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The story of majorizability as Karamata's condition of convergence for Abel summable series

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Abstract

The greatest Serbian mathematician, Jovan Karamata (1902–1967), gained worldwide fame working on problems related to theorems of a Tauberian nature. His simple and elegant 1930 proof of the Hardy–Littlewood theorem found its place in the well-known monographs by Titchmarsh, Knopp, Doetsch, Widder, Hardy and Favard. It is less known that the method used in this proof was mentioned for the first time at a conference of the Academy of Natural Sciences of the Serbian Royal Academy of Sciences in Belgrade in 1929, where Karamata introduced the notion of majorizability as a new condition of convergence for Abel summable series. This fact holds the key to a historical insight into Karamata's famous proof of the Hardy–Littlewood theorem.

Résumé

Le plus grand mathématicien serbe, Jovan Karamata (1902–1967), est devenu célèbre dans le monde entier avec ses travaux concernant les théorèmes sur les problèmes de la nature Tauberian. Sa preuve du théorème de Hardy–Littlewood de 1930, simple et élégante, a trouvé sa place dans les monographies de Titchmarsh, Knopp, Doetch, Wider, Hardy et Favard. Il est moins connu que la méthode utilisée pour cette preuve était mentionnée pour la première fois pendant les conférences de l'Académie des sciences naturelles de l'Académie royale serbe des sciences en 1929. C'est là que Karamata a introduit la notion de majorabilité comme une nouvelle condition de convergence de séries Abel sommables. Ce fait est la clef pour la compréhension complète, et surtout de son aspect historique, de la fameuse preuve du théorème de Hardy–Littlewood donnée par Karamata.

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1. Introduction

The Serbian mathematician Jovan Karamata (1902–1967) returned to Belgrade in 1928 after completing a one-year study visit to Paris. This marked the beginning of his most important and prolific period of scientific work. His best-known and most original contributions, which made him renowned among mathematicians all over Europe, were written between 1929 and 1933. He continued his mathematical, scientific and teaching activities without interruption until the outbreak of the Second World War. In that period of the greatest strength of his creative power, Karamata worked at a fevered pace. Mathematics was his passion. His obsessive work habits enabled him to extensively explore problems in the theory of series, more precisely theorems of a Tauberian type, as well as to create the theory of regularly varying and slowly varying functions [Tomić, 1968, 1969].

But there is a part of Karamata's work connected with Tauberian theorems that is largely overlooked. That is his notion of majorizability as a condition of convergence for Abel summable series. The aim of this paper is to present his pioneering work on the majorizability of sequences and, in particular, to draw attention to the fact that the core of his famous method used in the proof of the Hardy–Littlewood Tauberian theorem appeared for the first time in Karamata [1931]. Some further developments, up to Erdős and Karamata [1956], his only paper primarily devoted to C-majorizability, are also presented. The fact that this part of Karamata's work has been neglected when compared to his other achievements in mathematics is probably due to the fact that [Karamata, 1931] appeared in Serbian.

2. Theorems of a Tauberian nature

In 1826 Abel formulated the theorem that if the series $\sum_{v=0}^{\infty} a_v$ is convergent with sum s, i.e., if $\lim_{n\to\infty} s_n = s$, where $s_n = \sum_{v=0}^{n} a_v$, and if we take

$$f(x) = \sum_{v=0}^{\infty} a_v x^v,$$

where the series is convergent for $0 < x < 1$, then $f(x) \to s$, when $x \to 1-0$ [Abel, 1826, 1881]. In other words, from the convergence of the series $\sum_{v=0}^{\infty} a_v$ to the sum s follows its summability (later called Abel summability) to the same sum s. The converse is not always true.[1]

There are many different methods of summation and for each of them there is a "limit theorem," determining the final rate of divergence for which a series is summable by the method. Less obvious is the inability to sum divergent series that diverge too slowly. For the essence of Tauberian theory is that, if a summable series diverges too slowly, in the sense of satisfying the Tauberian condition, then it converges. In order to fully understand Karamata's work on theorems of a Tauberian type, it is necessary to refer to Cesàro's method of summability [Cesàro, 1890]. The series $\sum_{v=0}^{\infty} a_v$ is summable in the sense of Cesàro and its sum is s if there is a limit

$$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \frac{\sum_{v=0}^{n} s_v}{n+1} = s,$$

[1] For example, the series $\sum_{n=0}^{\infty} (-1)^n$ is divergent, while $\lim_{x\to 1-} f(x) = \lim_{x\to 1-} \sum_{n=0}^{\infty} (-1)^n x^n = \lim_{x\to 1-} \frac{1}{1+x} = \frac{1}{2}$, which means that the divergent series $\sum_{n=0}^{\infty} (-1)^n$ is Abel summable and its sum is $\frac{1}{2}$.

where $s_n = \sum_{v=0}^{n} a_v$. This definition of summability was first given by Frobenius [1880], but Cesàro in 1890 generalized it and today Frobenius's definition is considered as Cesàro summability of the first order (C_1), and is often designated as C-summability (using C for C_1). Frobenius, in the same paper, showed that if a series is summable in Cesàro's sense it is also Abel summable. In 1910 Hardy showed that if the series $\sum_{v=0}^{\infty} a_v$ is Cesàro summable and meets the additional condition that $va_v = O(1)$, then the series is convergent. Landau replaced that condition with the one-sided condition $na_n \geqslant -K$ [Landau, 1910].

A systematic development of divergent series theory was begun by Emile Borel, who in 1899, departing from Thomas Jan Stieltjes' work of 1882, gave his definition of integral summability, which he immediately applied to the theory of differential equations. The pivotal moment in the development of divergent series theory came in 1897. In that year an Austrian mathematician, Alfred Tauber, proved that convergence follows from Abel summability together with the supplementary condition $va_v = o(1)$ [Tauber, 1897]. By replacing Tauber's condition with the much more general $va_v = O(1)$, Littlewood proved the following theorem:

If the given series $\sum_{v=0}^{\infty} a_v$ is Abel summable, i.e., if

$$\lim_{x\to 1-0} f(x) = \lim_{x\to 1-0} \sum_{v=0}^{\infty} a_v x^v = s,$$

and if the condition

$$va_v = O(1)$$

holds, then

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{v=0}^{n} a_v = s;$$

i.e., the given series is convergent [Littlewood, 1910]. Hardy and Littlewood replaced Tauber's condition with the one-sided condition $va_v \leqslant O(1)$ [Hardy and Littlewood, 1914, 1920]. Such inverse theorems, containing a condition that implies a certain slowness of the possible divergence, were called Tauberian theorems by Hardy and Littlewood.

In spite of the efforts of many mathematicians (Landau, Hardy, R. Schmidt, etc.), the proof of what is now known as the Littlewood or Hardy–Littlewood Tauberian theorem remained very complicated until 1930, when a paper written by Karamata appeared in the journal *Mathematische Zeitschrift* [Karamata, 1930d]. This two-page article created a sensation in mathematical circles, making Karamata world-renowned immediately [Nikolić, 1999, 2003]. Here Karamata, devising a new method, provided a new proof of the theorem, enabling other applications and results to follow, e.g., Tauberian theorems with remainder. The estimate of the remainder represents a numerically important addition to the Hardy–Littlewood theorem. The order of the remainder $|s_n - s| = o(1/\log n)$ was estimated by Postnikov [1951], Freud [1952], and Korevaar [1953]. A complete survey of Tauberian theorems with remainder was given by Ganelius [1971]. In reaction to Karamata's result, K. Knopp observed,

> Previously known proofs of Littlewood's theorem were very complicated, in spite of the amount of research devoted to it, till in 1930 Jovan Karamata found a surprisingly simple proof, [Knopp, 1951, p. 501]

while E.C. Titchmarsh wrote,

> We shall give an extremely elegant proof which has recently been obtained by Karamata. [Titchmarsh, 1939, p. 226]

Karamata's proof, invaluable to the mathematical world, was also reproduced in the well-known monographs by Knopp [1951], Doetsch [1937], Widder [1946], Hardy [1949] and Favard [1962–1963]. Furthermore, it is said that the sensational brevity of the proof diminished the enthusiasm of Hardy and Littlewood for Tauberian theory. It is also worth mentioning that on the occasion of its sixtieth anniversary, the editorial board of *Mathematische Zeitschrift* quoted Karamata's paper in its selection of the 50 most important papers, out of many thousands published [Mathematische Zeitschrift, 1979; Nikolić, 1998, pp. 354–360].

Although Norbert Wiener gave in Wiener [1932] a general theory of Tauberian theorems based upon deep theorems from the theory of Fourier transforms, Karamata's method did not lose its significance. In his paper Wiener gave Karamata's work full mention and credit. Karamata, for his part, studied Wiener's new theory and immediately wrote two papers on it [1933a,b].[2] Wiener's theory was presented in a very precise and concise way and reduced to its simplest form by Pitt [1938].[3] It is not easy to arrive at Karamata's result through Fourier transforms because the auxiliary propositions in the general theory require many explanations. This is why Karamata's proof remains classical and why Wiener says [Wiener, 1932, p. 51], "… the elegant method of Karamata requires only one textbook theorem: the Weierstrass approximation theorem."[4]

A complete bibliography on Tauberian theory from 1889 to 1970 is given in Zeller and Beckmann [1970].

3. Karamata's condition of convergence

From a historical perspective, the paper "Théorèmes inverses de sommabilité I et II" [Karamata, 1931]—originally entitled by its author "On a statement of Landau's," but renamed by Mihailo Petrović and Bogdan Gavrilović, as the paper's reviewers—holds the key to Karamata's proof of the Hardy–Littlewood theorem. It was submitted to the Academy of Natural Sciences of the Serbian Royal Academy of Sciences in Belgrade on the 1st and the 19th April 1929 [Nikolić, 1997]. Tomić and Aljančić describe events leading up to the paper's submission:

> When Landau's monograph *Darstellung und Begründung einiger neuer Ergebnisse der Funktionentheorie* appeared in 1929, Karamata immediately set about studying the Hardy–Littlewood Tauberian theorem. He remembered, he told us, that H. Poincaré almost always made his discoveries on a characteristic special case first. Karamata immediately noticed that the Hardy–Littlewood theorem is almost obvious for the special case of polynomials. That brought to mind the Weierstrass approximation theorem of a function by polynomials. In fact, it all seemed extremely simple to him. He appended a number of historical remarks and similar theorems and asked his Professor Mihailo Petrović, who knew Landau well, for Landau's opinion. Landau *rejected everything* (emphasized by the author) but the Hardy–Littlewood theorem and immediately sent the result to

[2] Wiener's theory, since it has a wide range of applications (from integral equations to the distribution of primes), can be used for obtaining results outside the area of Tauberian theorems. An abstract version was given by Beurling [1945]. Among those who have transferred these abstract ideas to functional spaces we mention I.M. Gelfand, R. Godement, and S.L. Segal. See Korevaar [2004, Ch. V, Sect. 5].

[3] Pitt later wrote the first book on Tauberian theory, in which he also gave due credit to Karamata [Pitt, 1958, Ch. IV]. See also Bingham and Hayman [2008, pp. 263–268].

[4] See also Tomić [2001, p. 9].

Mathematische Zeitschrift. In his reply Landau expressed regret that he had not put off the writing of his monograph until some time later, because the chapter on inverse theorems would have looked differently if he had. [Tomić and Aljančić, 1990, p. 1]

This suggests that Jovan Karamata gave Mihailo Petrović the German version of his paper (which had originally been written in Serbian) and that the part that Landau rejected, i.e., the introduction of majorizability as a Tauberian condition, was actually the principal aim of the paper. In order to test this thesis we shall analyze the paper.

By comparing various Tauberian conditions,

(1) $\frac{1}{n}\sum_{v=1}^{n} va_v \to 0, \quad n \to \infty$ (Tauber),

(2) $a_n = o\left(\frac{1}{n}\right)$ (Tauber),

(3) $a_n = O\left(\frac{1}{n}\right)$ (Littlewood),

(4) $na_n \geqslant -M\ (na_n \leqslant M), \quad n = 1, 2, 3 \ldots$ (Hardy–Littlewood),

under which the convergence of the series follows from its Abel summability, Karamata stated that condition (2) is contained in conditions (1) and (4), condition (3) in condition (4), and that conditions (1) and (4) are not comparable.

The aim of the first part of the paper was to derive a new condition that would contain all four conditions mentioned above. For that purpose Karamata defined and introduced a new notion: majorizability-V, where V designates an arbitrary method of summability.

A sequence of numbers $\{a_v\}$, is one-sidedly majorizable-V (from the left or from the right side) if there exists a sequence of numbers $\{A_v\}$ which is summable-V, and such that

$$a_n \leqslant A_n \quad (\text{or } a_n \geqslant A_n)$$

holds for all $n = 1, 2, 3, \ldots$

In the same way, the sequence $\{a_v\}$ is two-sidedly majorizable-V if there are two V-summable series $\{A'_v\}$ and $\{A''_v\}$ that satisfy

$$A'_n \leqslant a_n \leqslant A''_n \quad \text{for all } n = 1, 2, 3, \ldots$$

Finally, Karamata defined absolute majorizability-V for the sequence of absolute values of the element of the given sequence.

Karamata's original symbolic notation of majorizability and summability was as follows:

$V - \lim A_n = A$ means summability-V of a sequence A_n

$V - \{a_v\} \ll V - \{A_v\}$ or $V - \{a_v\} \gg V - \{A_v\}$ means majorizability-V from left (right)

$V - \{|a_v|\} \ll V - \{A_v\}$ means absolute majorizability-V

$V' \subset V''$ means that all V'-summable sequences are V''-summable (but not the converse).

In order to show that the application of majorizability is a more general Tauberian condition than positivity, Karamata formulated and proved the following, as a "general and almost obvious" statement (p. 13).

Statement 2. Let V' and V'' be two methods of summability such that $V' \subset V''$ and that they are identical for all positive sequences. If a sequence is summable-V'', it is summable-V' if and only if it is one-sidedly or two-sidedly majorizable-V'. (And it is already sufficient that it is absolutely majorizable-V'.)

Karamata called such statements, in which the additional Tauberian condition V''-summability implies V'-summability, or V-summability implies convergence, inverse statements and designated them by $V'' - V'$ and $V - K$.

The identity of summabilities V' and V'' for all positive series is the essential condition that will be used in the special cases of Cesàro and Abel summability $C_2 - C$ and $A - C$ and in the theorem of convergence of series summable in the sense of Cesàro and Abel ($C - K$ and $A - K$).

The identity of summabilities of C and C_2 (Cesàro summability of first and second order) for all positive sequences was proved by Landau [1910], and this is the Landau statement that Karamata wrote about and that he put in the title of his paper. By combining that statement and Statement 2, Karamata formulated one inverse statement $C_2 - C$ with his condition of majorizability (p. 14).

Statement 3. If a sequence a_n is summable-C_2, in order for it to be summable-C, it is necessary and sufficient that it is majorizable-C.

The same statement holds, continued Karamata, if instead of summability C_2 we put summability A, and his proof is based upon the identity of the summabilities A and C for all positive sequences. In proving that identity, Karamata applied for the first time his own method, which was "shorter and easier" than the "rather difficult" Hardy–Littlewood proof [Karamata, 1931, p. 14].

Using these inverse statements, Karamata formulated and proved the inverse statements $C - K$ and $A - K$ for the sequence $s_n = \sum_{v=1}^{n} a_v$, i.e., the series $\sum_{v=1}^{\infty} a_v$, with a new condition more general than the conditions (1)–(4). These new statements are as follows (p. 16, 18).

Statement 6. If the series $\sum_{v=1}^{\infty} a_v$ is summable-C, in order for it to be convergent, it is necessary and sufficient that the sequence $\{va_v\}$ is majorizable-C.

Statement 7. If the series $\sum_{v=1}^{\infty} a_v$ is summable-A, in order for it to be convergent, it is necessary and sufficient that the sequence $\{va_v\}$ is majorizable-C.

Following the structure of the paper—the sequence of formulations of the theorems, the textual part between them, and especially the summary in the French language—it is possible to conclude that the main aim of the first part was to formulate a new condition, majorizability, that would contain frequently used conditions of Tauber and Hardy–Littlewood, and under which the convergence of the series results from its Abel summability (Statement 7). Karamata himself is clear about this in the summary:

> ...Dans cette note, l'auteur s'occupe en particulier des théorèmes inverses $C - K$ et $A - K$ donnant une condition nouvelle qui embrasse comme cas particulier certains théorèmes connus, à savoir les théorèmes $C - K$ de Kronecker–Hardy–Landau et les théorèmes analogues $A - K$ de Tauber–Hardy–Littlewood. Dans ce but l'auteur introduit la notion de majorabilité-V d'une suite... [Karamata, 1931, pp. 21–22]

The proof of identity of A- and C-summabilities for positive sequences using Weierstrass's theorem, which is the keystone for his method of proving the Hardy–Littlewood theorem, was not especially emphasized in comparison to other results whose proofs were clear and logically connected in a very precise way. But it is a fact that Karamata's method of proving the Hardy–Littlewood theorem appeared for the first time in the first part of Karamata [1931], written at the beginning of 1929. During the same year he applied the method in his talks at the First Congress of Mathematicians of the Slavonic Countries in Warsaw [Karamata, 1930a] and at the First Congress of Romanian Mathematicians [Karamata, 1930b].

The main aim of the second part of Karamata [1931] is to show the equivalence of the most general Schmidt condition and Karamata's condition of convergence of the Abel summable series. But, as Karamata mentioned several times in the paper, he did not succeed. He gave two more conditions, which are in his opinion the most convenient and the best suited for the inverse statements of summability. These are Landau's conditions, given in Landau [1913],

$$s_n = \sum_{v=1}^{n} a_v = O(1) \quad \text{when} \quad n \to \infty$$

and

$$\lim_{n=\infty} \sup \max_{n<\mu\leqslant n\lambda} |s_\mu - s_n| = w(\lambda) \to 0 \quad \text{when } 1 < \lambda \to 1,$$

with its generalization given in Schmidt [1925]:

$$\lim_{n=\infty} \sup\{s_\mu - s_n\} \leqslant 0, \quad \text{for all } \mu \geqslant n \quad \text{for which is } \mu \sim n. \tag{$*$}$$

Karamata states that the Schmidt condition is the most general and that it contains all the conditions mentioned so far. However, he did not succeed in completely investigating the relation between that condition and his condition of one-sided majorizability-C for the sequence na_n. He says "it is easy to show that Schmidt's condition $(*)$ is fulfilled whenever the condition of one-sided majorizability for the sequence na_n is satisfied; however, it seems the converse is also the case, i.e., that the two conditions are identical, but this question remains open" [Karamata, 1931, p. 123]. He then gives the necessary condition for one-sided majorizability-C of the sequence a_n,

$$\lim_{n=\infty} \sup \frac{1}{n} \sum_{v=n+1}^{\mu} a_v \leqslant 0 \quad \text{for all } \mu \quad \text{for which } n < \mu \sim n,$$

which he finds is probably necessary and sufficient (which would imply the identity of Schmidt's and Karamata's condition), but he does not succeed in proving it [Karamata, 1931, p. 126].

This statement, continued Karamata, enables us to prove that Schmidt's condition is always satisfied when his (i.e., Karamata's) condition is satisfied. Since he did not find the converse, i.e., that Schmidt's condition is contained in his condition [Karamata, 1931, p. 127], he proves easily and almost parenthetically the statement $A - K$ with Schmidt's condition $(*)$, in a simpler way than Schmidt himself or Vijayaraghavan in [1926]. He proves also the same $A - K$ statement with Landau's condition $s_n = O(1)$, which he generalizes to $s_n < O(1)$. On the basis of Vijayaraghavan's statement that from

$$(1 - r) \sum_{v=1}^{\infty} s_v r^v = O(1) \quad \text{when} \quad r \to 1$$

and

$$s_v - s_n < O(1) \quad \text{for all } n < v < 3n \to \infty$$

follows $s_n < O(1)$ when $n \to \infty$, he finds that this last condition is needless and that it can be omitted [Karamata, 1931, p. 133]. Finally, he proves Vijayaraghavan's statement in a new way as a consequence of several, as he described them, elementary statements.

As the above shows, the second part of Karamata [1931] contains a set of important and difficult theorems proved in a different and simple way. Consequently, like the first part, it too can be considered very significant.

After introducing and defining majorizability as a condition of convergence for Abel and Cesàro summable series in Karamata [1931], Karamata mentioned it in later papers only a few times. The first occasion derives from his presentation at the First Congress of Mathematicians of Romania held in Cluj, 9–12 May, 1929, only a month after the first presentation of Karamata [1931]. His presentation at the Congress was published in the journal *Mathematica* (*Cluj*) under the title "Sur certains "Tauberian theorems" de M.M. Hardy et Littlewood," [Karamata, 1930b]. This paper reached the editorial board on 30th January 1930. Karamata's more famous paper "Sur une mode de croissance régulière des fonctions," [Karamata, 1930c], had arrived three days earlier, on 27th January. In the latter Karamata introduced and defined the notion of a slowly increasing (varying) function, denoting it by $L(x)$. The intent of the paper was to generalize the Tauberian conditions in some inverse theorems of the Tauberian type for the Laplace transform. This result was not recognized and fully evaluated until 1966, when the large Feller monograph on probability theory and its applications appeared [Feller, 1971, p. XIII.5].[5]

The paper Karamata [1931b] was a natural continuation of the studies of $A - C$ statements started in Karamata [1931] with extension to $A(p) - C(p)$. Karamata defined the processes of summabilities $A(p)$ and $C(p)$, which generalize the summability in the sense of both Abel and Cesàro, in the following way:

$$A(p):\ \frac{\sum_{v=0}^{\infty} s_v p_v r^v}{\sum_{v=0}^{\infty} p_v r^v} \to s, \quad \text{when } r \to 1$$

and

$$C(p):\ \frac{\sum_{v=0}^{n} s_v}{\sum_{v=0}^{n} p_v} \to s, \quad \text{when } n \to \infty.$$

Using his own method, but also the properties of regularly and slowly varying functions, he proved equalities analogous to the above, and from them extracted the conclusion on equivalence of $A(p)$ and $C(p)$ methods of summability for all positive sequences. Then he showed that the condition of positivity of sequences could be replaced by the more general condition of one-sided majorizability-$C(p)$, and that the equivalence of the $A(p)$ and $C(p)$ methods still holds.

He next mentioned majorizability in the paper "Über einen Konvergenzsatz des Herrn Knopp" [Karamata, 1935], where he considered similar problems of equivalence of certain conditions of convergence in inverse theorems. He said that he had shown that the condition of majorizability-C was of O-type [Karamata, 1931, p. 143], and that it was contained in Schmidt's O-type condition of convergence

$$s_{n'} - s_n \to 0 \quad \text{for all } n' \sim n \to \infty.$$

He stated,

> It seems that these two conditions of convergence are not equivalent after all; if this were the case we would have in the case of convergence of O-type in majorizability-C an analogue to Kronecker's condition (as necessary and sufficient [author's remark]).
> [Karamata, 1935, p. 425]

[5] For contemporary theory of slowly and regularly varying functions see Bingham and Teugels [1972], Seneta [1976], Bingham et al. [1987], Geluk and de Haan [1987], and "Chapter IV: Karamata's Heritage: Regular Variation" in Korevaar [2004, 177–234].

He considered the question of equivalence of majorizability and Schmidt's Tauberian conditions again in a joint paper with Paul Erdős "Sur la majorabilité C des suites de nombres réel" [Erdős and Karamata, 1956], his only paper completely devoted to the notion of majorizability. In this paper a new notion—majorizability constant—is introduced in the definition of majorizability-C

> A sequence a_i, $i = 1, 2, \ldots$, which is majorizable-C with majorizability constant A, is considered finite if there is a sequence A_i, $i = 1, 2, \ldots$, such that
>
> $$a_i \leqslant A_i, \quad i = 1, 2, \ldots,$$
>
> and
>
> $$\sigma_n \stackrel{\text{def}}{=} \frac{1}{n}\sum_{i=1}^{n} A_i \to A, \quad \text{when } n \to \infty$$
>
> [Erdős and Karamata, 1956, p. 37].

Using the above, they proved various necessary and sufficient conditions concerning the terms of the series a_n that ensure majorizability-C. These conditions are given through the following three theorems:

Theorem 1. The sequence a_i is majorizable-C if and only if

$$\sum_{i=n+1}^{n+k} a_i < o(n), \quad \text{for all } k = o(n), \tag{1}$$

and for all $\varepsilon > 0$ and $m \geqslant (1+\varepsilon)n$

$$A(\varepsilon) \stackrel{\text{def}}{=} \lim_{m,n\to 0} \sup \frac{1}{m-n}\sum_{i=n+1}^{m} a_i \leqslant A < \infty. \tag{2}$$

For such $A(\varepsilon)$, since it is nonincreasing,

$$A^* \stackrel{\text{def}}{=} \lim_{\varepsilon=0} A(\varepsilon) \tag{3}$$

is the smallest majorizability constant.

Theorem 2. The sequence A_i is majorizable-C if and only if the condition (1) is satisfied and with

$$W(\varepsilon) \stackrel{\text{def}}{=} \lim_{n=\infty} \sup \max_{n<n'\leqslant(1+\varepsilon)n} \left\{ \frac{1}{n}\sum_{i=n+1}^{n'} a_i \right\}, \tag{4}$$

it is the case that

$$W(\varepsilon) = O(\varepsilon); \tag{5}$$

in that case we also have

$$\lim_{\varepsilon=0} \frac{W(\varepsilon)}{\varepsilon} = A^*. \tag{6}$$

Theorem 3. Every sequence a_i that satisfies the condition (1) will be majorizable-C if and only if there exists the sequence $m(n) > n$ such that

$$m(n) \sim n \quad \text{when } n \to \infty,$$

and

$$\frac{1}{m-n}\sum_{i=n+1}^{m} a_i \leqslant M \quad \text{for all } n. \tag{7}$$

They show that in these three theorems the condition (1) can be omitted if it is supposed that the sequence a_i is bounded from below, i.e., $a_n \geqslant -M$ for every n, and that condition (1) follows from the conditions (2), (5) with (4) or (7) [Erdős and Karamata, 1956, p. 43].

The first two theorems found their place in Bingham et al. [1987, p. 190, Exercise 17] through the following problem:

> Let (a_n) be a real sequence. There exists (A_n), A with
>
> $$a_n \leqslant A_n, \quad n^{-1}\sum_{1}^{n} A_k \to A$$
>
> if and only if $\liminf_{\varepsilon\downarrow 0} W(\varepsilon)/\varepsilon < \infty$, where
>
> $$W(\varepsilon) \stackrel{\text{def}}{=} \lim_{n\to\infty}\sup \max_{n<n'\leqslant(1+\varepsilon)n} n^{-1}\sum_{i=n+1}^{n'} a_i.$$
>
> In that case the least value of A is
>
> $$A^* = \lim_{\varepsilon\downarrow 0}\omega(\varepsilon)/\varepsilon = \sup_{\varepsilon>0}\omega(\varepsilon)/\varepsilon,$$
>
> where
>
> $$\omega(\varepsilon) \stackrel{\text{def}}{=} \lim_{n\to\infty}\sup n^{-1}\sum_{i=n+1}^{[(1+\varepsilon)n]} a_i, \quad (\varepsilon > 0),$$
>
> and A^* is attainable [Erdős and Karamata, 1956].

In Bingham and Goldie [1982, p. 522], this problem is given as a consequence of a theorem, where it is stated that it extends Theorem 2, and the given expression for A^* is corrected.

Further in Erdős and Karamata [1956, pp. 45–46], it is stated that the majorizability-C of the sequence nu_n, as a condition of convergence (referring to Karamata [1931]) in theorems of a Tauberian nature, is contained in Schmidt's condition

$$\lim_{n=\infty}\sup \max_{n<n'\leqslant(1+\varepsilon)n}\left\{\sum_{v=n+1}^{n'} u_v\right\} \stackrel{\text{def}}{=} \omega(\varepsilon) \to 0 \quad \text{when } \varepsilon \to 0.$$

It is concluded that the condition of majorizability of the sequence nu_n is more restrictive ("La majorabilité C de la suite nu_n est une condition plus restrictive..." [Erdős and Karamata, 1956, p. 45]), since with Schmidt's condition it is sufficient to have

$$\omega(\varepsilon) \to 0 \quad \text{with } \varepsilon,$$

while the majorizability-C of the sequence nu_n requires

$$\omega(\varepsilon) = O(\varepsilon), \quad \text{when } \varepsilon \to 0.$$

This is the consequence of condition (5) and the inequality

$$\sum_{v=n+1}^{n'} u_v \leqslant \frac{n}{n+1} \max_{n<m\leqslant n'} \left\{ \sum_{v=n+1}^{m} v u_v \right\},$$

which according to condition (4) with $nu_n = a_n$ reduces to $\omega(\varepsilon) \leqslant W(\varepsilon)$.

The work of Erdős and Karamata on the majorizability concept served as the background for further research on one-sided Tauberian conditions and relevant Tauberian theorems according to Wiener's Tauberian theory.[6]

In the last part of Erdős and Karamata [1956] the role of the concept of C-majorizability in the proof of the Prime Number Theorem is shown. The concept was used in Karamata [1957] in the proofs of several statements relating to problems concerning prime numbers. Particularly important is, as designated by Karamata, Statement C [Karamata, 1957, p. 29], which contains asymptotic relations known in the theory of numbers,

$$\Psi(x) \stackrel{\text{def}}{=} \sum_{n\leqslant x} \Lambda(n) = x + o(x),$$

$$M(x) \stackrel{\text{def}}{=} \sum_{n\leqslant x} \mu(n) = o(x),$$

$$L(x) \stackrel{\text{def}}{=} \sum_{n\leqslant x} \lambda(n) = o(x), \quad x \to \infty,$$

(where $\Lambda(n), \mu(n)$, and $\lambda(n)$ are the von Mangoldt, Möbius, and Liouville symbols), which are equivalent to the basic Prime Number Theorem.[7] The proof of the statement is based on the fact that the series $\Lambda(n)$ is C-majorizable with majorizability constant 2, which is proved in Erdős and Karamata [1956, pp. 366–367].

It is interesting to mention a mistake of Karamata's in Tauberian theory relating to the Prime Number Theorem. In Karamata [1930b], Karamata proved a theorem (Theorem II) in a case of $A(p)$ summability, but in a similar theorem (Theorem III) he changed to $C(p)$ summability and his proof was not correct. The mistake, which had implications for proofs of the Prime Number Theorem, was noticed by Doetsch, who drew attention to it in Doetsch [1937, pp. 253–255, 414 (footnote 156)].

A survey of summability methods in number theory is given in Karamata [1958].

4. Conclusion

Karamata's [1931] paper is significant for two reasons. First, the core of Karamata's famous method for proving the Hardy–Littlewood theorem appeared there for the first time; and, second, it contains the introduction of a new Tauberian condition called major-

[6] See Bingham et al. [1987, p. 167] and the paper Bingham and Goldie [1983] which originates in Erdős and Karamata [1956] and leads on to Bingham [1984], where the two-sided Tauberian condition $sn = O(1)$ is replaced by the best possible one-sided Tauberian condition

$$\lim_{h\downarrow 0} \liminf_{x\to\infty} \inf_{u\in[0,h]} \frac{1}{h\sqrt{x}} \sum_{x\leqslant n<x+u\sqrt{x}} s_n > -\infty,$$

leading from Borel to Euler summability.

[7] For discussions of these equivalences, see Hardy [1949, 12.11, Appendix]; Hardy and Wright [1979, Chs. 16, 17]; Bingham et al. [1987, pp. 287–297].

izability. The former was separately published in *Mathematische Zeitschrift* [Karamata, 1930d] and brought Karamata immediate fame and recognition. The latter, due to the fact that the paper was written and published in Serbian, was neglected and, in spite of Karamata using it in some later papers, remained almost unknown until Bingham's papers from the 1980s.

Although Tomić and Aljančić [1990] state that Karamata wrote a German version of the original paper and gave it to Mihailo Petrović, who sent it to Edmund Landau, we do not know whether Landau had in his hands the German version of both parts of the paper, since we have not been able to find the letter or Karamata's paper that Petrović sent to Landau. Nevertheless, it is possible that Landau received a German translation of the first part of the paper and that he was so surprised and delighted with the brevity and clarity of the new proof of the Hardy–Littlewood theorem that he sent *Mathematische Zeitschrift* only that result, omitting the material on the new idea of majorizability. Given that it appears to have been in Landau's nature to remove or ignore "unproved assertions or half substantiated claims" [Hardy and Heilbronn, 1938, p. 309], and that the notion of majorizability was not made fully clear until the publication of Erdős and Karamata [1956], this conjecture is not implausible.

From a mathematical point of view, Karamata's majorizability as a condition of convergence for Abel summable series is a very successful idea. It is logically correct, fit for the purpose, and very simply postulated, since operating with it does not demand complicated asymptotics, which is not the case with the other conditions of convergence. It also should be emphasized that although the problem of equivalence of the condition of majorizability and the Schmidt condition was not solved at the time it was postulated, it was completely decided in Erdős and Karamata [1956], written almost 30 years later. The condition of majorizability remains more general than the Tauber and Littlewood conditions, although it is not the most general one, as it (wrongly) seemed to Karamata at the beginning of his research on this topic. Nevertheless, with the publication of Erdős and Karamata [1956], Karamata, who before the Second World War was somewhat disappointed because of his "failure," was able calmly to close this particular chapter of his scientific work.

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