 + p_2^2 \right) d\tau \right] \tag{3.28}$$

This path integral representation is an integration over the canonical phase-space coordinates q_1, q_2 and p_1, p_2.

In this system we have obtained the quantization for a singular system of a charged particle moving in a plane under the influence of a perpendicular constant magnetic field.

The integrability conditions dH_o' and dH_4' are identically satisfied, and the system is integrable. Hence, the canonical phase-space coordinates (q_1, q_2) and (p_1, p_2) are obtained in terms of the parameters τ and q_3.

The path integral is obtained directly as an integration over the canonical phase-space coordinates (q_1, q_2) and (p_1, p_2) without using any gauge fixing condition.

Chapter 4

Relativistic Spinless Particle in an External Electromagnetic Field

In chapter three we covered the path integral quantization of a charged particle in a constant magnetic field.

In this chapter we will use the same formalism to quantize a system of a relativistic spinless particle in an external electromagnetic field A_μ.

The motion of a relativistic spinless particle of a charge g and mass m is described by the singular Lagrangian [28].

$$L = \frac{\dot{x}^2}{2e} + \frac{em^2}{2} + g\dot{x}_\mu A^\mu, \qquad \mu = 0, 1, 2, 3. \tag{4.1}$$

The singularity of the Lagrangian (4.1) follows from the fact that the rank of the Hessian matrix (1.16) is four.

The generalized momenta (1.31) and (1.32) read as

$$p_\mu = \frac{\partial L}{\partial \dot{x^\mu}} = \frac{\dot{x_\mu}}{e} + gA_\mu, \tag{4.2}$$

$$p_e = \frac{\partial L}{\partial \dot{e}} = 0 = -H_e. \tag{4.3}$$

Since the rank of the Hessian matrix is four, we solve (4.2) for $\dot{x_\mu}$ in terms of p_μ as

$$\dot{x_\mu} = ep_\mu - egA_\mu \equiv \omega_\mu. \tag{4.4}$$

The canonical Hamiltonian H_0 is

$$H_0 = -L|_{\dot{x}_\mu \equiv \omega_\mu} + p_a \dot{q_a} + p_\mu \dot{q_\mu}, \tag{4.5}$$

Making use of eqs. (4.1), (4.2) and (4.3), then eq. (4.5) becomes

$$H_0 = \frac{e}{2}\,(p^2 - m^2) - e\,g\,p_\mu A^\mu + \frac{e}{2}\,g^2 A^2. \tag{4.6}$$

The corresponding set of the (HJPDE) according to eqs. (1.36) and (1.37) is

$$H_0' = p_0 + \frac{e}{2}\,(p^2 - m^2) - e\,g\,p_\mu A^\mu + \frac{e}{2}\,g^2 A^2 = 0, \tag{4.7}$$

$$H_e' = p_e + H_e = p_e = 0. \tag{4.8}$$

The total differential equations (1.41), (1.42), and (1.43) are written as

$$dx^{\mu} = \frac{\partial H_0'}{\partial p_{\mu}}\, d\tau + \frac{\partial H_e'}{\partial p_{\mu}}\, de, \tag{4.9}$$

$$dp_{\mu} = -\frac{\partial H_0'}{\partial x^{\mu}}\, d\tau - \frac{\partial H_e'}{\partial x^{\mu}} de, \tag{4.10}$$

$$dp_e = -\frac{\partial H_0'}{\partial e}\, d\tau - \frac{\partial H_e'}{\partial e}\, de. \tag{4.11}$$

Substituting eqs. (4.7) and (4.8) in to eqs. (4.9 - 4.11), then the total differential equations are

$$dx^{\mu} = (e\, p^{\mu} - e\, g\, A^{\mu})\ d\tau, \tag{4.12}$$

$$dp_{\mu} = e\, g\, (p_{\nu} - gA_{\nu})\ \frac{\partial A^{\nu}}{\partial x^{\mu}}\, d\tau, \tag{4.13}$$

$$dp_e = \left(-\frac{1}{2}(p^2 - m^2) + g\, p_{\mu} A^{\mu} - \frac{1}{2}\, g^2 A^2\right)\ d\tau. \tag{4.14}$$

To check whether the set of eqs. (4.12 - 4.14) are integrable or not, let us consider the total variations of H_0' and H_e'.

In fact

$$dH_e' = dp_e = \left[-\frac{1}{2}(p^2 - m^2) + g\, p_{\mu} A^{\mu} - \frac{1}{2}\, g^2 A^2\right] d\tau, \tag{4.15}$$

is not identically zero.

Since dH_e' is not identically zero, we have a new constraint H_e'',

$$H_e'' = \left[(p^2 - m^2) - 2\, g\, p_{\mu} A^{\mu} + g^2 A^2\right] \equiv 0. \tag{4.16}$$

The total differential of H_0' and H_e'' vanish identically, such that,

$$dH_e'' \equiv 0, \tag{4.17}$$

and the equations of motions (4.12 - 4.14) are integrable.

Since the equations of motion are integrable, the canonical phase space coordinates (q_μ, p_μ) are obtained in terms of the parameters (τ, e).

The action in eq. (1.45) reads as

$$dZ = \left[p_\mu \dot{x^\mu} - \frac{e}{2}\,(p^2 - m^2) + e\,g\,p_\mu A^\mu - \frac{e}{2}\,g^2 A^2\right] d\tau. \tag{4.18}$$

Making use of eqs. (4.18) and (1.63) the canonical path integral quantization for the system (4.1) is obtained as

$$\begin{aligned}\left\langle x_\mu, e, \tau\,;\, x'_\mu, e', \tau'\right\rangle &= \int \prod_\mu dx^\mu dp_\mu \\ &\times \exp\left[i \int \left\{p_\mu \dot{x^\mu} - \frac{e}{2}\,(p^2 - m^2) + e\,g\,p_\mu A^\mu - \frac{e}{2}\,g^2 A^2\right\} d\tau\right].\end{aligned} \tag{4.19}$$

In this chapter we have obtained the quantization for another singular system of a relativistic spinless particle in an external electromagnetic field A_μ.

The integrability conditions dH_o' and dH_e'' are satisfied, the system is integrable. Hence, the canonical phase-space coordinates

x^μ and p_μ are obtained in parameters τ and e. The path integral is obtained directly as an integration over the canonical phase-space coordinates x^μ and p_μ without using any gauge fixing condition.

The advantage of this path integral formalism is that we have no need to enlarge the initial phase-space by introducing unphysical auxiliary field, no need to introduce Lagrange multipliers, no need to use delta functions in the measure as well as to use gauge fixing conditions; all that is needed is the set of Hamilton-Jacobi partial differential equations and the equations of motions. If the system is integrable, then one can construct the reduced canonical phase-space.

In this case the path integral is obtained directly as an integration over the canonical reduced phase-space coordinates x^μ , p_μ.

Chapter 5

Conclusion

This work is aimed at study of the quantization of constrained systems using both Dirac approach and the canonical approach. The two methods, represent the Hamiltonian treatment of the constrained systems. Where Dirac's approach hinges on introducing primary constraints, then constructing the total Hamiltonian by adding the primary constraints, multiplied by Lagrangian multipliers, to the usual Hamiltonian. The consistency conditions are checked on the primary constrained. All other constraints are obtained from these conditions. These constraints are classified into two types: First and second-class constraints, the distinction between these two types is quite important in Dirac's method. The equations of motion, obtained using Poisson brackets, are in ordinary differential equation forms.

The gauge fixing conditions, which are not an easy task in this approach, are necessary in order to determine the unknown

Lagrange multipliers.

The canonical method is an alternative formulation for studying singular systems. A physically important result, in this method, is that we first exhibit the fact that a singular system can be treated as a system with many independent variable. In other words, the equations of motion are not ordinary differential equation but total differential ones in many variables. In general mathematically speaking, it is not possible to solve the equations of motion for singular systems unless they satisfy the integrability conditions. If these conditions are not identically satisfied, it will be considered as new constraints. This process will continue until we obtain a complete system and the path integral quantization can be constructed as an integration over the canonical phase space coordinates (q_a, p_a).
The gauge fixing condition are not necessary in the canonical formulation since one dose not need to introduce Lagrange multipliers. This makes the canonical method simpler than Dirac's method.

Path integral quantization of three different examples, free relativistic spinless particle, relativistic spinless particle in an external electromagnetic field and a charged particle moving in constant magnetic field are obtained by using the canonical method. In this method, the integrability conditions are satisfied, so these systems are integrable, and hence the path integral is obtained directly as an integration over the canonical phase space.

The advantage of the canonical path integral quantization is that we have, no need to enlarge the initial phase space by introducing unphysical auxiliary fields, no need to introduce Lagrange multipliers, no need to use delta function in the measure as well as no need to use gauge-fixing conditions, all that is needed is a set of Hamilton-Jacobi partial differential equations and the equations of motion. If the system is integrable, then one can construct the reduced canonical phase-space. In this case the path integral is obtained directly as an integration over the canonical reduced phase space coordinates.

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