




2017

On Independence, Matching, and Homomorphism Complexes

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Digital Object Identifier: <https://doi.org/10.13023/ETD.2017.119>

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Recommended Citation

Hough, Wesley K., "On Independence, Matching, and Homomorphism Complexes" (2017). *Theses and Dissertations--Mathematics*. 42.

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ON INDEPENDENCE, MATCHING, AND HOMOMORPHISM COMPLEXES

DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics in the College of Arts and Sciences at the University of Kentucky

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2017

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ABSTRACT OF DISSERTATION

ON INDEPENDENCE, MATCHING, AND HOMOMORPHISM COMPLEXES

First introduced by Forman in 1998, discrete Morse theory has become a standard tool in topological combinatorics. The main idea of discrete Morse theory is to pair cells in a cellular complex in a manner that permits cancellation via elementary collapses, reducing the complex under consideration to a homotopy equivalent complex with fewer cells. In chapter 1, we introduce the relevant background for discrete Morse theory.

In chapter 2, we define a discrete Morse matching for a family of independence complexes that generalize the matching complexes of suitable “small” grid graphs. Using this matching, we determine the dimensions of the chain spaces for the resulting Morse complexes and derive bounds on the location of non-trivial homology groups. Furthermore, we determine the Euler characteristic for these complexes and prove that several of their homology groups are non-zero.

In chapter 3, we introduce the notion of a homomorphism complex for partially ordered sets, placing particular emphasis on maps between chain posets and the Boolean algebras. We extend the notion of folding from general graph homomorphism complexes to the poset case, and we define an iterative discrete Morse matching for these Boolean complexes. We provide formulas for enumerating the number of critical cells arising from this matching as well as for the Euler characteristic. We end with a conjecture on the optimality of our matching derived from connections to 3-equal manifolds.

KEYWORDS: discrete Morse theory, independence complexes,
matching complexes, homomorphism complexes,
Boolean algebras

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ON INDEPENDENCE, MATCHING, AND HOMOMORPHISM COMPLEXES

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Dedicated to LEGOs, the Nintendo Company, Cheeto, Cairo, and sweet tea.

ACKNOWLEDGMENTS

I never would have made it this far without the support of the following people: Diane and Brian Edgerton and Joe and Elaine Hough, the parents who taught me that I could do anything; Monica Mead, Brandon Hough, Corinne Edgerton, Katie Heiny, Olivia Edgerton, Hope DePaoli, and Alexanne Hough, the siblings who put up with my nonsense; Dorothy Hough, Carol and Ron Koenes, Marilyn and Gail Edgerton, and Jerrie Bracken, the grandparents who gave me food and rest during holiday breaks; Katie Crowe, the one who consented to serve as my first student; Randy Wolfe, the one who said, “I think you should try this math thing.”; Rhea Hoppes, the one who taught me to talk good; Julie Bickel, Rachel Lemon, and Hannah Byers, the ones who taught me how to have fun; Spencer Sholty, Ann Bladen, Becky Lara, and Rachel Jacobs, the Non-Sporting Group; Yefim Katsov, Nancy Rodgers, Barb Wahl, Haris Skiadas, and Carl Jagels, the ones who taught me what math is all about; Kate Ponto, the one who sold me on U.K. for grad school; Luke and Lola Davidson, the ones who keep me in check; Clark and Michelle Embree, the ones who are always available for a trip to Target; Alex Happ, Isaiah Harney, Luis Sordo Vieira, and George Lytle, the Gentlemen Who Lunch; Chase Russell and Kasey Bray, two of the best office mates a guy could ask for; Neville Fogarty and Sarah Nelson, the ones who paved the way and set an example; Carolyn Yarnall, Kayla Blyman, Erica Whitaker, Katie Paullin, and Rafael González D’León, the ones who provided valuable advice and support during the job search; Sheri Rhine and Christine Levitt, the ones who cheered from the sidelines; Bert Guillou, the one who taught me topology; and, last but not least, Ben Braun, the one who guided me through the whole thing. I don’t know where I’d be without any one of them.

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Chapter 1 - Tools from Discrete Morse Theory

1.1 Notation & Overview

This dissertation primarily concerns the applications of discrete Morse theory to cellular complexes generated from combinatorial graphs. The main idea of discrete Morse theory, first introduced by Robin Forman in [13], is to pair faces in a polyhedral complex in a manner that allows them to be cancelled via elementary collapses, reducing the complex under consideration to a homotopy-equivalent complex with fewer cells that is cellular (but possibly non-polyhedral) and is often much easier to study. The remainder of this chapter discussed the definitions and background theorems relevant to understanding discrete Morse theory.

Recall that a combinatorial graph G consists of a vertex set $V(G)$ and an edge set $E(G)$ of pairs of elements in $V(G)$. A graph is typically represented by drawing points in the plane to represent $V(G)$ and then connecting two such points with a line segment if they appear in an edge together, i.e. when they are *adjacent*. A classic example is the complete graph K_n with $n \in \mathbb{Z}_{>0}$, whose vertex set is $[n] := \{1, 2, \dots, n\}$ and where every pair of vertices forms an edge. A combinatorial graph is *directed* when its edges are ordered pairs instead of 2-element sets, i.e. when $E(G) \subseteq V(G) \times V(G)$. A directed edge is usually indicated in the geometric realization with an arrow leading from the first element to the second.

Recall that a relation \leq on a set P is a partial ordering if it is reflexive ($x \leq x$ for all $x \in P$), antisymmetric (if $x \leq y$ and $y \leq x$, then $x = y$ for all $x, y \in P$), and transitive (if $x \leq y$ and $y \leq z$, then $x \leq z$: for all $x, y, z \in P$). A *partially ordered set (poset)* is any set P equipped with a partial ordering. An element y of P *covers* $x \in P$ if $x < y$ and there exists no element c such that $x < c < y$. We often graphically represent a poset by its Hasse diagram $|P|$, a directed combinatorial graph with $V(|P|) = P$ and directed edges (x, y) exactly corresponding to the cover relations in P .

Definition 1.1.1. A *partial matching* on a poset P is a subset $\mu \subseteq P \times P$ such that $(a, b) \in \mu$ implies b covers a and each $a \in P$ belongs to at most one element in μ . For any such pair $(a, b) \in \mu$, we define $d(b) := a$. Then, we call a partial matching on P *acyclic* if there does not exist a cycle

$$b_1 > d(b_1) < b_2 > d(b_2) < \dots < b_n > d(b_n) < b_1$$

with $n \geq 2$ and all $b_i \in P$ being distinct. Also, given an acyclic partial matching μ on P , we say that the unmatched elements of P are *critical*.

Definition 1.1.2. An *abstract simplicial complex* is a family Δ of sets that is closed under taking subsets. Equivalently, if $F \in \Delta$ and $T \subseteq F$, then $T \in \Delta$. A set $F \in \Delta$ of cardinality $i + 1$ is called a *i -face* or *i -cell*. The *geometric realization* $|\Delta|$ is a collection of standard simplices (one i -simplex for each i -face) with identifications corresponding to subset inclusion. We do not distinguish between Δ and $|\Delta|$. Also, the *simplicial join* of two abstract simplicial complexes Δ, Γ is the abstract simplicial complex $\Delta * \Gamma := \{\sigma \cup \tau \mid \sigma \in \Delta, \tau \in \Gamma\}$.

Observe that subset containment induces a partial ordering on the faces of a simplicial complex, which allows us to discuss the face poset of our complex. We denote the face poset of a complex X by $\mathcal{F}(X)$. Note that a cellular map $\phi : X \rightarrow Y$ between two CW complexes induces an order-preserving map $\mathcal{F}(\phi) : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$.

The following theorem asserts that an acyclic partial matching on the face poset of a polyhedral cell complex is exactly the pairing needed to produce the homotopy equivalence promised by discrete Morse theory.

Theorem 1.1.3. (Main Theorem of Discrete Morse Theory) Let Δ be a polyhedral cell complex, and let μ be an acyclic partial matching on the face poset of Δ . Let c_i denote the number of critical i -dimensional cells of Δ . The space Δ is homotopy equivalent to a cell complex Δ_c with c_i cells of dimension i for each $i \geq 0$, plus a single 0-dimensional cell in the case where the empty set is paired in the matching.

It is often useful to create acyclic partial matchings on different sections of a poset and then combine them to form a larger acyclic partial matching. This process is formalized via the following theorem, referred to as the *Cluster Lemma* in [15] and the *Patchwork Theorem* in [17].

Theorem 1.1.4. If $\phi : P \rightarrow Q$ is an order-preserving poset map and, for each $q \in Q$, each subposet $\phi^{-1}(q)$ carries an acyclic partial matching μ_q , then the union of the μ_q 's is an acyclic partial matching on P .

In essence, this (surprisingly small) set of tools enables us to take a combinatorial structure of interest and quickly reduce a related topological space based almost entirely on relations inherent in the combinatorial object itself.

1.2 Independence & Matching Complexes

A *matching* on a simple graph G is a subset of edges that are pairwise disjoint, i.e. no two edges in the matching meet at the same vertex. Since deletion of edges preserves

the property of being a matching, we can construct a simplicial complex whose faces correspond exactly to the matchings on G . We call this complex the *matching complex of G* , denoted $M(G)$.

Matching complexes are far more interesting and mysterious than what is suggested by their simple definition. Bouc, Shareshian, and Wachs [7, 21, 23] were surprised to discover that $M(K_n)$ has factors of $\mathbb{Z}/3\mathbb{Z}$ in the smallest non-trivial integral homology groups for most n . Jonsson [15] was able to explicitly calculate almost all of the homology groups for $n \leq 14$, but the sheer number of faces when $n > 14$ causes calculation issues beyond the capabilities of all but the best supercomputers.

Due to the computational complexity of $M(K_n)$, we focus our attention on a particular subgraph of K_{2n} , namely the grid graph $G(2, n)$ defined by $V = \{1, 2\} \times [n]$ and where two vertices (x_0, y_0) and (x_1, y_1) are adjacent when their Euclidean distance is exactly 1.

Bousquet-Mélou, Linusson, and Nevo [8] examined the independence complexes of $G(2, n)$ and other small grid graphs. They also introduced a certain family of decision trees as a way to apply discrete Morse theory to $\text{Ind}(G)$ for any simple graph G , and their key result is that a so-called *matching tree* on G yields an acyclic partial matching on the face poset of $\text{Ind}(G)$ where the sets located at non-root leaves are critical faces of the reduced complex.

We gain traction on the $M(G(2, n))$ problem by rephrasing matchings in terms of independent sets and then using these matching tree techniques to instead study $\text{Ind}(L(G(2, n)))$, where $L(G(2, n))$ denotes the line graph of $G(2, n)$. We define a very specific matching tree for the face poset of $M(G(2, n))$ that has a highly convenient recursive structure that allows us to easily count the number and dimension of faces in the reduced complex as well as obtain formulas for the Euler characteristic and discover homological information beyond the capabilities of a computer.

1.3 Poset Homomorphism Complexes

Given two combinatorial graphs G and H , we can discuss the homomorphism complex $\text{Hom}(G, H)$, a cellular complex generated by all possible maps from G to H . Many properties of a combinatorial graph G can be encoded in the homomorphism complex merely by choosing H suitably. Since partially ordered sets can be realized as directed combinatorial graphs, we investigate how the graph homomorphism complex is affected by the additional structure of the partial ordering. In particular, we consider the complex $\text{Hom}(C_{n+1}, B_n)$ where C_{n+1} denotes the chain poset on $n + 1$ elements and

where B_n denotes the Boolean algebra on n letters. The homomorphisms in question correspond exactly to permutations on n letters and are related via transposition of adjacent elements, and so we can precisely count the number and dimension of cells in $\text{Hom}(C_{n+1}, B_n)$ as well as given an explicit formula for the Euler characteristic. Moreover, this permutation interpretation enables us to define a specific acyclic partial matching on the face poset of $\text{Hom}(C_{n+1}, B_n)$ that allows us to explicitly count the number of critical cells which agree with calculations by other authors regarding the 3-equal manifolds.

Chapter 2 - Independence and Matching Complexes

2.1 Introduction

Definition 2.1.1. A *matching* on a simple graph G is a subgraph $H = (V(G), S)$ where the elements of $S \subseteq E(G)$ are pairwise non-adjacent. We make no distinction between a matching and its edge set S . The *matching complex of G* , denoted $M(G)$, is the simplicial complex with vertex set $E(G)$ and faces given by the matchings on G .

It is useful to reframe matchings in the language of independent sets as follows.

Definition 2.1.2. An *independent set* in a simple graph G is a set $T \subseteq V(G)$ such that no two vertices in T are adjacent in G . The *independence complex of G* , denoted $\text{Ind}(G)$, is the abstract simplicial complex with vertex set $V(G)$ and faces given by the independent sets in G .

Definition 2.1.3. Given a simple graph G , its *line graph $L(G)$* has vertex set $E(G)$ with two vertices of $L(G)$ adjacent if they are adjacent edges in G .

First, we observe that $M(G) = \text{Ind}(L(G))$ for a finite simple graph G . It is also clear from these definitions that $\text{Ind}(A \uplus B)$ equals $\text{Ind}(A) * \text{Ind}(B)$ and $M(A \uplus B)$ equals $M(A) * M(B)$ for graphs A and B . Here \uplus denotes disjoint union.

For the path on n vertices, denoted Pa_n , and the cycle on n vertices, denoted C_n , the homotopy type of the matching and independence complexes are known [15, Section 11.4]. However, matching and independence complexes quickly become quite complicated, e.g. [2, 4, 7, 9, 10, 11, 16, 21, 22, 23]. Jonsson [15] provides a thorough survey regarding these and other simplicial complexes arising from graphs with special emphasis on the matching complex for complete graphs and complete bipartite graphs.

We focus our attention on $G(2, n)$, the $2 \times n$ grid graph with $V = \{1, 2\} \times [n]$ and where two vertices (x_0, y_0) and (x_1, y_1) are adjacent when their Euclidean distance is exactly 1.

Definition 2.1.4. We define $\Gamma_n := G(2, n + 2)$ and $D_n := L(\Gamma_n)$. For example, Γ_3 and D_3 are isomorphic the graphs in Figure 2.1. The indexing shift is chosen so that n is the number of interior rungs on the ladder of Γ_n as well as the number of interior vertices of degree 4 in D_n .

Figure 2.1: Γ_3 and D_3



In an unpublished manuscript [14], Jonsson establishes basic results regarding the matching complexes for Γ_n and more general grid graphs. For example, Jonsson shows that the homotopical depth of $M(\Gamma_n)$ is $\lceil 2n/3 \rceil$, which implies that this skeleton of the complex is a wedge of spheres. However, Jonsson states [14, page 3] that “it is probably very hard to determine the homotopy type of” matching complexes of grid graphs.

2.2 Matching Trees and the Comb Algorithm

In [8], Bousquet-Mélou, Linusson, and Nevo introduced *matching trees* as a way to apply discrete Morse theory to $\text{Ind}(G)$ for a simple graph $G = (V, E)$. For $A, B \subseteq V$ such that $A \cap B = \emptyset$, let

$$\Sigma(A, B) := \{I \in \text{Ind}(G) : A \subseteq I \text{ and } B \cap I = \emptyset\} .$$

For a vertex $p \in V(G)$, let $N(p)$ denote the neighbors of p in G . A *matching tree* $\tau(G)$ for G is a directed tree constructed according to the following algorithm.

Algorithm 2.2.1 (Matching Tree Algorithm (MTA)). Begin by letting $\tau(G)$ be a single node labeled $\Sigma(\emptyset, \emptyset)$, and consider this node a sink until after the first iteration of the following loop:

WHILE $\tau(G)$ has a leaf node $\Sigma(A, B)$ that is a sink with $|\Sigma(A, B)| \geq 2$,
DO ONE OF THE FOLLOWING:

1. If there exists a vertex $p \in V \setminus (A \cup B)$ such that $|N(p) \setminus (A \cup B)| = 0$, create a directed edge from $\Sigma(A, B)$ to a new node labeled \emptyset . Refer to p as a *free vertex* of $\tau(G)$. Since $p \notin A \cup B$, neither p nor any of its neighbors are in A . Moreover, $|N(p) \setminus (A \cup B)| = 0$ implies that all neighbors of p are in B . Consequently, given $\sigma \in \Sigma(A, B)$, we may pair σ and $\sigma \cup \{p\}$ in the face poset of $\text{Ind}(G)$.
2. If there exist vertices $p \in V \setminus (A \cup B)$ and $v \in N(p)$ such that $N(p) \setminus (A \cup B)$ contains only v , create a directed edge from $\Sigma(A, B)$ to a new node labeled $\Sigma(A \cup \{v\}, B \cup N(v))$. Then, v is a *matching vertex* of $\tau(G)$ with respect to p . Note that neither p nor any of its neighbors are in A , and all of p 's neighbors

(except for v) are in B . Performing Step 3 (described below) with v implies that the branch with $\Sigma(A, B \cup \{v\})$ has p as a free vertex, so we can perform Step 1 on that branch.

3. Choose a vertex $v \in V \setminus (A \cup B)$ and created two directed edges from $\Sigma(A, B)$ to new nodes labeled $\Sigma(A, B \cup \{v\})$ and $\Sigma(A \cup \{v\}, B \cup N(v))$. Refer to v as a *splitting vertex* of $\tau(G)$.

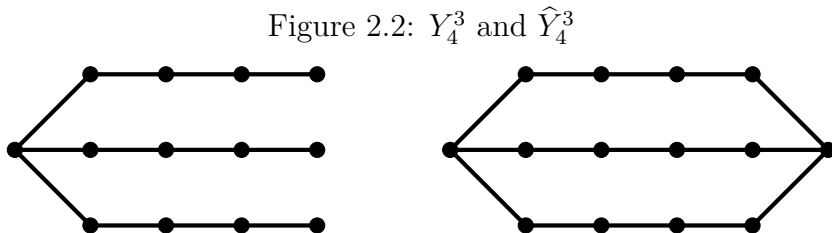
The node $\Sigma(\emptyset, \emptyset)$ is called the *root* of the matching tree, while any non-root node of outdegree 1 in $\tau(G)$ is called a *matching site* of $\tau(G)$ and any non-root node of outdegree 2 is called a *splitting site* of $\tau(G)$. Note that the empty set is always matched at the last node of the form $\Sigma(\emptyset, B)$.

A key observation from [8] is that a matching tree on G yields an acyclic partial matching on the face poset of $\text{Ind}(G)$ as follows.

Theorem 2.2.2 ([8], Section 2). A matching tree $\tau(G)$ for G yields an acyclic partial matching on the face poset of $\text{Ind}(G)$ whose critical cells are given by the non-empty sets $\Sigma(A, B)$ labeling non-root leaves of $\tau(G)$. In particular, for such a set $\Sigma(A, B)$, the set A yields a critical cell in $\text{Ind}(G)$.

Now, we will use matching trees to produce a Morse matching on the face poset of $M(\Gamma_n) = \text{Ind}(D_n)$. In fact, our techniques actually apply to independence complexes of a larger class of graphs that include D_n ; however, before introducing these graphs, we define two families of related graphs.

Definition 2.2.3. First, for $m \geq 1$ and $n \geq 1$, let Y_n^m denote the extended star graph with a central vertex of degree m and paths of n edges emanating outward. We call each of these paths a *tendrils*. We further define \hat{Y}_n^m to be two vertices connected by m parallel paths each having $n + 1$ edges. (In both cases, we ignore the degenerate cases $m = 0$ and $n = 0$.) As examples, Y_4^3 and \hat{Y}_4^3 are isomorphic to the graphs in Figure 2.2. Also, observe that $Y_n^1 \cong Pa_{n+1}$, $Y_n^2 \cong Pa_{2n+1}$, $\hat{Y}_n^1 \cong Pa_{n+2}$, and $\hat{Y}_n^2 \cong C_{2n+2}$

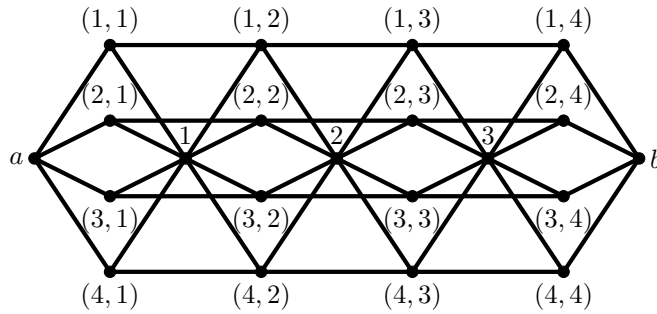


We will impose a specific labeling on these \widehat{Y}_n^m graphs: the leftmost vertex is a , the rightmost vertex is b , and the k -th vertex away from a on the j -th path is (j, k) .

Definition 2.2.4. Let Δ_n^m denote the (labeled) graph \widehat{Y}_{n+1}^m with n additional vertices labeled $\{1, \dots, n\}$ and edges $\{k, (j, k)\}$ and $\{k, (j, k+1)\}$ for each $j \in [m]$ and each $k \in [n]$. For example, Δ_3^4 is depicted in Figure 2.3.

In accordance with this numbering scheme, we define $\Delta_0^m := \widehat{Y}_1^m$ and $\Delta_{-1}^m := K_1$ where K_1 denotes an isolated vertex with no loops.

Figure 2.3: Labeled Δ_3^4



The indexing convention is chosen so that n is the number of interior vertices of degree $2m$. These vertices are special in that vertex j “cones” over the vertices of the form (i, j) and $(i, j+1)$ for each $1 \leq j \leq n$. It is straightforward to verify that $\Delta_n^2 = D_n$, and hence Δ_n^m is a family generalizing the D_n graphs.

We begin by determining the homotopy type of $\text{Ind}(Y_n^m)$ and $\text{Ind}(\widehat{Y}_n^m)$. Since Y_n^m is a tree for $m \geq 1$ and $n \geq 0$, we know by work of Ehrenborg and Heteyi [12] that $\text{Ind}(Y_n^m)$ is either contractible or homotopy equivalent to a single sphere.

Lemma 2.2.5. For $m \geq 1$ and $n \geq 0$,

$$\text{Ind}(Y_n^m) \simeq \begin{cases} * & \text{if } n = 3k \\ S^{mk} & \text{if } n = 3k + 1 \\ S^{m(k+1)-1} & \text{if } n = 3k + 2 \end{cases} .$$

Proof. Case 1: Suppose that $n = 3k$. We use induction on m . If $m = 1$, then $Y_n^1 \cong Pa_{3k+1}$; hence, $\text{Ind}(Y_n^1)$ is contractible [17, Prop 11.16]. Suppose the induction hypothesis holds for $\ell < m$. Select a tendril of Y_n^m and label the vertices 1 through n starting at the leaf. We consider a matching tree on $\text{Ind}(Y_n^m)$. Perform Step 2 of the MTA with $p = 1$ and $v = 2$. Repeat with $p = 4$ and $v = 5$ and so on modulo 3. Since $n = 3k$, we will eventually perform Step 2 with $p = n - 2$ and $v = n - 1$. The

remaining subgraph of Y_n^m from which we may select vertices is isomorphic to Y_n^{m-1} . Since $\text{Ind}(Y_n^{m-1})$ is contractible by assumption, by induction $\text{Ind}(Y_n^m)$ is contractible as well.

Case 2: Suppose that $n = 3k + 1$ or $n = 3k + 2$. Let a be the vertex of degree m in Y_n^m . We again consider a matching tree on $\text{Ind}(Y_n^m)$. We apply Step 3 of the MTA with $v = a$. At the $\Sigma(\{a\}, N(a))$ and $\Sigma(\emptyset, \{a\})$ nodes, the remaining subgraphs of Y_n^m from which we may select vertices are isomorphic to an m -fold disjoint union of Pa_{n-1} 's and an m -fold disjoint union of Pa_n 's respectively. When $n = 3k + 1$, the union of Pa_n 's is contractible [17, Prop 11.16], and each subcomplex $\text{Ind}(Pa_{n-1})$ contributes $\lfloor \frac{n-2}{3} \rfloor + 1 = k$ vertices toward a single critical cell. In total, the vertex a and the vertices from each $\text{Ind}(Pa_{n-1})$ factor combine to form a single critical cell of dimension mk . When $n = 3k + 2$, the union of the Pa_{n-1} 's is contractible [17, Prop 11.16], and each subcomplex $\text{Ind}(Pa_n)$ contributes $\lfloor \frac{n-1}{3} \rfloor + 1 = k + 1$ vertices toward a single critical cell. In total, the vertices from each $\text{Ind}(Pa_n)$ factor combine to form a single critical cell of dimension $m(k + 1) - 1$. This gives the result. \square

Lemma 2.2.6. For $m \geq 2$ and $n \geq 1$,

$$\text{Ind}(\widehat{Y}_n^m) \simeq \begin{cases} S^{mk} & \text{if } n = 3k \\ S^{mk} & \text{if } n = 3k + 1 \\ S^{mk+1} \vee S^{m(k+1)-1} & \text{if } n = 3k + 2 \end{cases} .$$

Proof. In \widehat{Y}_n^m , label the two vertices of degree m as a and b respectively. We consider a matching tree on $\text{Ind}(\widehat{Y}_n^m)$. First, we apply Step 3 of the MTA with $v = b$. At the $\Sigma(\{b\}, N(b))$ and $\Sigma(\emptyset, \{b\})$ nodes, the remaining subgraphs of \widehat{Y}_n^m from which we may select vertices are isomorphic to Y_{n-1}^m and Y_n^m respectively. For $n = 3k$ and $n = 3k + 1$, the result is immediate from applying Lemma 2.2.5 as one of the branches will produce contractible information.

For the $n = 3k + 2$ case with $m \geq 3$, Lemma 2.2.5 only shows that two cells of the appropriate dimension exist, but they may not necessarily form a wedge. This is sufficient for our purposes, but we prove that the two cells do, in fact, form a wedge for sake of completeness. Given the matching tree defined above for $\text{Ind}(\widehat{Y}_n^m)$, let τ denote the cell of dimension $mk + 1$, and let σ denote the cell of dimension $m(k + 1) - 1$. In the style of [20, Theorem 2.2], we argue that the feasibility domain of σ (see [20, Def 2.1]) is such that τ and σ must form a wedge. Suppose there exists a generalized alternating path from σ to τ as per [20, Def 2.1]. Our choice of matching tree implies $b \in \tau$ while $b \notin \sigma$. Let x_i be the last element in the alternating path with $b \notin x_i$, so $b \in x_{i+1}$. If $x_i \prec x_{i+1}$, then x_i and x_{i+1} are matched in the matching tree and so b was

designated as a free vertex during some application of Step 1 of the MTA. This is not possible as b is included in $A \cup B$ in all tree nodes except for the root. If $x_i > x_{i+1}$, then $x_{i+1} \subseteq x_i$ as sets. This contradicts that $b \notin x_i$ and $b \in x_{i+1}$. Consequently, no such generalized alternating path can exist between σ and τ . The feasibility region of σ does not contain τ , and so σ and τ form a wedge per [20, Theorem 2.2]. \square

We now develop a specific matching tree for $\text{Ind}(\Delta_n^m)$.

Algorithm 2.2.7 (Comb Algorithm (CA)). Fix $m \geq 2, n \geq 1$ and use the labeling of the vertices of Δ_n^m from Definition 2.2.4.

1. Perform Step 3 of the MTA for $v = 1$, which produces two leaves $\Sigma(\{1\}, N(1))$ and $\Sigma(\emptyset, \{1\})$ respectively.
2. For each $k \in \{2, \dots, n\}$, inductively perform Step 3 of the MTA for $v = k$ on the leaf $\Sigma(\emptyset, \{1, 2, \dots, k-1\})$, successively producing leaves of the form

$$\Sigma(\{k\}, N(k) \cup \{1, 2, \dots, k-1\}) \quad \text{and} \quad \Sigma(\emptyset, \{1, 2, \dots, k\}).$$

3. At the $\Sigma(\{1\}, N(1))$ leaf, we may perform Step 1 of the MTA with $p = a$.
4. For each $k \in \{2, \dots, n-1\}$, consider the leaf

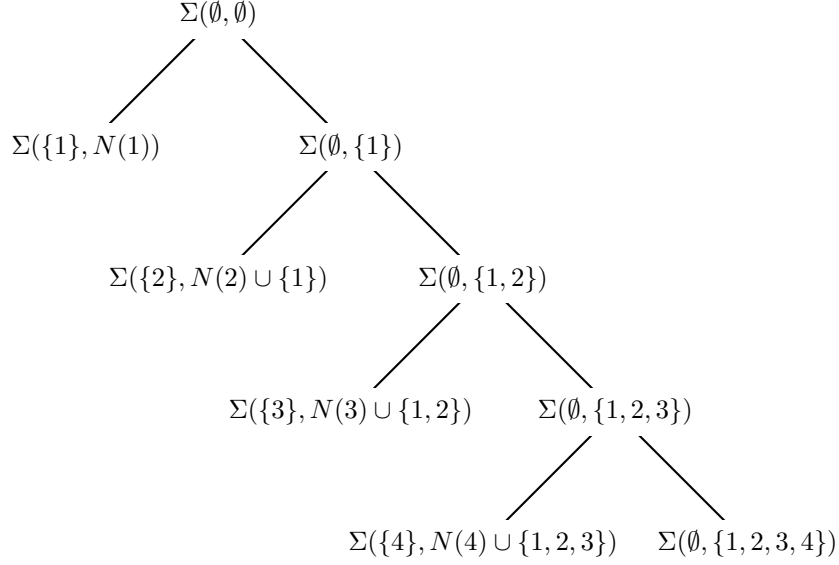
$$\Sigma(\{k\}, N(k) \cup \{1, 2, \dots, k-1\}).$$

The remaining subgraph of Δ_n^m from which we may query vertices is isomorphic to $Y_{k-1}^m \uplus \Delta_{n-(k+1)}^m$. Since $\text{Ind}(Y_{k-1}^m)$ is known, we can determine the number and dimension of critical cells below this node by inductively applying this algorithm to $\Delta_{n-(k+1)}^m$.

5. At the $\Sigma(\{n\}, N(n) \cup \{1, 2, \dots, n-1\})$ leaf, we may perform Step 1 of the MTA with $p = b$.
6. At the $\Sigma(\emptyset, \{1, 2, \dots, n\})$ leaf, the remaining subgraph of Δ_n^m from which we may query vertices is isomorphic to \widehat{Y}_{n+1}^m . Since $\text{Ind}(\widehat{Y}_{n+1}^m)$ is known, we can determine the number and dimension of critical cells arising below this node.

We call this process for generating a matching tree for $\text{Ind}(\Delta_n^m)$ the ‘‘Comb Algorithm’’ because of the visual shape of the resulting matching tree. Steps 1 and 2 produce the backbone of the comb, while Steps 3 through 6 produce the teeth. For example, applying Steps 1 and 2 of the comb algorithm to $\text{Ind}(\Delta_4^m)$ leads to the (partial) matching tree in Figure 2.4.

Figure 2.4: Example of the Comb Algorithm



2.3 Chain Spaces after the Comb Algorithm

Definition 2.3.1. Denote by X_n^m the cellular complex arising from the Comb Algorithm applied to $\text{Ind}(\Delta_n^m)$ for $m \geq 2$ and $n \geq 1$. Since we cannot apply the Comb Algorithm to $\text{Ind}(\Delta_0^m)$, we define X_0^m to be S^0 in agreement with the fact that $\Delta_0^m \cong \widehat{Y}_1^m$. Now, for fixed $m \geq 2$ and arbitrary $d \geq 1$, let C_n^d be the number of d -dimensional cells in X_n^m .

Since the Comb Algorithm will always pair the empty set with a 0-cell, we insist that $C_n^{-1} = 0$. Also, we set C_n^0 to be one less than the number of 0-dimensional cells in X_n^m to avoid including the extra 0-cell generating by the empty set pairing. Furthermore, the overall context implies that $C_n^d = 0$ if $d < 0$ or $n < 0$.

Proposition 2.3.2. Suppose $0 \leq n \leq 3$. Then, $C_n^d = 0$ for all $d \geq 0$ except the data in Table 2.1.

Table 2.1: Initial conditions of the Comb Algorithm recursion

	C_0^0	C_1^1	C_1^{m-1}	C_2^m	C_3^2	C_3^m
$m = 2$	1	2	2	1	2	2
$m \geq 3$	1	1	1	1	1	1

Proof. Fix $m \geq 2$. We separately consider $\text{Ind}(\Delta_n^m)$ for $n \in \{0, 1, 2, 3\}$.

Case 1: Suppose $n = 0$. Then $\Delta_0^m \cong \widehat{Y}_1^m$, which implies $\text{Ind}(\Delta_0^m) \simeq S^0$ by Lemma 2.2.6. Consequently, $C_0^0 = 1$ while $C_0^d = 0$ for all other d .

Case 2: Suppose $n = 1$. We apply Step 1 followed by Step 3 of the CA to $\text{Ind}(\Delta_1^m)$. At the $\Sigma(\emptyset, \{1\})$ node, the remaining graph from which we may select vertices is isomorphic to \widehat{Y}_2^m . Thus, $\text{Ind}(\Delta_1^m) \simeq \text{Ind}(\widehat{Y}_2^m)$, from which we can apply Lemma 2.2.6. So, $C_1^1 = C_1^{m-1} = 1$ if $m \geq 3$, and $C_1^1 = 2$ if $m = 2$. In either case, $C_1^d = 0$ for all other d .

Case 3: Suppose $n = 2$. First, apply the Comb Algorithm to $\text{Ind}(\Delta_2^m)$. We note that Step 5 subsumes Step 4 in this particular instance. Now, Steps 3 and 5 imply that no critical cells are picked out below the nodes $\Sigma(\{1\}, N(1))$ and $\Sigma(\{2\}, N(2) \cup \{1\})$. Consequently, Step 6 implies that $\text{Ind}(\Delta_2^m) \simeq \text{Ind}(\widehat{Y}_3^m) \simeq S^m$ via Lemma 2.2.6. Thus, $C_2^m = 1$ while $C_2^d = 0$ for all other d .

Case 4: Suppose $n = 3$. First, apply the Comb Algorithm to $\text{Ind}(\Delta_3^m)$. Now, Steps 3 and 5 imply that no critical cells are picked out below the nodes $\Sigma(\{1\}, N(1))$ and $\Sigma(\{3\}, N(3) \cup \{1, 2\})$. Per Step 4, at the $\Sigma(\{2\}, N(2) \cup \{1\})$ leaf, the remaining subgraph of Δ_3^m from which we may query vertices is isomorphic to $Y_1^m \uplus \Delta_0^m$. We already know that $\text{Ind}(Y_1^m)$ and $\text{Ind}(\Delta_0^m)$ are both homotopy equivalent to S^0 , thus each has one critical 0-cell with one vertex. Consequently, $\text{Ind}(Y_1^m \uplus \Delta_0^m)$ must have a single critical cell consisting of two vertices, so it is homotopy equivalent to S^1 . Therefore, the Comb Algorithm generates a 2-cell below the $\Sigma(\{2\}, N(2) \cup \{1\})$ node. At the node $\Sigma(\emptyset, \{1, 2, 3\})$ generated in Step 6, the remaining subgraph of Δ_3^m from which we may query vertices is isomorphic to \widehat{Y}_4^m . Since $\text{Ind}(\widehat{Y}_4^m) \simeq S^m$, the Comb Algorithm generates an m -cell below this node. In total, we have $C_3^2 = C_3^m = 1$ if $m > 2$, otherwise $C_3^2 = 2$. In either case, $C_3^d = 0$ for all other d . \square

Theorem 2.3.3. Using Proposition 2.3.2 as initial conditions, we have

$$C_n^d = C_{n-3}^{d-2} + C_{n-4}^{d-(m+1)} + C_{n-3}^{d-m}, \quad (2.1)$$

when $n \geq 4$ for fixed $m \geq 2$. In this formula, a summand is zero if the subscript or superscript is negative.

Proof. Assume $n \geq 4$ and $d \geq 0$. Applying the Comb Algorithm to $\text{Ind}(\Delta_n^m)$ generates factors of the form $\text{Ind}(Y_{k-1}^m \uplus \Delta_{n-(k+1)}^m)$ for $1 \leq k \leq n$, each of which is homeomorphic to $\text{Ind}(Y_{k-1}^m) * \text{Ind}(\Delta_{n-(k+1)}^m)$. We let $C_n^d(k)$ be the number of d -dimensional cells in X_n^m produced by the Comb Algorithm below the node $\Sigma(\{k\}, N(k) \cup \{1, 2, \dots, k-1\})$, that is, the cells referenced in Step 4 of the Comb Algorithm. We use $C_n^d(\emptyset)$ to denote the number of d -dimensional cells arising from Step 6 of the Comb Algorithm. It is clear that $C_n^d = \sum_{k=1}^n C_n^d(k) + C_n^d(\emptyset)$.

First, whenever $k - 1 \equiv 0 \pmod{3}$, $\text{Ind}(Y_{k-1}^m)$ is contractible and, consequently, so is $\text{Ind}(Y_{k-1}^m) * \text{Ind}(\Delta_{n-(k+1)}^m)$. Thus, $C_n^d(k) = 0$ when $k - 1 \equiv 0 \pmod{3}$, and so we may assume that $k = 3\ell$ or $k = 3\ell + 2$ for some non-negative integer ℓ . Also, note that $\text{Ind}(Y_{k-1}^m) * \text{Ind}(\Delta_{n-(k+1)}^m)$ is contractible for $k = n$ since $\text{Ind}(\Delta_{-1}^m)$ is contractible, i.e. $C_n^d(n) = 0$. These observations subsume Steps 3 and 5 of the Comb Algorithm.

Next, we consider $C_n^d(2)$. Such a d -cell must correspond to the set of $d + 1$ vertices consisting of the vertex 2, a single vertex contributed from $\text{Ind}(Y_1^m)$, and $d - 1$ vertices contributed from $\text{Ind}(\Delta_{n-3}^m)$. Therefore, the d -cells coming from $\text{Ind}(Y_1^m) * \text{Ind}(\Delta_{n-3}^m)$ are in bijective correspondence with the $(d - 2)$ -cells of $\text{Ind}(\Delta_{n-3}^m)$. Hence, $C_n^d(2)$ equals C_{n-3}^{d-2} . Note that if $d < 2$, then $C_n^d(2) = 0$.

Similarly, we consider $C_n^d(3)$. The $d + 1$ vertices corresponding to such a d -cell consist of the vertex 3, m vertices contributed from $\text{Ind}(Y_2^m)$, and $d - m$ vertices contributed from $\text{Ind}(\Delta_{n-4}^m)$, provided $d - m > 0$. Therefore, the d -cells coming from $\text{Ind}(Y_2^m) * \text{Ind}(\Delta_{n-4}^m)$ are in bijective correspondence with the $(d - (m + 1))$ -cells of $\text{Ind}(\Delta_{n-4}^m)$. Hence, $C_n^d(3) = C_{n-4}^{d-(m+1)}$. Note that if $d < m + 1$, then $C_n^d(3) = 0$.

Lastly, we simultaneously consider $C_n^d(k)$ for $k \in \{4, 5, \dots, n, \emptyset\}$. As before, we can disregard $k \equiv 1 \pmod{3}$ and $k = n$. We first consider the case when $k = 3\ell$ for some positive integer ℓ , which implies that $\text{Ind}(Y_{k-1}^m) \simeq S^{m\ell-1}$. A d -cell contributed from the factor $\text{Ind}(Y_{k-1}^m) * \text{Ind}(\Delta_{n-(k+1)}^m)$ consists of (1) the vertex k , (2) $m\ell$ vertices from $\text{Ind}(Y_{k-1}^m)$, and (3) $d - m\ell$ vertices from $\text{Ind}(\Delta_{n-(k+1)}^m)$, provided that $d - m\ell > 0$. We observe that a similar factor of $\text{Ind}(Y_{(k-1)-3}^m \uplus \Delta_{n-(k+1)}^m)$ is generated when the Comb Algorithm is applied to $\text{Ind}(\Delta_{n-3}^m)$. It is straightforward to show that the difference in dimension of the critical cell in $\text{Ind}(Y_{k-1}^m)$ from that of the critical cell in $\text{Ind}(Y_{k-4}^m)$ is m . This implies that the $d - m\ell$ vertices from $\text{Ind}(\Delta_{n-(k+1)}^m)$ that generate a given d -cell in the factor $\text{Ind}(Y_{k-1}^m) * \text{Ind}(\Delta_{n-(k+1)}^m)$ for $\text{Ind}(\Delta_n^m)$ also generate a cell of dimension $d - m$ in the factor $\text{Ind}(Y_{k-4}^m \uplus \Delta_{n-(k+1)}^m)$ for $\text{Ind}(\Delta_{n-3}^m)$ and vice versa. Consequently, $C_n^d(k) = C_{n-3}^{d-m}(k - 3)$, provided $d \geq m$. A similar argument holds for the case when $k \equiv 2 \pmod{3}$.

Next, we see that $C_n^d(\emptyset) = C_{n-3}^{d-m}(\emptyset)$ if $d \geq m$. This observation follows because the difference in dimensions of the critical cells in $\text{Ind}(\widehat{Y}_{n+1}^m)$ from those of the critical cells in $\text{Ind}(\widehat{Y}_{n-2}^m)$ is m while the number of critical cells is constant modulo 3.

Hence, we must have

$$\sum_{k=4}^n C_n^d(k) + C_n^d(\emptyset) = \sum_{k=1}^{n-3} C_{n-3}^{d-m}(k) + C_{n-3}^{d-m}(\emptyset) = C_{n-3}^{d-m},$$

which gives

$$C_n^d = \sum_{k=1}^n C_n^d(k) + C_n^d(\emptyset) = C_{n-3}^{d-2} + C_{n-4}^{d-(m+1)} + C_{n-3}^{d-m}.$$

□

Now that we have a recursive formula that gives the number of critical cells generated by the Comb Algorithm, we can manipulate this formula to get a recursive formula for the reduced Euler characteristic of both $\text{Ind}(\Delta_n^m)$ and X_n^m . We denote this reduced Euler characteristic by χ_n^m . Note that since C_n^0 is one less than the number of zero-dimensional cells in X_n^m , we have $\chi_n^m = \sum_{d \geq 0} (-1)^d C_n^d$.

Corollary 2.3.4. Given the initial conditions from Proposition 2.3.2, when $m \geq 2$ and $n \geq 4$, we have

$$\chi_n^m = (1 + (-1)^m) \chi_{n-3}^m + (-1)^{m+1} \chi_{n-4}^m. \quad (2.2)$$

Proof. Fix m and n as above. Using formula (2.1) for C_n^d , we obtain

$$\begin{aligned} \chi_n^m &= \sum_{d \geq 0} (-1)^d (C_{n-3}^{d-2} + C_{n-4}^{d-(m+1)} + C_{n-3}^{d-m}) \\ &= \left(\sum_{d \geq 0} (-1)^d C_{n-3}^{d-2} \right) + \left(\sum_{d \geq 0} (-1)^d C_{n-4}^{d-(m+1)} \right) + \left(\sum_{d \geq 0} (-1)^d C_{n-3}^{d-m} \right) \\ &= \left(\sum_{d \geq 0} (-1)^{d-2} C_{n-3}^{d-2} \right) + \left((-1)^{m+1} \sum_{d \geq 0} (-1)^{d-(m+1)} C_{n-4}^{d-(m+1)} \right) \\ &\quad + \left((-1)^m \sum_{d \geq 0} (-1)^{d-m} C_{n-3}^{d-m} \right) \\ &= \left(\sum_{d \geq 0} (-1)^d C_{n-3}^d \right) + \left((-1)^{m+1} \sum_{d \geq 0} (-1)^d C_{n-4}^d \right) + \left((-1)^m \sum_{d \geq 0} (-1)^d C_{n-3}^d \right) \\ &= \chi_{n-3}^m + (-1)^{m+1} \chi_{n-4}^m + (-1)^m \chi_{n-3}^m \\ &= (1 + (-1)^m) \chi_{n-3}^m + (-1)^{m+1} \chi_{n-4}^m \end{aligned}$$

The fourth equality above is obtained by reindexing and noting that $C_{n-4}^{d-(m+1)} = 0$ for $d < m$ and $C_{n-3}^{d-m} = 0$ for $d < m - 1$. □

Corollary 2.3.5. When m is even, χ_n^m satisfies the recursion $a_n = a_{n-3} - a_{n-2} - a_{n-1}$ with initial conditions $a_0 = 1$, $a_1 = -2$, and $a_2 = 1$, and hence has generating function

$$\frac{1 - x}{1 + x + x^2 - x^3}.$$

(This sequence is the A078046 entry in the OEIS [1].)

Proof. Assume that $m \geq 2$ is even. First, observe that $\chi_0^m = 1$, $\chi_1^m = -2$, and $\chi_2^m = 1$ by Proposition 2.3.2, so both relations have the same initial conditions. We can easily verify that $\chi_3^m = 2 = 1 - (-2) - 1 = a_0 - a_1 - a_2 = a_3$. Now, for fixed n , assume that χ_ℓ^m satisfies both relations for $\ell < n$. Since m is even, we have that $\chi_n^m = 2 \cdot \chi_{n-3}^m - \chi_{n-4}^m = \chi_{n-3}^m + (\chi_{n-3}^m - \chi_{n-4}^m)$. By assumption, $\chi_{n-1}^m = \chi_{n-4}^m - \chi_{n-3}^m - \chi_{n-2}^m$, which implies that $\chi_{n-3}^m - \chi_{n-4}^m = -\chi_{n-2}^m - \chi_{n-1}^m$. Therefore, we obtain by substituting that $\chi_n^m = \chi_{n-3}^m + (\chi_{n-3}^m - \chi_{n-4}^m) = \chi_{n-3}^m - \chi_{n-2}^m - \chi_{n-1}^m$. Consequently, χ_n^m satisfies both relations by induction. \square

Remark 2.3.6. When m is odd, $\chi_n^m = \chi_{n-4}^m$. It is easy to verify that $\chi_0^m = 1$, $\chi_1^m = 0$, $\chi_2^m = -1$, and $\chi_3^m = 1$ from Proposition 2.3.2. Therefore, $\chi_n^m \in \{-1, 0, 1\}$ depending on the value of n modulo 4.

For the special case $m = 2$, the dimensions of C_n^d have an interesting enumerative interpretation. In particular, the sequence A201780 in OEIS [1] is the Riordan array of

$$\left(\frac{(1-x)^2}{1-2x}, \frac{x}{1-2x} \right)$$

which can be alternatively defined by

$$T(j, k) = 2 \cdot T(j-1, k) + T(j-1, k-1) \quad (2.3)$$

with initial conditions $T(0, 0) = 1$, $T(1, 0) = 0$, $T(2, 0) = 1$, and $T(j, k) = 0$ if $k < 0$ or $j < k$.

Proposition 2.3.7. When $m = 2$, formula (2.1) reduces to $C_n^d = 2C_{n-3}^{d-2} + C_{n-4}^{d-3}$. We can convert between our C_n^d array and the above Riordan array by the relations

$$C_n^d = T(n-d+2, 3d-2n) \quad \text{and} \quad T(j, k) = C_{2(j-2)+k}^{3(j-2)+k}.$$

Proof. The initial conditions of C_d^n are realized as entries in this Riordan array as follows. First, it is clear that we have $C_0^0 = 1 = T(2, 0)$. It is straightforward to obtain the following:

$$\begin{aligned} C_1^1 &= 2 \\ &= 2(2 \cdot 0 + 1) + 0 \\ &= 2(2 \cdot T(0, 1) + T(0, 0)) + T(1, 0) \\ &= 2 \cdot T(1, 1) + T(1, 0) \\ &= T(2, 1) \end{aligned}$$

$$\begin{array}{ll}
C_2^2 = 1 & C_2^3 = 2 \\
= 2 \cdot 0 + 1 & = 2 \cdot 1 + 0 \\
= 2 \cdot T(1, 2) + T(1, 1) & = 2 \cdot T(2, 0) + T(2, -1) \\
= T(2, 2) & = T(3, 0)
\end{array}$$

Now, define expressions $J_n^d := n - d + 2$ and $K_n^d := 3d - 2n$, which means that $T(J_n^d, K_n^d) = T(n - d + 2, 3d - 2n)$. It is straightforward to verify that applying the relation (2.3) to this entry gives

$$\begin{aligned}
T(J_n^d, K_n^d) &= T(n - d + 2, 3d - 2n) \\
&= 2 \cdot T(n - d + 1, 3d - 2n) + T(n - d + 1, 3d - 2n - 1) \\
&= 2 \cdot T(J_{n-3}^{d-2}, K_{n-3}^{d-2}) + T(J_{n-4}^{d-3}, K_{n-4}^{d-3})
\end{aligned}$$

Thus, the recursion applied to $T(n - d + 2, 3d - 2n)$ matches that of C_d^n . The proof of the second half of the claim is similar and omitted. \square

2.4 Homological Properties after the Comb Algorithm

In this section, we consider some of the homological implications resulting from the Comb Algorithm.

Theorem 2.4.1. Fix $m \geq 2$ and $n \geq 0$. Define

$$d_n^{min} := \begin{cases} \left\lfloor \frac{2n+2}{3} \right\rfloor & \text{if } n = 3k \text{ or } n = 3k + 1 \\ 2 \left\lfloor \frac{n-1}{3} \right\rfloor + m & \text{if } n = 3k + 2 \end{cases} .$$

Then, $C_n^d = 0$ if $0 \leq d < d_n^{min}$, excluding the base 0-cell. When $m = 2$, these two formulas coincide for $n = 3k + 2$.

Proof. By Proposition 2.3.2, the claim holds for the base cases of $n \in \{0, 1, 2, 3\}$. We proceed by strong induction. For $n \geq 4$, suppose that the claim is true for all $0 \leq i < n$. For fixed j , consider the leaf $\Sigma(\{j\}, N(j) \cup \{1, 2, \dots, j-1\})$ from the Comb Algorithm applied to $\text{Ind}(\Delta_n^m)$. Steps 3 and 4 of the Comb Algorithm allow us to assume that $j \in \{2, \dots, n\}$. If $j < n$, then the remaining subgraph of Δ_n^m from which we may query vertices is isomorphic to $Y_{j-1}^m \uplus \Delta_{n-(j+1)}^m$, which corresponds to a subcomplex of $\text{Ind}(\Delta_n^m)$ of the form $\text{Ind}(Y_{j-1}^m) * \text{Ind}(\Delta_{n-(j+1)}^m)$. Moreover, by

Lemma 2.2.5, $\text{Ind}(Y_{j-1}^m)$ is contractible when $j \equiv 1 \pmod{3}$. Since joins respect homotopy equivalences, $\text{Ind}(Y_{j-1}^m) * \text{Ind}(\Delta_{n-(j+1)}^m)$ is contractible when $j \equiv 1 \pmod{3}$, thus we may further assume that j is of the form 3ℓ or $3\ell + 2$ for some non-negative integer ℓ . Observe that when $j = 3\ell$ or $j = 3\ell + 2$, $\text{Ind}(Y_{j-1}^m)$ is homotopy equivalent to $S^{m\ell-1}$ or $S^{m\ell}$ respectively. We let δ_j denote the dimension of this sphere.

Still considering $j \in \{2, \dots, n-1\}$, we have $n - (j+1) < n$, and so the induction hypothesis holds for $\text{Ind}(\Delta_{n-(j+1)}^m)$. We now count the minimum number of vertices in a critical cell in the matching tree below the node $\Sigma(\{j\}, N(j) \cup \{1, 2, \dots, j-1\})$. We have the vertex j itself, $\delta_j + 1$ vertices from $\text{Ind}(Y_{j-1}^m)$, and $d_{n-(j+1)}^{\min} + 1$ vertices from $X_{n-(j+1)}^m$. This total number of vertices corresponds to a cell of dimension $\delta_j + d_{n-(j+1)}^{\min} + 2$ below the node $\Sigma(\{j\}, N(j) \cup \{1, 2, \dots, j-1\})$. In the special case $j = n$, the remaining subgraph of Δ_n^m from which we may query vertices is isomorphic to \widehat{Y}_{n+1}^m , so we can also expect the subcomplex $\text{Ind}(\widehat{Y}_{n+1}^m)$ to contribute one or two cells of the appropriate dimension per Lemma 2.2.6.

Next, we explicitly calculate d_n^{\min} for each value of $n \pmod{3}$.

Case 1: Suppose that $n = 3k$. The proposed d_n^{\min} is $\lfloor \frac{2n+2}{3} \rfloor = \lfloor \frac{6k+2}{3} \rfloor = 2k$.

Subcase 1a: If $j = 3\ell$, then we have $n - (j+1) = 3(k - \ell - 1) + 2$, which implies $d_{n-(j+1)}^{\min} = 2(k - \ell - 1) + m$. Thus,

$$\begin{aligned} \delta_j + d_{n-(j+1)}^{\min} + 2 &= (m\ell - 1) + 2(k - \ell - 1) + m + 2 \\ &= 2k + (m - 2)\ell + (m - 1). \end{aligned}$$

Subcase 1b: If $j = 3\ell + 2$, then we have $n - (j+1) = 3(k - \ell - 1)$, which implies $d_{n-(j+1)}^{\min} = 2(k - \ell - 1)$. Thus,

$$\begin{aligned} \delta_j + d_{n-(j+1)}^{\min} + 2 &= (m\ell) + 2(k - \ell - 1) + 2 \\ &= 2k + (m - 2)\ell. \end{aligned}$$

By Lemma 2.2.6, the cell contributed by the subcomplex $\text{Ind}(\widehat{Y}_{n+1}^m)$ is of dimension mk . Observe that each of these cellular dimensions is no less than $2k$ since $m \geq 2$. Hence, none of the cells in X_n^m are of dimension smaller than $\lfloor \frac{2n+2}{3} \rfloor$. Furthermore, when $j = 2$, we have that the factor $\text{Ind}(Y_1^m) * \text{Ind}(\Delta_{n-3}^m)$ produces a cell of dimension exactly $2k$.

Case 2: Suppose that $n = 3k + 1$. The proposed d_n^{\min} is $\lfloor \frac{2n+2}{3} \rfloor = \lfloor \frac{6k+4}{3} \rfloor = 2k + 1$.

Subcase 2a: If $j = 3\ell$, then we have $n - (j + 1) = 3(k - \ell)$, which implies that $d_{n-(j+1)}^{min} = 2(k - \ell)$. Thus,

$$\begin{aligned}\delta_j + d_{n-(j+1)}^{min} + 2 &= (m\ell - 1) + 2(k - \ell) + 2 \\ &= 2k + (m - 2)\ell + 1.\end{aligned}$$

Subcase 2b: If $j = 3\ell + 2$, then we have $n - (j + 1) = 3(k - \ell - 1) + 1$, which implies that $d_{n-(j+1)}^{min} = 2(k - \ell) - 1$. Thus,

$$\begin{aligned}\delta_j + d_{n-(j+1)}^{min} + 2 &= (m\ell) + 2(k - \ell) - 1 + 2 \\ &= 2k + (m - 2)\ell + 1.\end{aligned}$$

By Lemma 2.2.6, the cells contributed by the subcomplex $\text{Ind}(\widehat{Y}_{n+1}^m)$ are of dimensions $mk + 1$ and $m(k + 1) - 1$. Observe that each of these cellular dimensions is no less than $2k + 1$ since $m \geq 2$. Therefore, none of the cells in X_n^m are of dimension smaller than $\lfloor \frac{2n+2}{3} \rfloor$. Furthermore, when $j = 2$, we have that the factor $\text{Ind}(Y_1^m) * \text{Ind}(\Delta_{n-3}^m)$ produces a cell of dimension exactly $2k + 1$.

Case 3: Suppose that $n = 3k + 2$. The proposed d_n^{min} is $2 \lfloor \frac{n-1}{3} \rfloor + m = 2k + m$.

Subcase 3a: If $j = 3\ell$, then we have $n - (j + 1) = 3(k - \ell) + 1$, which implies $d_{n-(j+1)}^{min} = 2(k - \ell) + 1$. Thus,

$$\begin{aligned}\delta_j + d_{n-(j+1)}^{min} + 2 &= (m\ell - 1) + 2(k - \ell) + 1 + 2 \\ &= 2k + (m - 2)\ell + 2.\end{aligned}$$

Because $j = 3\ell$ and $j \geq 2$, we have $\ell \geq 1$, which implies that

$$2k + (m - 2)\ell + 2 \geq 2k + (m - 2) + 2 = 2k + m.$$

Subcase 3b: If $j = 3\ell + 2$, then we have $n - (j + 1) = 3(k - \ell - 1) + 2$, which implies $d_{n-(j+1)}^{min} = 2(k - \ell - 1) + m$. Thus,

$$\begin{aligned}\delta_j + d_{n-(j+1)}^{min} + 2 &= (m\ell) + 2(k - \ell - 1) + m + 2 \\ &= 2k + (m - 2)\ell + m.\end{aligned}$$

By Lemma 2.2.6, the subcomplex $\text{Ind}(\widehat{Y}_{n+1}^m)$ produces a cell of dimension $m(k + 1)$. Observe that each of these cellular dimensions is at least $2k + m$ since $m \geq 2$. Therefore, none of the cells in X_n^m are of dimension smaller than $2 \lfloor \frac{n-1}{3} \rfloor + m$. Furthermore, when $j = 2$, we have that the factor $\text{Ind}(Y_1^m) * \text{Ind}(\Delta_{n-3}^m)$ produces a cell of dimension exactly $2k + m$.

For all three cases, the dimension of any cell generated by the Comb Algorithm applied to $\text{Ind}(D_n^m)$ is never smaller than d_n^{min} . As a final observation, when $m = 2$ and $n = 3k + 2$, we have $\lfloor \frac{2n+2}{3} \rfloor = 2k + 2 = 2 \lfloor \frac{n-1}{3} \rfloor + m$. \square

Remark 2.4.2. Theorem 2.4.1 shows that X_n^m is at least d_n^{min} -connected. After a suitable adjustment of notation, this agrees with results of Jonsson [14, Proposition 2.7] regarding the connectivity of $\text{Ind}(\Delta_n^2)$.

Theorem 2.4.3. Fix $m \geq 2$ and $n \geq 0$. Define

$$d_n^{max} := \begin{cases} \left\lfloor \frac{3n+2}{4} \right\rfloor & \text{if } m = 2 \\ n+1 + (m-3) \cdot \left\lfloor \frac{n+2}{3} \right\rfloor & \text{otherwise} \end{cases}.$$

Then, $C_n^d = 0$ if $d > d_n^{max}$.

Proof. By Proposition 2.3.2, the claim holds for the bases cases of $n \in \{0, 1, 2, 3\}$. We proceed by strong induction. For $n \geq 4$, suppose the claim is true for all $0 \leq i < n$. For fixed j , we will be considering the maximum dimension of a cell produced below the node $\Sigma(\{j\}, N(j) \cup \{1, 2, \dots, j-1\})$ from the Comb Algorithm applied to $\text{Ind}(\Delta_n^m)$. As before, we may assume $j \in \{2, \dots, n\}$. If $j = n$, the remaining subgraph of Δ_n^m from which we may query vertices is isomorphic to \widehat{Y}_{n+1}^m . If $j < n$, then the remaining subgraph is $Y_{j-1}^m \uplus \Delta_{n-(j+1)}^m$, which corresponds to a subcomplex of $\text{Ind}(\Delta_n^m)$ of the form $\text{Ind}(Y_{j-1}^m) * \text{Ind}(\Delta_{n-(j+1)}^m)$. We will again use the notation δ_j from the proof of Theorem 2.4.1.

Again considering $j \in \{2, \dots, n-1\}$, we have $n - (j+1) < n$, and so the induction hypothesis holds for $\text{Ind}(\Delta_{n-(j+1)}^m)$. We now count the maximum number of vertices in a critical cell in the matching tree below the node $\Sigma(\{j\}, N(j) \cup \{1, 2, \dots, j-1\})$. We have the vertex j itself, $\delta_j + 1$ vertices from $\text{Ind}(Y_{j-1}^m)$, and $d_{n-(j+1)}^{max} + 1$ vertices from $X_{n-(j+1)}^m$. This total number of vertices corresponds to a cell of dimension $\delta_j + d_{n-(j+1)}^{max} + 2$ below the node $\Sigma(\{j\}, N(j) \cup \{1, 2, \dots, j-1\})$. As before, in the special case $j = n$, we expect the subcomplex corresponding to $\text{Ind}(\widehat{Y}_{n+1}^m)$ to contribute one or two cells of the appropriate dimension per Lemma 2.2.6.

Next, we explicitly calculate d_n^{max} for the two cases of m .

Case 1: Suppose that $m = 2$. The proposed d_n^{max} is $\left\lfloor \frac{3n+2}{4} \right\rfloor$.

Subcase 1a: If $j = 3\ell$, then we have

$$\begin{aligned} \delta_j + d_{n-(j+1)}^{max} + 2 &= (2\ell - 1) + \left\lfloor \frac{3(n - (3\ell + 1)) + 2}{4} \right\rfloor + 2 \\ &= \left\lfloor \frac{3n - \ell + 3}{4} \right\rfloor \leq d_n^{max} \end{aligned}$$

since $\ell \geq 1$ as a consequence of $j \geq 2$.

Subcase 1b: If $j = 3\ell + 2$, then we have

$$\begin{aligned}\delta_j + d_{n-(j+1)}^{max} + 2 &= 2\ell + \left\lfloor \frac{3(n - (3\ell + 3) + 2)}{4} \right\rfloor + 2 \\ &= \left\lfloor \frac{3n - \ell + 1}{4} \right\rfloor \leq d_n^{max}.\end{aligned}$$

We now consider the contribution of the subcomplex corresponding to $\text{Ind}(\widehat{Y}_{n+1}^m)$. When $n = 3k$, $d_n^{max} = \left\lfloor \frac{9k+2}{4} \right\rfloor \geq 2k$ while the \widehat{Y}_{n+1}^m contribution has dimension $2k$. When $n = 3k + 1$, $d_n^{max} = \left\lfloor \frac{9k+5}{4} \right\rfloor \geq 2k + 1$ while the \widehat{Y}_{n+1}^m contributions have dimension $2k + 1$. When $n = 3k + 2$, $d_n^{max} = \left\lfloor \frac{9k+8}{4} \right\rfloor \geq 2k + 2$ while the \widehat{Y}_{n+1}^m contribution has dimension $2k + 2$. So, all things considered, no cells of X_n^m exceed the proposed maximum dimension.

Case 2: Suppose that $m \geq 3$. The proposed d_n^{max} is $n + 1 + (m - 3) \left\lfloor \frac{n+2}{3} \right\rfloor$.

Subcase 2a: If $j = 3\ell$, then we have $\delta_j + d_{n-(j+1)}^{max} + 2$ is equal to the following

$$\begin{aligned}& (m\ell - 1) + \left(n - (3\ell + 1) + 1 + (m - 3) \left\lfloor \frac{n - (3\ell + 1) + 2}{3} \right\rfloor \right) + 2 \\ &= n + 1 + (m - 3) \left(\left\lfloor \frac{n - 3\ell + 1}{3} \right\rfloor + \ell \right) \\ &= n + 1 + (m - 3) \left\lfloor \frac{n + 1}{3} \right\rfloor \leq d_n^{max}.\end{aligned}$$

Subcase 2b: If $j = 3\ell + 2$, then we have $\delta_j + d_{n-(j+1)}^{max} + 2$ is equal to the following

$$\begin{aligned}& (m\ell) + \left(n - (3\ell + 3) + 1 + (m - 3) \left\lfloor \frac{n - (3\ell + 3) + 2}{3} \right\rfloor \right) + 2 \\ &= n + (m - 3) \left(\left\lfloor \frac{n - 3\ell - 1}{3} \right\rfloor + \ell \right) \\ &= n + (m - 3) \left\lfloor \frac{n - 1}{3} \right\rfloor \leq d_n^{max}.\end{aligned}$$

We now consider the contribution of the subcomplex corresponding to $\text{Ind}(\widehat{Y}_{n+1}^m)$. When $n = 3k$, $d_n^{max} = 3k + 1 + (m - 3) \left\lfloor \frac{3k+2}{3} \right\rfloor = mk + 1$ while the \widehat{Y}_{n+1}^m contribution has dimension mk . When $n = 3k + 1$, $d_n^{max} = (3k + 1) + 1 + (m - 3) \left\lfloor \frac{(3k+1)+2}{3} \right\rfloor = m(k + 1) - 1$ while the \widehat{Y}_{n+1}^m contributions have dimension $mk + 1$ and $m(k + 1) - 1$ respectively. When $n = 3k + 2$, $d_n^{max} = (3k + 2) + 1 + (m - 3) \left\lfloor \frac{(3k+2)+2}{3} \right\rfloor = mk + m$ while the \widehat{Y}_{n+1}^m contribution has dimension $mk + m$. So, all things considered, no cells of X_n^m exceed the proposed maximum dimension.

Observe that, in both Case 1 and Case 2, if $j = 3$, then the $\text{Ind}(Y_2^m) * \text{Ind}(\Delta_{n-4}^m)$ factor produces at least one cell of dimension exactly d_n^{max} . \square

Using Theorems 2.3.2 and 2.3.3 we can create data tables containing dimensions of the integral cellular chain spaces of X_n^m for reasonable values of n and m . The reader may refer to the Appendix for the tables displaying the data for the $0 \leq n \leq 20$ and $2 \leq m \leq 5$. For $m \geq 4$, it is interesting that gaps appear in the dimensions of the chain spaces for low values of d relative to n . For example, the Comb Algorithm eliminates all cells of dimensions $\lfloor \frac{2n+2}{3} \rfloor + 1$ through $\lfloor \frac{2n+2}{3} \rfloor + (m-3)$ when $n = 3k$ or $n = 3k + 1$. Furthermore, we can explicitly determine the lowest non-vanishing homology for $n = 3k$ and $n = 3k + 1$ when $m \geq 4$; see Jonsson [14, Lemma 2.3 and Proposition 2.7] for analogous results when $m = 2$.

Theorem 2.4.4. Suppose that $m \geq 4$, and let $d_n = \lfloor \frac{2n+2}{3} \rfloor$. If $n = 3k$ or $n = 3k + 1$, then $H_{d_n}(X_n^m; \mathbb{Z}) \cong \mathbb{Z}$. If $n = 3k + 2$, then $H_{d_n}(X_n^m; \mathbb{Z})$ is trivial.

Proof. We consider three cases, one for each value of $n \pmod 3$.

Case 1: Suppose $n = 3k$. We know that $C_n^\ell = 0$ for $\ell < d_n$ from our cellular dimension range. We argue by induction on k that $C_n^{d_n} = 1$ while $C_n^{d_n+1} = 0$, which proves the claim for $n = 3k$. Begin by recalling that $C_0^0 = 1$ and $C_0^1 = 0$, which provides a base case.

Now, assume that $C_{3\ell}^{d_{3\ell}} = 1$ while $C_{3\ell}^{d_{3\ell}+1} = 0$ for $0 \leq \ell < k$. We know that

$$C_n^{d_n} = C_{3k-3}^{d_{3k}-2} + C_{3k-4}^{d_{3k}-m-1} + C_{3k-3}^{d_{3k}-m}$$

by our cellular recursion. Observe that $d_{3k} - 2 = 2k - 2 = d_{3k-3}$, so $C_{3k-3}^{d_{3k}-2} = 1$ by the induction hypothesis. Since $d_{3k} - m - 1 < d_{3k} - 2 = d_{3k-4}$, it follows that $C_{3k-4}^{d_{3k}-m-1} = 0$. Similarly, $d_{3k} - m < d_{3k} - 2 = d_{3k-3}$, so $C_{3k-3}^{d_{3k}-m} = 0$. Hence, $C_n^{d_n} = 1$.

Our cellular recursion also gives

$$C_n^{d_n+1} = C_{3k-3}^{d_{3k}-1} + C_{3k-4}^{d_{3k}-m} + C_{3k-3}^{d_{3k}-m+1}.$$

Observe that $d_{3k} - 1 = 2k - 1 = d_{3k-3} + 1$, so $C_{3k-3}^{d_{3k}-1} = 0$ by the induction hypothesis. Now, we note that $d_{3k} - m < d_{3k} - 2 = d_{3k-4}$ still, which implies $C_{3k-4}^{d_{3k}-m} = 0$. Similarly, $d_{3k} - m + 1 < d_{3k} - 2 = d_{3k-3}$, so $C_{3k-3}^{d_{3k}-m+1} = 0$. Hence, $C_n^{d_n+1} = 0$. By induction, we conclude that $C_{3k}^{d_{3k}} = 1$ while $C_{3k}^{d_{3k}+1} = 0$ for all k , from which the result follows.

Case 2: Suppose $n = 3k + 1$; this argument is similar to that of the previous case. We argue by induction on k that $C_n^{d_n} = 1$ while $C_n^{d_n+1} = 0$. We obtain our base case by recalling that $C_1^1 = 1$ and $C_1^2 = 0$ for $m \geq 4$. Next, we know that

$$C_n^{d_n} = C_{3k-2}^{d_{3k+1}-2} + C_{3k-3}^{d_{3k+1}-m-1} + C_{3k-3}^{d_{3k+1}-m}$$

by our cellular recursion. Observe that $d_{3k+1} - 2 = 2k - 1 = d_{3k-2} = d_{3(k-1)+1}$, so $C_{3k-2}^{d_{3k+1}-2} = 1$ by the induction hypothesis. Now, $d_{3k+1} - m - 1 = 2k - m < 2k - 2$,

which is precisely d_{3k-3} , implying that $C_{3k-3}^{d_{3k+1}-m-1} = 0$. Similarly, we see that $d_{3k+1}-m$ equal $2k - m + 1 < 2k - 1 = d_{3k-2}$, so $C_{3k-2}^{d_{3k+1}-m} = 0$. Hence, $C_n^{d_n} = 1$.

We also know that

$$C_n^{d_n+1} = C_{3k-2}^{d_{3k+1}-1} + C_{3k-3}^{d_{3k+1}-m} + C_{3k-2}^{d_{3k+1}-m+1}$$

by our cellular recursion. Observe that $d_{3k+1} - 1 = d_{3k+1} - 2 + 1 = d_{3k-2} + 1$, so $C_{3k-2}^{d_{3k+1}-1} = 0$ by the induction hypothesis. Now, $d_{3k+1} - m = 2k - m + 1 < 2k - 2$, which is again d_{3k-3} . Therefore, $C_{3k-3}^{d_{3k+1}-m} = 0$. Similarly, $d_{3k+1}-m+1 < 2k-1 = d_{3k-2}$, so $C_{3k-2}^{d_{3k+1}-m+1} = 0$. Hence, $C_n^{d_n+1} = 0$. By induction, we conclude that $C_{3k+1}^{d_{3k+1}} = 1$ while $C_{3k+1}^{d_{3k+1}+1} = 0$ for all k , from which the result follows.

Case 3: Suppose $n = 3k + 2$. Recall from Theorem 2.4.1 that for $n = 3k + 2$ and $m \geq 3$, the minimum dimension of critical cells produced by the Comb Algorithm is $2 \lfloor \frac{n-1}{3} \rfloor + m$. It is easy to check that $\lfloor \frac{2n+2}{3} \rfloor = 2k + 2 < 2k + m = 2 \lfloor \frac{n-1}{3} \rfloor + m$. Therefore, $C_n^{d_n} = 0$ when $n = 3k + 2$, i.e. $H_{d_n}(X_n^m; \mathbb{Z})$ is trivial. \square

For other homology groups, the Comb Algorithm provides less comprehensive results. For example, when $m = 2$, that is, when X_n^m is homotopy equivalent to the matching complex on the $2 \times (n + 2)$ grid graph, a direct analysis of the chain space dimensions on a data table yields the following.

Observation 2.4.5. X_n^2 has non-trivial free integral homology in dimension $\lfloor \frac{9n+9}{13} \rfloor$ for $0 \leq n \leq 99$, except for $n \in \{48, 61, 74, 84, 87, 90, 94, 97\}$. This arises because the rank of the chain space of X_n^2 in dimension $\lfloor \frac{9n+9}{13} \rfloor$ exceeds the sum of the ranks of the chain spaces in dimensions $\lfloor \frac{9n+9}{13} \rfloor - 1$ and $\lfloor \frac{9n+9}{13} \rfloor + 1$ for these values of n . Consequently, even if we were to try to further match away the critical cells in dimension $\lfloor \frac{9n+9}{13} \rfloor$, there are not enough cells in the adjacent dimensions to completely pair them all away.

As an interesting side note, when $m = 2$, the values of d_n^{min} and d_n^{max} imply that X_n^2 is a wedge of spheres for $n \in \{0, 1, 2, 3, 4, 5, 7, 8, 11\}$.

As n grows larger, the data suggest that the rank of the $\lfloor \frac{9n+9}{13} \rfloor$ -dimensional chain space ceases to “typically” exceed the sum of the ranks of the neighboring chain spaces. This suggests that the behavior of $\text{Ind}(\Delta_n^m)$ for “small” values of n , including many values of n for which by-hand computations appear prohibitive, is not indicative of the general behavior of these complexes.

In conclusion, the topology of $\text{Ind}(\Delta_n^m)$ remains generally mysterious. It would be of interest to investigate the following two questions.

1. Does torsion occur in the homology of $\text{Ind}(\Delta_n^m)$? If so, for which p does $\mathbb{Z}/p\mathbb{Z}$ appear as a summand?
2. There is a natural action of the symmetric group \mathfrak{S}_m on $\text{Ind}(\Delta_n^m)$. What is the \mathfrak{S}_m -module structure of $H_*(\text{Ind}(\Delta_n^m); \mathbb{C})$?

Chapter 3 - Poset Homomorphism Complexes

3.1 Introduction

Definition 3.1.1. Given two combinatorial graphs G and H , a *graph homomorphism* from G to H is a map $\phi : V(G) \rightarrow V(H)$ such that $\{u, v\} \in E(G)$ implies $\{\phi(u), \phi(v)\} \in E(H)$. We write $\phi : G \rightarrow H$ for economy of space.

Lovász was the first to introduce the notion of the graph homomorphism complex $\text{Hom}(G, H)$ and use it to detect information about the chromatic number of the underlying graph G . Since then, homomorphism complexes have played a prominent role in the topological investigation of graphs. See [19] for an excellent survey of such results. Usually, one realizes a graph property of interest as a particular kind of graph homomorphism, generates the corresponding homomorphism complex, and then uses topological techniques on the resulting space to infer conclusions about the underlying graph property. For example, a proper n -coloring of a graph G corresponds precisely to a graph homomorphism $\phi : G \rightarrow K_n$, where K_n denotes the complete graph on n vertices, as before. Therefore, information about the chromatic number of G can be detected by studying $\text{Hom}(G, K_n)$ for various n .

Given two graphs G and H , we construct the graph homomorphism complex $\text{Hom}(G, H)$ as follows.

Definition 3.1.2. Let $\Delta^{V(H)}$ be a simplex whose set of vertices is $V(H)$. Let $C(G, H)$ denote the direct product $\prod_{x \in V(G)} \Delta^{V(H)}$, i.e. the copies of $\Delta^{V(H)}$ are indexed by vertices of G . Then, $\text{Hom}(G, H)$ is the subcomplex of $C(G, H)$ defined by the condition that: $\sigma = \prod_{x \in V(G)} \sigma_x \in \text{Hom}(G, H)$ if and only if for any $x, y \in V(G)$, if $(x, y) \in E(G)$, then (σ_x, σ_y) is a complete bipartite subgraph of H , i.e. $\sigma_x \times \sigma_y \subseteq E(H)$.

Remark 3.1.3. Observe that $\text{Hom}(G, H)$ is a polyhedral complex whose cells are indexed by all functions $\eta : V(G) \rightarrow 2^{V(H)} \setminus \{\emptyset\}$, such that if $(x, y) \in E(G)$, then $\eta(x) \times \eta(y) \subseteq E(H)$. The closure of a cell η consists of all cells $\tilde{\eta}$ satisfying $\tilde{\eta}(v) \subseteq \eta(v)$ for all $v \in V(G)$. Also, given a graph homomorphism $\phi : G \rightarrow G'$, we will denote the topological map induced by composition by $\phi_H : \text{Hom}(G', H) \rightarrow \text{Hom}(G, H)$. Note that ϕ_H is cellular on the first barycentric subdivision.

The construction of a complex of morphisms is not restricted merely to graphs; another interesting category to work with is that of posets and poset morphisms,

particularly poset maps that are strictly order-preserving. We now extend the above definitions to the poset case.

Definition 3.1.4. Given two partially ordered sets P and Q , a strict order-preserving *poset homomorphism* from P to Q is any map $\phi : P \rightarrow Q$ such that $x <_P y$ implies $\phi(x) <_Q \phi(y)$.

Definition 3.1.5. We define $\text{Hom}(P, Q)$ analogously to its graph counterpart. Let Δ^Q be a simplex whose set of vertices is Q . Let $C(P, Q)$ denote the direct product $\prod_{x \in P} \Delta^Q$, i.e. the copies of Δ^Q are indexed by elements of P . Then, $\text{Hom}(P, Q)$ is the subcomplex of $C(P, Q)$ defined by the following condition: $\sigma = \prod_{x \in P} \sigma_x \in \text{Hom}(P, Q)$ if and only if for any $x, y \in P$, if $x \leq_P y$, then $x' \leq_Q y'$ for any $x' \in \sigma_x$ and any $y' \in \sigma_y$.

3.2 Folding

In the graph context, the homotopy type of $\text{Hom}(H, G)$ is preserved by a “folding” operation on either H or G , defined as follows.

Definition 3.2.1. Let $\mathcal{N}(v)$ denote the neighborhood of v in G . Then, $G - v$ is called a **fold** of G if there exists $u \in V(G)$ with $u \neq v$ such that $\mathcal{N}(u) \supseteq \mathcal{N}(v)$. We let $i : G - v \hookrightarrow G$ denote the inclusion homomorphism.

Theorem 3.2.2. (Kozlov [18]) Let $G - v$ be a fold of G , and let H be some graph. Then, $Bd \text{Hom}(G, H)$ collapses onto $Bd \text{Hom}(G - v, H)$, whereas $\text{Hom}(H, G)$ collapses onto $\text{Hom}(H, G - v)$. Here, Bd denotes the barycentric subdivision.

Therefore, there exists a suitable neighborhood condition that allows us to essentially ignore certain vertices in either of our graphs. We extend the definition of a fold to the poset case.

Definition 3.2.3. Let $\mathcal{U}(x)$ denote the set of elements that cover x , and let $\mathcal{D}(x)$ denote the set of elements covered by x . We say that $P - x$ is a **fold** of P if there exists $y \in P$ with $x \neq y$ such that $\mathcal{U}(y) \supseteq \mathcal{U}(x)$ and $\mathcal{D}(y) \supseteq \mathcal{D}(x)$. Observe that $\mathcal{U}(x) \cup \mathcal{D}(x)$ is equal to the graph-theoretic neighborhood of x .

This modified notion of folds and poset neighborhoods yields the following.

Theorem 3.2.4. Let $P - x$ be a fold of P with Q some poset. Then $Bd \text{Hom}(P, Q)$ collapses onto $Bd \text{Hom}(P - x, Q)$, whereas $\text{Hom}(Q, P)$ collapses onto $\text{Hom}(Q, P - x)$.

In order to prove this result, we will use the idea and language of closure operators.

Definition 3.2.5. An order-preserving map ϕ from a poset P to itself is a *descending closure operator* if $\phi^2 = \phi$ and $\phi(x) \leq x$ for any $x \in P$. Similarly, ϕ is an *ascending closure operator* if $\phi^2 = \phi$ and $\phi(x) \geq x$ for any $x \in P$.

It is well-known in topological combinatorics that ascending and descending closure operators induce strong deformation retracts.

Definition 3.2.6. Given a poset P , $\Delta(P)$ is the *order complex* of P , the simplicial complex whose simplices are the chains of P . Moreover, $\mathcal{F}(\Delta(P)) = Bd P$.

Theorem 3.2.7. (Kozlov [18]) Let P be a poset, and let ϕ be a descending closure operator. Then $\Delta(P)$ collapses onto $\Delta(\phi(P))$. By symmetry, the same is true for an ascending closure operator.

Afforded this language, we need only slightly modify Kozlov's original proof for the graphical case.

Proof. (of Theorem 3.2.4)

First, we show that $Bd \text{Hom}(P, Q)$ collapses onto $Bd \text{Hom}(P - x, Q)$. Identify $\mathcal{F}(\text{Hom}(P - x, Q))$ with the subposet of $\mathcal{F}(\text{Hom}(P, Q))$ consisting of all η such that $\eta(x) = \eta(y)$. Let X be the subposet consisting of all $\eta \in \mathcal{F}(\text{Hom}(P, Q))$ satisfying $\eta(x) \supseteq \eta(y)$. Then, $\mathcal{F}(\text{Hom}(P - x, Q)) \subseteq X \subseteq \mathcal{F}(\text{Hom}(P, Q))$. Consider order-preserving maps

$$\mathcal{F}(\text{Hom}(P, Q)) \xrightarrow{\alpha} X \xrightarrow{\beta} \mathcal{F}(\text{Hom}(P - x, Q)),$$

defined by

$$\alpha\eta(z) = \begin{cases} \eta(y) \cup \eta(x), & \text{for } z = x; \\ \eta(z), & \text{otherwise;} \end{cases} \quad \beta\eta(z) = \begin{cases} \eta(y), & \text{for } z = x; \\ \eta(z), & \text{otherwise;} \end{cases}$$

for all $z \in P$. Maps α and β are well defined because $P - x$ is a fold of P . Clearly $\beta \circ \alpha = \mathcal{F}(i_Q)$, α is an ascending closure operator, and β is a descending closure operator. Since the image of $\mathcal{F}(i_Q)$ is $\mathcal{F}(\text{Hom}(P - x, Q))$, the statement follows from Theorem 3.2.7.

Next, we show that $\text{Hom}(Q, P)$ collapses onto $\text{Hom}(Q, P - x)$ by presenting a sequence of elementary collapses. Let $Q = \{q_1, q_2, \dots, q_t\}$. For $\eta \in \mathcal{F}(\text{Hom}(Q, P))$, let $1 \leq i(\eta) \leq t$ be the minimal index such that $x \in \eta(q_{i(\eta)})$. Write $\mathcal{F}(\text{Hom}(Q, P))$ as a disjoint union $A \cup B \cup \mathcal{F}(\text{Hom}(Q, P - x))$, defined as follows: for $\eta \in A \cup B$, we have $\eta \in A$ if $y \notin \eta(q_{i(\eta)})$, and we have $\eta \in B$ otherwise.

There is a bijection $\phi : A \rightarrow B$ which adds y to $\eta(q_{i(\eta)})$ without changing the other values of η . Adding y to $\eta(q_{i(\eta)})$ yields an element in $\mathcal{F}(\text{Hom}(Q, P))$ since $P - x$ is a

fold of P . Clearly, $\phi(\alpha)$ covers α for all $\alpha \in A$. We take the set $\{(\alpha, \phi(\alpha)) \mid \alpha \in A\}$ to be our collection of the elementary collapses. These are ordered lexicographically after the pairs of integers $(i(\alpha), -\dim \alpha)$.

Let us see that these collapses can be performed in this lexicographic order. Take $\eta > \alpha$, $\eta \neq \phi(\alpha)$. Assume $i(\eta) = i(\alpha)$. If $\eta \in B$, then $\eta = \phi(\tilde{\alpha})$, $i(\tilde{\alpha}) = i(\alpha)$, and $\dim \tilde{\alpha} > \dim \alpha$. Otherwise $\eta \in A$ and $\dim \eta > \dim \alpha$. The third possibility is that $i(\eta) < i(\alpha)$. In either case, η has been removed before α . \square

Consequently, certain poset elements with an appropriate neighborhood property can be ignored when examining $\text{Hom}(P, Q)$.

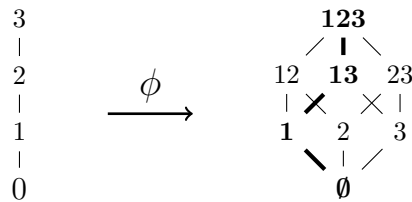
3.3 Maximal Chains in the Boolean Algebras

Definition 3.3.1. Let C_n be the chain on n elements with labels in $\{0, 1, 2, \dots, n-1\}$.

The complex $\text{Hom}(C_n, P)$ can be thought of as the complex of n -chains in P . In particular, when P is graded of length n , $\text{Hom}(C_n, P)$ topologizes the space of maximal chains in P . A particularly interesting example of this type involves the Boolean algebras.

Definition 3.3.2. Let B_n denote the Boolean algebra on n elements, i.e the power set of $[n]$ ordered by set inclusion. Recall that B_n has rank $n + 1$. We use $\text{Hom}(B_n)$ to denote $\text{Hom}(C_{n+1}, B_n)$.

Figure 3.1: A homomorphism from C_4 into B_3



Note that, for any homomorphism $\phi \in \text{Hom}(B_n)$, $\phi(0) = \emptyset$ and $\phi(n) = [n]$. These homomorphisms are in bijective correspondence with permutations on n letters, and this correspondence sends a homomorphism ϕ to the permutation whose i -th entry is $\phi(i) \setminus \phi(i-1)$ for $1 \leq i \leq n$. We will not distinguish between such a homomorphism and its corresponding permutation. Also, we will use the convention that $\phi_i = \phi(i)$ for these permutations.

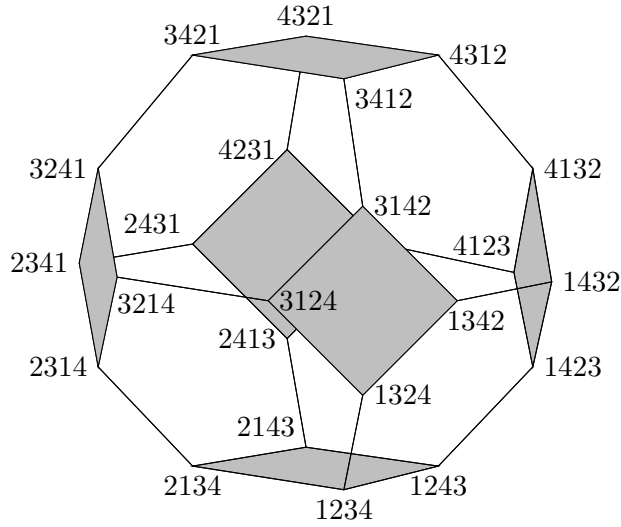
Two vertices in $\text{Hom}(B_n)$ are adjacent if they differ by a single transposition of adjacent entries, that is $\{\sigma, \tau\}$ form a 1-cell in $\text{Hom}(B_n)$ if $\sigma_i = \tau_{i+1}$ and $\sigma_{i+1} = \tau_i$ for

some $1 \leq i \leq n - 1$ while $\sigma_j = \tau_j$ for $j \notin \{i, i + 1\}$. Observe that this implies that $\text{Hom}(B_n)$ lies in the boundary of the permutohedron of order n . See Figure 3.2 below. Without loss of generality, assume that σ is lexicographically earlier than τ under the usual ordering of the positive integers. Then, we can denote the above transposition by $(\sigma_{i+1}\sigma_i)$, written in decreasing order for later convenience. We will denote the corresponding 1-cell as

$$\sigma_1 \dots \sigma_{i-1} (\sigma_{i+1}\sigma_i) \sigma_{i+2} \dots \sigma_n.$$

This notation intuitively extends to all i -cells for general i . Note that the number of transpositions in a given cell equals its dimension, and so $0 \leq i \leq \lfloor n/2 \rfloor$.

Figure 3.2: $\text{Hom}(B_4)$



Definition 3.3.3. Entries that are not included in a cell's transpositions will be called *free*, while two elements are *joined* if they appear in a transposition together.

Example 3.3.4. The homomorphisms 35142 and 31542 are adjacent in $\text{Hom}(B_5)$ via the transposition (51), and so the corresponding 1-cell is $3(51)42 = \{31542, 35142\}$. The expression $(64)5(32)(71)$ denotes the 3-cell in $\text{Hom}(B_7)$ with vertex set

$$\{4652317, 4652371, 4653217, 4653271, 6452317, 6452371, 6453217, 6453271\}.$$

Also, 2, 3, and 4 are free while 1 and 5 are joined in the first example. The only free element in the second example is 5, while each pair of 6 and 4, 3 and 2, and 7 and 1 are all joined.

We use this parenthesized permutation notation to obtain the following result.

Theorem 3.3.5. The unreduced Euler characteristic of $\text{Hom}(B_n)$ is

$$\chi_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n!}{2^k} \binom{n-k}{k}. \quad (3.1)$$

Proof. Fix $n \geq 1$. We now count the number of k -cells in $\text{Hom}(B_n)$ for $0 \leq k \leq \lfloor n/2 \rfloor$. Since the dimension of a cell in $\text{Hom}(B_n)$ equals the number of transpositions in its parenthesized notation, a k -cell will have transpositions in k different locations. If we consider a free element as a block of size 1 and a transposition as a block of size 2, then we see that the relative order of free elements and transpositions for a given k -cell corresponds to a composition of n using parts of size 1 and 2. We know there are $\binom{n-k}{k}$ such compositions. (There will be $n-k$ blocks, and we choose k of them to be of size 2.) Now, for each such block arrangement, there are $\frac{n!}{2^k}$ ways to order the elements into the blocks where the order in a block of size 2 is irrelevant. In total, there are $\frac{n!}{2^k} \binom{n-k}{k}$ cells of size k in $\text{Hom}(B_n)$. \square

Theorem 3.3.6. Formula 3.1 satisfies the recursion $\chi_n = n \cdot \chi_{n-1} - \binom{n}{2} \cdot \chi_{n-2}$ with initial conditions $\chi_1 = \chi_2 = 1$.

Proof. It is easy to verify the initial conditions. Let $X_n = \frac{\chi_n}{n!}$ for arbitrary $n \geq 1$, and we prove instead that $X_n = X_{n-1} - \frac{1}{2}X_{n-2}$. Multiplying by $n!$ gives the result.

Case 1: Suppose that $n = 2j$ for some integer j . From formula 3.1, we know that

$$X_{n-1} = \sum_{k=0}^{j-1} (-1)^k \frac{1}{2^k} \binom{n-1-k}{k}.$$

We also have

$$-\frac{1}{2}X_{n-2} = \sum_{k=0}^{j-1} (-1)^{k+1} \frac{1}{2^{k+1}} \binom{n-2-k}{k} = \sum_{k=1}^j (-1)^k \frac{1}{2^k} \binom{n-1-k}{k-1},$$

where the second equality is obtained by reindexing. Thus, $X_{n-1} - \frac{1}{2}X_{n-2}$ equals

$$1 + \sum_{k=1}^{j-1} (-1)^k \frac{1}{2^k} \binom{n-1-k}{k} + \sum_{k=1}^{j-1} (-1)^k \frac{1}{2^k} \binom{n-1-k}{k-1} + (-1)^j \frac{1}{2^j} \binom{n-1-j}{j-1}.$$

Since $\binom{n-1-k}{k-1} + \binom{n-1-k}{k} = \binom{n-k}{k}$, we have

$$X_{n-1} - \frac{1}{2}X_{n-2} = 1 + \sum_{k=1}^{j-1} (-1)^k \frac{1}{2^k} \binom{n-k}{k} + (-1)^j \frac{1}{2^j} \binom{n-1-j}{j-1}.$$

Observe that $(-1)^0 \frac{1}{2^0} \binom{n-0}{0} = 1$. Also, $\binom{n-1-j}{j-1} = 1 = \binom{n-j}{j}$ since $j = n/2$. Thus,

$$X_{n-1} - \frac{1}{2}X_{n-2} = \sum_{k=0}^j (-1)^k \frac{1}{2^k} \binom{n-k}{k} = X_n.$$

Case 2: Suppose that $n = 2j + 1$ for some integer j . Formula 3.1 gives

$$X_{n-1} = \sum_{k=0}^j (-1)^k \frac{1}{2^k} \binom{n-1-k}{k}.$$

We also have

$$-\frac{1}{2}X_{n-2} = \sum_{k=0}^{j-1} (-1)^{k+1} \frac{1}{2^{k+1}} \binom{n-2-k}{k} = \sum_{k=1}^j (-1)^k \frac{1}{2^k} \binom{n-1-k}{k-1},$$

where the second equality is obtained by reindexing. Therefore,

$$X_{n-1} - \frac{1}{2}X_{n-2} = 1 + \sum_{k=1}^j (-1)^k \frac{1}{2^k} \binom{n-1-k}{k} + \sum_{k=1}^j (-1)^k \frac{1}{2^k} \binom{n-1-k}{k-1}.$$

Since $\binom{n-1-k}{k-1} + \binom{n-1-k}{k} = \binom{n-k}{k}$, we have

$$X_{n-1} - \frac{1}{2}X_{n-2} = 1 + \sum_{k=1}^j (-1)^k \frac{1}{2^k} \binom{n-k}{k}.$$

Observe that $(-1)^0 \frac{1}{2^0} \binom{n-0}{0} = 1$. Thus,

$$X_{n-1} - \frac{1}{2}X_{n-2} = \sum_{k=0}^j (-1)^k \frac{1}{2^k} \binom{n-k}{k} = X_n.$$

Thus, the result holds in both cases. \square

Corollary 3.3.7. Using the Division Algorithm, write $n = 4q + r$ where $0 \leq r \leq 3$ and q is a non-negative integer. Then,

$$\chi_n = \begin{cases} (-1/4)^q \cdot n! & r \in \{0, 1\} \\ (1/2)(-1/4)^q \cdot n! & r = 2 \\ 0 & r = 3 \end{cases}$$

Proof. It is straightforward to verify that $\chi_1 = \chi_2 = 1$, $\chi_3 = 0$, and $\chi_4 = -6$. Let $X_n = \chi_n/n!$ as before, and recall that $X_n = X_{n-1} - \frac{1}{2}X_{n-2}$ for $n \geq 3$. Then, we have $X_1 = 1$, $X_2 = 1/2$, $X_3 = 0$, and $X_4 = -1/4$. Now, for $n \geq 5$, we see that

$$\begin{aligned} X_n &= X_{n-1} - \frac{1}{2}X_{n-2} \\ &= (X_{n-2} - \frac{1}{2}X_{n-3}) - \frac{1}{2}(X_{n-3} - \frac{1}{2}X_{n-4}) \\ &= X_{n-2} - X_{n-3} + \frac{1}{4}X_{n-4} \\ &= (X_{n-3} - \frac{1}{2}X_{n-4}) - X_{n-3} + \frac{1}{4}X_{n-4} \\ &= -\frac{1}{4}X_{n-4}. \end{aligned}$$

Hence, $X_n = (-1/4)^q X_r$ where $n = 4q + r$ as in the statement. Technically, X_0 is undefined, but we set it equal to 1 to ensure that $X_4 = (-1/4)^1 X_0 = -1/4$ compatibly. Multiplying by $n!$ proves the claim. \square

Another convenient way to view a k -cell of $\text{Hom}(B_n)$ is as a block system consisting of $k+1$ blocks; the first k blocks consist of a consecutive run of free entries terminating in a transposition, and the terminal $(k+1)$ -st block is just another consecutive sequence of free entries (which we abbreviate as *c.f.s.*) without a transposition. For a given block, the consecutive free sequence may be empty.

Example 3.3.8. The 1-cell $3(51)42$ from Example 3.3.4 has two blocks from this new viewpoint: the first block is $3(51)$, and the second is 42 . Similarly, the 3-cell $(64)5(32)(71)$ has 4 blocks: the first three, namely (64) , $5(32)$, and (71) , are nontrivial, and the fourth/final block is empty.

We will use this block system interpretation to define an acyclic partial matching on the face poset of $\text{Hom}(B_n)$ and then enumerate the critical cells remaining in the reduced complex. We find the following definition useful.

Definition 3.3.9. An element of a *c.f.s.* is a *peak* if it is larger than its adjacent free elements, even if those elements do not exist.

Example 3.3.10. The element 5 is a peak for each of 12534 , 12345 , and $125(43)$ while 4 is a peak only for the first.

Now, we define a partial matching on $\mathcal{F}(\text{Hom}(B_n))$. The main idea is to first partition $\mathcal{F}(\text{Hom}(B_n))$ using the *c.f.s.* of the terminal block and then pair cells using appropriate peaks. The following construction defines the first partition, while the second defines how to inductively sub-partition the remaining poset blocks.

Construction 3.3.11. (Initial Partition)

1. Consider an arbitrary cell σ of $\text{Hom}(B_n)$ with parenthesized notation as before. Recall that all transpositions are written in decreasing order. Now, ignoring any parentheses in σ , let j denote the largest index such that $\sigma_j \neq j$. We let $j = 0$ for the identity permutation. Observe that $j \in J := \{0, 2, 3, \dots, n\}$ as $j = 1$ implies that only the first element is out of order, which is impossible.
2. Define the poset map $\phi : \mathcal{F}(\text{Hom}(B_n)) \rightarrow C_J$ by $\phi(\sigma) = j$. Here, C_J is the chain poset on the set J with ordering inherited from the integers. This map is order-preserving because releasing a transposition (removing its parentheses and writing the joined entries in either order) does not increase the value of j .

3. Now, fix $j \in J$ and suppose that $\sigma \in \phi^{-1}(j)$. We may now think of σ as living in $\text{Hom}(B_j)$ by removing the ascending run of $j+1, j+2, \dots, n$ in the terminal block of σ .
4. Ignoring any parentheses, let r_0 be the index such that $\sigma_{r_0} = j$. Observe that $r_0 < j$.
5. Define the poset map $\rho_0 : \phi^{-1}(j) \rightarrow C_{R_j}^{op}$ by $\rho_0(\sigma) = r_0$ where $R_j = [j-1]$. Recall that P^{op} denotes the *opposite poset* of P , the same elements with every relation reversed. If j is in a transposition, releasing the transposition in either order will not decrease the value of r_0 . If j is not in a transposition, then releasing another transposition will leave r_0 fixed. Thus, ρ_0 is order-preserving.
6. For $\sigma \in \rho_0^{-1}(r_0) \subseteq \phi^{-1}(j)$, define a poset map $\psi_0 : \rho_0^{-1}(r_0) \rightarrow \{a < b\}$ as follows: If j is a peak of its block with σ_{r_0+1} free OR if j and σ_{r_0+1} are in a transposition together, then $\psi_0(\sigma) := a$. Otherwise, $\psi_0(\sigma) := b$. The fact that ψ_0 is order-preserving is proven in Claim 3.3.12.
7. On $\psi_0^{-1}(a) \subseteq \rho_0^{-1}(r_0) \subseteq \phi^{-1}(j)$, pair the cells with free elements j and σ_{r_0+1} with the corresponding parent cells with joined j and σ_{r_0+1} .

Claim 3.3.12. The poset map ψ_0 as defined in Step 6 of Construction 3.3.11 is order-preserving.

Proof. Suppose that $\sigma < \tau$ in $\rho_0^{-1}(r_0)$ with $\psi_0(\tau) = a$. Either (1) j is a peak of its block with τ_{r_0+1} free, or (2) j and τ_{r_0+1} are joined. If τ has property (1), so must σ as freedom of elements is inherited downward. If τ has property (2), then σ must have property (1) or (2) as transpositions are inherited upward in the poset. Since $\psi_0^{-1}(a)$ is down-closed, ψ_0 is order-preserving. \square

Remark 3.3.13. This pairing defined above is perfect and acyclic on $\psi_0^{-1}(a)$. No cycle of the form described in 1.1.1 exists since a second up-move in such a cycle would correspond to the j element being joined to two different elements simultaneously, which is not allowed. It is important to note that, for elements of $\psi_0^{-1}(b)$, j must be followed immediately by a transposition.

In Construction 3.3.14 below, we factor the subposet $\psi_0^{-1}(b) \subseteq \rho_0^{-1}(r_0) \subseteq \phi^{-1}(j)$ and then perform an iterative version of 3.3.11. Suppose $i \geq 1$ and consider the fiber

$$\Omega_i^{(r_0, r_1, \dots, r_{i-1})} := \psi_{i-1}^{-1}(b) \subseteq \rho_{i-1}^{-1}(r_{i-1}) \subseteq \psi_{i-2}^{-1}(b) \subseteq \dots \subseteq \rho_0^{-1}(r_0) \subseteq \phi^{-1}(j),$$

where the values of the r_k 's are chosen below.

Construction 3.3.14. (Iterative Partitioning)

1. Ignoring the parentheses in σ , define r_i to be the index such that $\sigma(r_i) = j - i$.
2. Define a poset map $\rho_i : \Omega_i^{(r_0, r_1, \dots, r_{i-1})} \rightarrow C_{R_j}^{op}$ by $\rho_i(\sigma) = r_i$. If $j - i$ is joined, releasing the transposition in either order will not decrease the value of r_i . If $j - i$ is not joined, then releasing another transposition will leave r_i fixed. Thus, ρ_i is order-preserving.
3. Consider the following properties:
 - (i) $j - i$ is a peak of its block with $\sigma_{r_{i+1}}$ free.
 - (ii) $j - i$ and $\sigma_{r_{i+1}}$ are joined.
 - (iii) $\sigma_{r_{i-1}}$ is free and less than $j - i$.
 - (iv) $\sigma_{r_{i-1}}$ is joined with $\sigma_{r_{i-2}}$.

Now, define another poset map $\psi_i : \rho_i^{-1}(r_i) \rightarrow \{a < b\}$ such that $\psi_i(\sigma) := a$ if σ satisfies property (i), properties (ii) and (iii), or properties (ii) and (iv). Otherwise $\psi_i(\sigma) := b$. The fact that ψ_i is order-preserving is proven in Claim 3.3.15.

4. Observe that each $\sigma \in \psi_i^{-1}(a) \subseteq \rho_i^{-1}(r_i) \subseteq \Omega_i^{(r_0, r_1, \dots, r_{i-1})}$ has either property (iii) or (iv). Otherwise, σ would not actually be in $\Omega_i^{(r_0, r_1, \dots, r_{i-1})}$. Also, note that every cell with property (i) has exactly one parent with property (ii). So, pair each cell with with property (i) to the corresponding parent element with property (ii). Please note that This pairing is well-defined and perfect on $\psi_i^{-1}(a) \subseteq \rho_i^{-1}(r_i)$ inside $\Omega_i^{(r_0, r_1, \dots, r_{i-1})}$. This pairing is also acyclic since a second up-move in the face poset would correspond to the $j - i$ element being joined to two different elements simultaneously.
5. Repeat the above steps for $i + 1$. This is a finite process as there are finitely many entries in σ .

Claim 3.3.15. The poset map ψ_i as defined in Step 3 of Construction 3.3.14 is order-preserving.

Proof. Suppose that $\sigma < \tau$ in $\rho_i^{-1}(r_i) \subseteq \Omega_i^{(r_0, r_1, \dots, r_{i-1})}$ and $\psi_i(\tau) = a$. If τ has property (i), then so must σ . If τ has properties (ii) and (iii), then σ must have property (i) or both (ii) and (iii). If τ has properties (ii) and (iv), then σ must have property (i), both (ii) and (iii), or both (ii) and (iv). Since $\psi_i^{-1}(a)$ is down-closed, ψ_i is order-preserving. \square

By construction of the ρ_i 's and the ψ_i 's, the critical cells of $\text{Hom}(B_n)$ are the elements of

$$\psi_j^{-1}(b) \subseteq \rho_j^{-1}(r_j) \subseteq \psi_{j-1}^{-1}(b) \subseteq \cdots \subseteq \rho_0^{-1}(r_0) \subseteq \phi^{-1}(j)$$

for all possible permutations $\pi = r_0 r_1 \dots r_{j-1}$ of the set $[j-1]$. Each of these critical cells consists of blocks of an ascending c.f.s. terminating in a transposition, and there may or may not be an ascending c.f.s. without a transposition in the last block. Also, the largest element of each non-terminal block is the last element of the ascending c.f.s. and, in particular, is not in the transposition. Using these observations, we count the number of critical k -cells in $\text{Hom}(B_n)$ after applying the matching defined in Constructions 3.3.11 and 3.3.14.

Theorem 3.3.16. Fix $n \geq 1$, $k \geq 0$. The number of critical k -cells in $\text{Hom}(B_n)$ arising from the above acyclic matching is equal to

$$\sum_{j=0}^{n-3k} \left[\binom{n}{j} \sum_{\mathbf{c}} \left(\binom{n-j}{(c_1+2)(c_2+2)\cdots(c_k+2)} \times \prod_{i=1}^k \binom{c_i+1}{2} \right) \right]$$

where the inner sum is taken over all compositions $c = c_1 + \cdots + c_k$ of $n - j - 2k$ into exactly k parts.

Proof. Again, we consider a critical k -cell and view each transposition and the c.f.s. preceding as a block of information. We consider an additional transposition-less block of the final c.f.s. that may or may not exist.

First, we select j of the n entries to be in the special terminal block lacking a transposition; there are $\binom{n}{j}$ ways to do this. Observe that we need at least $3k$ entries set aside to at least fill out the transpositions and the required free entry preceding each. Thus, $0 \leq j \leq n - 3k$. Now, we know that will need $2k$ entries of the remaining $n - j$ to fill out the transpositions. Observe that the remaining $n - j - 2k$ entries will be free and need to be distributed among the k regular blocks. Notice that the lengths of the c.f.s.'s for the regular blocks will form a composition of $n - j - 2k$ using exactly k parts; there are $\binom{n-j-2k-1}{k-1}$ such compositions. Fix a composition of $n - j - 2k$ with parts $c_1 + c_2 + \cdots + c_k$. Now, block i is going to have a total size of $c_i + 2$, namely c_i free entries and 2 in the transposition. Sort the $n - j$ entries not in the terminal block among the regular blocks; there are $\binom{n-j}{(c_1+2)(c_2+2)\cdots(c_k+2)}$ ways to do this. Now, in block i , there are $c_i + 2$ entries, and we need to select 2 of those entries to be in the transposition. Since the largest value cannot be in the transposition and the non-transposition entries must be in ascending order, there are exactly $\binom{c_i+1}{2}$ ways to properly arrange the entries in block i . Therefore, there are $\binom{n-j}{(c_1+2)(c_2+2)\cdots(c_k+2)} \times \prod_{i=1}^k \binom{c_i+1}{2}$ critical cells

corresponding to a given composition. Summing across all compositions of $n - j - 2k$ into exactly k parts and all values of j gives the result. \square

The following table lists some of the initial values of the array generated by the summation in Theorem 3.3.16.

Table 3.1: Critical cells remaining after the inductive matching

	$k = 0$	1	2	3	4
$n = 3$	1	1			
4	1	7			
5	1	31			
6	1	111	20		
7	1	351	350		
8	1	1023	3541		
9	1	2815	27174	1680	
10	1	7423	175422	54600	
11	1	18943	1005312	986370	
12	1	47103	52779252	1306150	369600

3.4 Applications to 3-Equal Arrangements

The homomorphism complexes for these Boolean algebras have some interesting connections to the complement of 3-equal arrangements in real space.

Definition 3.4.1. For $2 \leq k \leq n$, let $V_{n,k}$ denote the set of points $(x_1, \dots, x_n) \in \mathbb{R}^n$ such that $x_{i_1} = x_{i_2} = \dots = x_{i_k}$ for some k -set of indices $1 \leq i_1 < i_2 < \dots < i_k \leq n$. Then, we have the k -equal manifold $M_{n,k} := \mathbb{R}^n - V_{n,k}$.

As mentioned previously, $\text{Hom}(B_n)$ can be thought of as a cubical subcomplex living entirely in the boundary of the permutohedron of order n .

Theorem 3.4.2. (Björner, [5]; see also [3]) Delete from the boundary of the permutohedron $\mathbf{Z}_n^{\text{perm}}$ every face that contains a d -dimensional permutohedron, $d \geq k - 1$, in its decomposition. Then, the remaining subcomplex has the homotopy type of the complement of the k -equal arrangement.

In the special case $k = 3$, one deletes all cells except those that are products of edges, keeping only the cubical faces, which produces $\text{Hom}(B_n)$. In [6], Björner & Welker obtain rank information about $M_{n,k}$ using the Goresky-MacPherson formula.

Theorem 3.4.3. (Björner & Welker, [6]) The cohomology groups of $M_{n,k}$ are free. Furthermore,

- (a) $M_{n,k}$ is $(k - 3)$ -connected.
- (b) $H^d(M_{n,k})$ is non-trivial if and only if $d = t \cdot (k - 2)$ for some integer t such that $0 \leq t \leq \lfloor n/k \rfloor$.
- (c) $\text{rank } H^{k-2}(M_{n,k}) = \sum_{i=k}^n \binom{n}{i} \cdot \binom{i-1}{k-1}$.

They also give a more explicit statement of (b) in Theorem 3.4.3 as follows, and the reader may refer to [6] for explicit formulas for the exact values of the \tilde{B} 's.

Theorem 3.4.4. (Björner & Welker, [6]) For each partition $\pi \neq \hat{0}$, let m be the number of non-singleton blocks and a_1, \dots, a_m their sizes. Then

$$\text{rank } \tilde{H}^d(M_{n,k}) = \sum_{\pi \in \Pi_{n,k}^{>\hat{0}}} \sum_{q_1 + \dots + q_m = d} \tilde{B}_{a_1,k}^{a_1-3-q_1} \dots \tilde{B}_{a_m,k}^{a_m-3-q_m}$$

Table 3.2, a simplified version of the one presented in [6], lists the Betti numbers of $H^d(M_{n,3})$ for some initial values of n and d .

Table 3.2: Betti numbers for $H^d(M_{n,3})$

	$d = 0$	1	2	3	4
$n = 3$	1	1			
4	1	7			
5	1	32			
6	1	111	20		
7	1	351	350		
8	1	1023	3542		
9	1	2815	27174	1680	
10	1	7423	175422	54600	
11	1	18943	1005312	986370	
12	1	47103	52779252	1306150	369600

Interestingly, the numbers obtained above in Table 3.2 by Björner & Welker correspond exactly with those generated by our critical cell summation in Table 3.1. In summary, we have produced a discrete Morse function on $\text{Hom}(C_{n+1}, B_n)$, which is homotopy-equivalent to $M_{n,3}$, and, for small values of n , the numbers of critical cells in each dimension arising from our $\text{Hom}(B_n)$ matching agree with the ranks of the cohomology groups calculated in [6].

A discrete Morse matching is *optimal* when the number of critical cells produced in each dimension agrees precisely with the dimension of the complex's homology in the corresponding dimensions. Since the cohomology of $M_{n,k}$ is free, homology and cohomology are precisely the same. We end with a final conjecture based on the above agreement in data.

Conjecture 3.4.5. The discrete Morse function defined in Constructions 3.3.11 and 3.3.14 is optimal and has all zero boundary maps in the corresponding Morse complex.

Appendix - Critical cells generated by the Comb Algorithm

The following tables contain the values of C_n^d (see 2.3.1) arising when the Comb Algorithm is applied to $\text{Ind}(\Delta_n^m)$ for the indicated values of m .

$$m = 2$$

$\begin{matrix} n \\ d \end{matrix}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	1																				
1		2																			
2			1	2																	
3					5																
4						4	4														
5							1	12													
6									13	8											
7										6	28										
8											1	38	16								
9													25	64							
10														8	104	32					
11															1	88	144				
12																	41	272	64		
13																		10	280	320	
14																			1	170	688
15																					61

$$m = 3$$

$\begin{matrix} n \\ d \end{matrix}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		
0	1																						
1		1																					
2			1																				
3				1																			
4					1																		
5						1																	
6							1																
7								1															
8									1														
9										1													
10											1												
11												1											
12													1										
13														1									
14															1								
15																1							
16																	1						
17																		1					
18																			1				
19																				1			
20																					1		
21																						1	
22																							1

$$m = 4$$

$\begin{matrix} n \\ d \end{matrix}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
0	1																					
1		1																				
2				1																		
3		1			1																	
4			1	1			1															
5					3			1														
6						2	2			1												
7				1				5			1											
8					2	1			3	3			1									
9						1	5				7			1								
10									7	3		4	4			1						
11								1		3	12			9			1					
12									3	1		15	6		5	5			1			
13										3	7		6	22			11			1		
14											1	15	4		26	10		6	6			
15												1		13	22		10	35			13	
16													4	1	4	42	10		40	15		7
17														6	9		34	50		15	51	
18															4	26	5	10	90	20		57
19															1	1	34	35		70	95	
20																5	1	21	90	15	20	165
21																	10	11	5	110	95	
22																		10	40	6	65	230
23																		1	5	70	51	15
24																			6	2	65	165
25																				15	13	31
26																					20	57
27																					1	15
28																						7

$$m = 5$$

$\begin{matrix} n \\ d \end{matrix}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
0	1																					
1		1																				
2				1																		
3					1																	
4		1					1															
5			1	1				1														
6					3					1												
7						2	2				1											
8								5					1									
9					1				3	3				1								
10						2	1				7					1						
11							1	5				4	4				1					
12									7	3				9					1			
13										3	12				5	5					1	
14								1				15	6				11					
15									3	1			6	22					6	6		
16										3	7				26	10					13	
17											1	15	4			10	35					7
18													13	22					40	15		
19											1			4	42	10				15	51	
20												4	1			34	50					57
21													6	9			10	90	20			
22														4	26	5			70	95		
23															1	34	35			20	165	
24														1			21	90	15			
25															5	1		5	110	95		
26																10	11		65	230		
27																	10	40	6		15	
28																		5	70	51		
29																	1		1	65	165	
30																		6	1			31
31																			15	13		
32																					20	57
33																						15
34																					1	
35																						7

Bibliography

- [1] The On-Line Encyclopedia of Integer Sequences, published electronically at <https://oeis.org>.
- [2] Michał Adamaszek. Special cycles in independence complexes and superfrustration in some lattices. *Topology Appl.*, 160(7):943–950, 2013.
- [3] Hélène Barcelo and Reinhard Laubenbacher. Perspectives on a -homotopy theory and its applications. *Disc. Math.*, 298:39–61, 2005.
- [4] Jonathan Ariel Barmak. Star clusters in independence complexes of graphs. *Adv. Math.*, 241:33–57, 2013.
- [5] Anders Björner. *Random Walks, Arrangements, Cell Complexes, Greedoids, and Self-Organizing Libraries*, pages 165–203. Springer Berlin Heidelberg, Berlin, Heidelberg, 2008.
- [6] Anders Björner and Volkmar Welker. The homology of “ k -equal” manifolds and related partition lattices. *Adv. Math.*, 110:277–313, 1995.
- [7] Serge Bouc. Homologie de certains ensembles de 2-sous-groupes des groupes symétriques. *J. Algebra*, 150(1):158–186, 1992.
- [8] Mireille Bousquet-Mélou, Svante Linusson, and Eran Nevo. On the independence complex of square grids. *J. Algebraic Combin.*, 27(4):423–450, 2008.
- [9] Benjamin Braun. Independence complexes of stable Kneser graphs. *Electron. J. Combin.*, 18(1):P118, (electronic), 2011.
- [10] Xun Dong and Michelle L. Wachs. Combinatorial Laplacian of the matching complex. *Electron. J. Combin.*, 9(1):Research Paper 17, 11 pp. (electronic), 2002.
- [11] Jonathan Earl, Kevin N. Vander Meulen, and Adam Van Tuyl. Independence Complexes of Well-Covered Circulant Graphs. *Exp. Math.*, 25(4):441–451, 2016.
- [12] Richard Ehrenborg and Gábor Hetyei. The topology of the independence complex. *European J. Combin.*, 27(6):906–923, 2006.
- [13] Robin Forman. Morse theory for cell complexes. *Adv. Math.*, 134(1):90–145, 1998.
- [14] Jakob Jonsson. Matching complexes on grids. unpublished manuscript available at <http://www.math.kth.se/~jakobj/doc/thesis/grid.pdf>.
- [15] Jakob Jonsson. *Simplicial complexes of graphs*, volume 1928 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2008.
- [16] Jakob Jonsson. More torsion in the homology of the matching complex. *Experiment. Math.*, 19(3):363–383, 2010.

- [17] Dmitry Kozlov. *Combinatorial algebraic topology*, volume 21 of *Algorithms and Computation in Mathematics*. Springer, Berlin, 2008.
- [18] Dmitry N. Kozlov. A simple proof for folds on both sides in complexes of graph homomorphisms. *Proc. Amer. Math. Soc.*, 135(5):1265–1270, 2005.
- [19] Dmitry N. Kozlov. Chromatic numbers, morphism complexes, and Stiefel-Whitney characteristic classes. In *Geometric combinatorics*, volume 13 of *IAS/Park City Math. Ser.*, pages 249–315. Amer. Math. Soc., Providence, RI, 2007.
- [20] Dmitry N. Kozlov. Discrete Morse theory and Hopf bundles. *Pacific J. Math.*, 249(2):371–376, 2011.
- [21] John Shareshian and Michelle L. Wachs. Top homology of hypergraph matching complexes, p -cycle complexes and Quillen complexes of symmetric groups. *J. Algebra*, 322(7):2253–2271, 2009.
- [22] Siniša T. Vrećica and Rade T. Živaljević. Cycle-free chessboard complexes and symmetric homology of algebras. *European J. Combin.*, 30(2):542–554, 2009.
- [23] Michelle L. Wachs. Topology of matching, chessboard, and general bounded degree graph complexes. *Algebra Universalis*, 49(4):345–385, 2003. Dedicated to the memory of Gian-Carlo Rota.

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